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Upper Bounds for the Modified Second Multiplicative Zagreb Index of Graph Operations

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Abstract. The modified second multiplicative Zagreb index of a connected graph G, denoted by $\prod_2^* (G)$, is defined as $\prod_2^* (G) = \prod_{uv\in E(G)} [d_G(u) + d_G(v)]^{[d_G(u)+d_G(v)]}$ where $d_G(z)$ is the degree of a vertex z in G. In this paper, we present some upper bounds for the modified second multiplicative Zagreb index of graph operations such as union, join, Cartesian product, composition and corona product of graphs are derived.

1. Introduction

Throughout this paper, we consider only simple connected graphs. Let G be such a graph on n vertices and m edges. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. Thus, $|V(G)| = n$ and $|E(G)| = m$. As usual, n is said to be the order and m the size of G. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The degree of a vertex $w \in V(G)$ is the number of vertices adjacent to w and is denoted by $d_G(w)$. We refer to [9] for unexplained terminology and notation.

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modeled by means of molecular-graph-based structure-descriptors, which are also referred to as topological indices [8, 14]. The first and second Zagreb indices, respectively, defined

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 \text{ and } M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$$

are widely studied degree-based topological indices, that were introduced by Gutman and Trinajstic′ [7] in 1972.

In 2013, G. H. Shirdel, H. Rezapour and A. M. Sayadi [13] introduced a new version of Zagreb index named hyper-Zagreb index which is defined for a graph G as

$$HM(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))^2.$$

Recently, Basavanagoud et al. [5] introduced a multiplicative version of index named modified second multiplicative Zagreb index and is defined as

$$\prod_2^* (G) = \prod_{uv\in E(G)} [d_G(u) + d_G(v)]^{[d_G(u)+d_G(v)]}$$

In this paper, we present some upper bounds for the modified second multiplicative Zagreb index of graph operations such as union, join, Cartesian product, composition and corona product of graphs are derived. Readers interested in more information on computing topological indices of graph operations can be referred to [1-4, 6, 10-12].

The following lemma is useful for proving our main results.

Lemma 1.1. *(Weighted AM-GM inequality) Let* $x_1, x_2, \ldots, x_n$ *be nonnegative numbers and also let* $w_1, w_2, \ldots, w_n$ *be nonnegative weights. Set* $w = w_1 + w_2 + \cdots + w_n$. *If* $w > 0$, *then the inequality*

$$\frac{w_1 x_1 + w_2 x_2 + \cdots + w_n x_n}{w} \geq \sqrt[w]{x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}}$$

holds with equality if and only if all the x_k with $w_k > 0$ are equal.

2. Main results

All the operations considered are binary, hence we deal with two finite and simple graphs, G_1 and G_2. For a given graph G_i, its vertex and edge sets will be denoted by $V(G_i)$ and $E(G_i)$, respectively, and their cardinalities by n_i and m_i, respectively, where $i = 1, 2$.

2.1. Union

A union $G_1 \cup G_2$ of the graphs G_1 and G_2 is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $\mathrm{E}(G_1) \cup \mathrm{E}(G_2)$. The degree $d_{G_1 \cup G_2}(u)$ of a vertex u is equal to the degree of u in the component G_i; $i = 1, 2$ that contains it.

The following theorem obvious by definition.

Theorem 2.1. *Let* G_1 *and* G_2 *be two simple graphs. Then*

$$\prod\nolimits_2^* (G_1 \cup G_2) = \prod\nolimits_2^* (G_1) \times \prod\nolimits_2^* (G_2)$$

2.2. Join

The join $G_1 + G_2$ of graphs G_1 and G_2 is a graph with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. If u is a vertex of $G_1 + G_2$ then

$$d_{G_1+G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 & if \;\; u \in V(G_1) \\ d_{G_2}(u) + n_1 & if \;\; u \in V(G_2). \end{cases}$$

Theorem 2.2. *The modified second multiplicative Zagreb index of* $G_1 + G_2$ *satisfies the following inequality:*

$$\prod\nolimits_2^* (G_1 + G_2) \leq \left[\frac{HM(G_1) + 4n_2^2 m_1 + 4n_2 M_1(G_1)}{M_1(G_1) + 2m_1 n_2}\right]^{M_1(G_1) + 2m_1 n_2} \left[\frac{HM(G_2) + 4n_1^2 m_2 + 4n_1 M_1(G_2)}{M_1(G_2) + 2m_2 n_1}\right]^{M_1(G_2) + 2m_2 n_1}$$
$$\times \left[\frac{n_2 M_1(G_1) + n_1 M_1(G_2) + 8m_1 m_2 + n_1 n_2 (n_1 + n_2)^2 + 4(n_1 + n_2)(n_2 m_1 + m_2 n_1)}{2n_2 m_1 + 2m_2 n_1 + n_1 n_2 (n_1 + n_2)}\right]^{2n_2 m_1 + 2m_2 n_1 + n_1 n_2 (n_1 + n_2)} \quad (1)$$

the equality holds in (1) if and only if G_1 and G_2 are regular graphs.

Proof. By definition of modified second multiplicative Zagreb index, we have

$$\prod\nolimits_2^* (G_1 + G_2) = \prod\nolimits_{uv \in E(G_1+G_2)} [d_{G_1+G_2}(u) + d_{G_1+G_2}(v)]^{[d_{G_1+G_2}(u) + d_{G_1+G_2}(v)]}$$
$$= \prod\nolimits_{uv \in E(G_1)} [d_{G_1}(u) + d_{G_1}(v) + 2n_2]^{[d_{G_1}(u) + d_{G_1}(v) + 2n_2]} \prod\nolimits_{uv \in E(G_2)} [d_{G_2}(u)$$
$$+ d_{G_2}(v) + 2n_1]^{[d_{G_2}(u) + d_{G_2}(v) + 2n_1]} \prod\nolimits_{u \in V(G_1)} \prod\nolimits_{v \in V(G_2)} [d_{G_1}(u) + d_{G_2}(v) + n_1$$
$$+ n_2]^{[d_{G_1}(u) + d_{G_2}(v) + n_1 + n_2]}.$$

By Lemma 1.1, we have

$$\prod_2^* (G_1+G_2) \le [\tfrac{1}{A_1}\sum_{uv\in E(G_1)} [d_{G_1}(u)+d_{G_1}(v)+2n_2]^2]^{A_1}[\tfrac{1}{A_2}\sum_{uv\in E(G_2)} [d_{G_2}(u)+d_{G_2}(v)$$
$$+2n_1]^2]^{A_2}[\tfrac{1}{A_3}\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} [d_{G_1}(u)+d_{G_2}(v)+n_1+n_2]^2]^{A_3} \quad (2)$$

where

$$A_1=\sum_{uv\in E(G_1)} [d_{G_1}(u)+d_{G_1}(v)+2n_2]=M_1(G_1)+2m_1n_2$$
$$A_2=\sum_{uv\in E(G_2)} [d_{G_2}(u)+d_{G_2}(v)+2n_1]=M_1(G_2)+2m_2n_1$$
$$A_3=\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} [d_{G_1}(u)+d_{G_2}(v)+n_1+n_2]=2n_2m_1+2m_2n_1+n_1n_2(n_1+n_2).$$

$$\prod_2^* (G_1+G_2) \le [\tfrac{1}{A_1}\sum_{uv\in E(G_1)} [(d_{G_1}(u)+d_{G_1}(v))^2+4n_2^2+4n_2(d_{G_1}(u)+d_{G_1}(v))]]^{A_1}$$
$$\times [\tfrac{1}{A_2}\sum_{uv\in E(G_2)} [(d_{G_2}(u)+d_{G_2}(v))^2+4n_1^2+4n_1(d_{G_2}(u)+d_{G_2}(v))]]^{A_2}$$
$$\times [\tfrac{1}{A_3}\sum_{u\in V(G_1)}\sum_{v\in V(G_2)} [d_{G_1}(u)^2+d_{G_2}(v)^2+2d_{G_1}(u)d_{G_2}(v)+(n_1+n_2)^2$$
$$+2(n_1+n_2)(d_{G_1}(u)+d_{G_2}(v))]]^{A_3}$$
$$=[\frac{1}{A_1}[HM(G_1)+4n_2^2m_1+4n_2M_1(G_1)]]^{A_1}[\frac{1}{A_2}[HM(G_2)+4n_1^2m_2+4n_1M_1(G_2)]]^{A_2}$$
$$\times [\frac{1}{A_3}[n_2M_1(G_1)+n_1M_1(G_2)+8m_1m_2+n_1n_2(n_1+n_2)^2+4(n_1+n_2)(n_2m_1$$
$$+m_2n_1)]]^{A_3}. \quad (3)$$

Substituting A_1, A_2 and A_3 value in (3), we get the inequality (1). The equality holds in (2) if and only if $d_{G_1}(u)+d_{G_1}(v)+2n_2=d_{G_1}(r)+d_{G_1}(s)+2n_2$ for any $uv,rs\in E(G_1)$, $d_{G_2}(u)+d_{G_2}(v)+2n_1=d_{G_2}(r)+d_{G_2}(s)+2n_1$ for any $uv,rs\in E(G_2)$ and $d_{G_1}(u)+d_{G_2}(v)+n_1+n_2=d_{G_1}(r)+d_{G_2}(s)+n_1+n_2$ for any $u,r\in V(G_1), v,s\in V(G_2)$ by Lemma 1.1. This implies that the equality holds in (1) if and only if G_1 and G_2 must be regular graphs.

2.3. Cartesian product

The Cartesian product $G_1\times G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1\times G_2)=V(G_1)\times V(G_2)$ and $(a,x)(b,y)$ is an edge of $G_1\times G_2$ if and only if $[a=b$ and $xy\in E(G_2)]$ or $[x=y$ and $ab\in E(G_1)]$. If (a,x) is a vertex of $G_1\times G_2$ then $d_{G_1\times G_2}((a,x))=d_{G_1}(a)+d_{G_2}(x)$.

Theorem 2.3. *The modified second multiplicative Zagreb index of $G_1\times G_2$ satisfies the following inequality:*

$$\prod_2^* (G_1\times G_2) \le [\tfrac{4m_2M_1(G_1)+n_1HM(G_2)+8m_1M_1(G_2)}{4m_1m_2+n_1M_1(G_2)}]^{4m_1m_2+n_1M_1(G_2)}$$
$$\times [\tfrac{4m_1M_1(G_2)+n_2HM(G_1)+8m_2M_1(G_1)}{4m_1m_2+n_2M_1(G_1)}]^{4m_1m_2+n_2M_1(G_1)} \quad (4)$$

the equality holds in (4) if and only if G_1 and G_2 are regular graphs.

Proof. We have,

$$\prod_2^* (G_1 \times G_2) = \prod_{(u_1,u_2)(v_1,v_2)\in E(G_1\times G_2)} [d_{G_1\times G_2}((u_1,u_2)) + d_{G_1\times G_2}\big((v_1,v_2)\big)]^{[d_{G_1\times G_2}((u_1,u_2))+d_{G_1\times G_2}((v_1,v_2))]}$$

$$= \prod_{u_1\in V(G_1)} \prod_{u_2v_2\in E(G_2)} [(d_{G_1}(u_1) + d_{G_2}(u_2)) + \big(d_{G_1}(u_1) + d_{G_2}(v_2)\big)]^{\left[\left(d_{G_1}(u_1)+d_{G_2}(u_2)\right)+\left(d_{G_1}(u_1)+d_{G_2}(v_2)\right)\right]} \times \prod_{u_2\in V(G_2)} \prod_{u_1v_1\in E(G_1)} [(d_{G_1}(u_1) + d_{G_2}(u_2)) + (d_{G_1}(v_1) + d_{G_2}(u_2))]^{[(d_{G_1}(u_1)+d_{G_2}(u_2))+(d_{G_1}(v_1)+d_{G_2}(u_2))]}$$

$$= \prod_{u_1\in V(G_1)} \prod_{u_2v_2\in E(G_2)} [2d_{G_1}(u_1) + (d_{G_2}(u_2) + d_{G_2}(v_2))]^{[2d_{G_1}(u_1)+(d_{G_2}(u_2)+d_{G_2}(v_2))]}$$

$$\times \prod_{u_2\in V(G_2)} \prod_{u_1v_1\in E(G_1)} [2d_{G_2}(u_2) + (d_{G_1}(u_1) + d_{G_1}(v_1))]^{[2d_{G_2}(u_2)+(d_{G_1}(u_1)+d_{G_1}(v_1))]}.$$

By Lemma 1.1, we have

$$\prod_2^* (G_1 \times G_2) \le [\tfrac{1}{B_1} \sum_{u_1\in V(G_1)} \sum_{u_2v_2\in E(G_2)} [2d_{G_1}(u_1) + (d_{G_2}(u_2) + d_{G_2}(v_2))]^2]^{B_1}$$
$$\times [\tfrac{1}{B_2} \sum_{u_2\in V(G_2)} \sum_{u_1v_1\in E(G_1)} [2d_{G_2}(u_2) + (d_{G_1}(u_1) + d_{G_1}(v_1))]^2]^{B_2} \tag{5}$$

where

$$B_1 = \sum_{u_1\in V(G_1)} \sum_{u_2v_2\in E(G_2)} [2d_{G_1}(u_1) + (d_{G_2}(u_2) + d_{G_2}(v_2))] = 4m_1m_2 + n_1M_1(G_2)$$
$$B_2 = \sum_{u_2\in V(G_2)} \sum_{u_1v_1\in E(G_1)} [2d_{G_2}(u_2) + (d_{G_1}(u_1) + d_{G_1}(v_1))] = 4m_1m_2 + n_2M_1(G_1).$$
$$\prod_2^* (G_1 \times G_2) \le [\tfrac{1}{B_1}[4m_2M_1(G_1) + n_1HM(G_2) + 8m_1M_1(G_2)]]^{B_1}$$
$$\times [\tfrac{1}{B_2}[4m_1M_1(G_2) + n_2HM(G_1) + 8m_2M_1(G_1)]]^{B_2}. \tag{6}$$

Substituting B_1 and B_2 value in (6), we get the inequality (4). Moreover, the equality holds in (5) if and only if $2d_{G_1}(u_1) + d_{G_2}(u_2) + d_{G_2}(v_2) = 2d_{G_1}(x_1) + d_{G_2}(x_2) + d_{G_2}(y_2)$ for any $u_1, x_1 \in V(G_1)$ and u_2v_2 , $x_2y_2 \in E(G_2)$ and $2d_{G_2}(u_2) + d_{G_1}(u_1) + d_{G_1}(v_1) = 2d_{G_2}(x_2) + d_{G_1}(x_1) + d_{G_1}(y_1)$ for any $u_2, x_2 \in V(G_2)$ and u_1v_1, $x_1y_1 \in E(G_1)$ by Lemma 1.1. One can easily see that the equality in (5) if and only if $d_{G_1}(u_1) = d_{G_1}(v_1)$ for any $u_1, v_1 \in V(G_1)$ and $d_{G_2}(u_2) = d_{G_2}(v_2)$ for any $u_2, v_2 \in V(G_2)$. Hence the equality holds in (4) if and only if both G_1 and G_2 are regular graphs.

2.4. Composition

The composition $G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_1, v_1) is adjacent to (u_2, v_2) whenever $[u_1$ is adjacent to $u_2]$ or $[u_1 = u_2$ and v_1 is adjacent to $v_2]$. If (a, b) is a vertex of $G_1[G_2]$ then $d_{G_1[G_2]}((a, b)) = n_2d_{G_1}(a) + d_{G_2}(b)$.

Theorem 2.4 *The modified second multiplicative Zagreb index of $G_1[G_2]$ satisfies the following inequality:*

$$\prod_2^* (G_1[G_2]) \le [\tfrac{n_2^4HM(G_1)+2m_1n_2M_1(G_2)+8m_1m_2^2+8n_2^2m_2M_1(G_1)}{n_2^3M_1(G_1)+4m_1m_2n_2}]^{n_2^3M_1(G_1)+4m_1m_2n_2}$$
$$\times [\tfrac{4n_2^2m_2M_1(G_1)+n_1HM(G_2)+8n_2m_1M_1(G_2)}{4n_1m_1m_2+n_1M_1(G_2)}]^{4n_1m_1m_2+n_1M_1(G_2)} \tag{7}$$

the equality holds in (7) if and only if G_1 and G_2 are regular graphs.

Proof. By definition of modified second multiplicative Zagreb index, we have

$$\prod_2^* (G_1[G_2]) = \prod_{(u_1,u_2)(v_1,v_2)\in E(G_1[G_2])} [d_{G_1[G_2]}((u_1,u_2)) + d_{G_1[G_2]}((v_1,v_2))]^{[d_{G_1[G_2]}((u_1,u_2))+d_{G_1[G_2]}((v_1,v_2))]}$$
$$= \prod_{u_1v_1\in E(G_1)} \prod_{u_2\in V(G_2)} \prod_{v_2\in V(G_2)} [(n_2 d_{G_1}(u_1) + d_{G_2}(u_2)) + (n_2 d_{G_1}(v_1) + d_{G_2}(v_2))]^{[(n_2 d_{G_1}(u_1)+d_{G_2}(u_2))+(n_2 d_{G_1}(v_1)+d_{G_2}(v_2))]} \times \prod_{u_1\in V(G_1)} \prod_{u_2v_2\in E(G_2)} [(n_2 d_{G_1}(u_1) + d_{G_2}(u_2)) + (n_2 d_{G_1}(u_1) + d_{G_2}(v_2))]^{[(n_2 d_{G_1}(u_1)+d_{G_2}(u_2))+(n_2 d_{G_1}(u_1)+d_{G_2}(v_2))]}.$$

By Lemma 1.1, we have

$$\prod_2^* (G_1[G_2]) \le [\tfrac{1}{C_1} \textstyle\sum_{u_1v_1\in E(G_1)} \sum_{u_2\in V(G_2)} \sum_{v_2\in V(G_2)} [n_2(d_{G_1}(u_1) + d_{G_1}(v_1)) + (d_{G_2}(u_2) + d_{G_2}(v_2))]^2]^{C_1} \times [\tfrac{1}{C_2} \sum_{u_1\in V(G_1)} \sum_{u_2v_2\in E(G_2)} [2n_2 d_{G_1}(u_1) + (d_{G_2}(u_2) + d_{G_2}(v_2))]^2]^{C_2} \tag{8}$$

where

$$C_1 = \sum_{u_1v_1\in E(G_1)} \sum_{u_2\in V(G_2)} \sum_{v_2\in V(G_2)} \left[n_2 \left(d_{G_1}(u_1) + d_{G_1}(v_1)\right) + \left(d_{G_2}(u_2) + d_{G_2}(v_2)\right)\right]$$
$$= n_2^3 M_1(G_1) + 4m_1m_2n_2$$
$$C_2 = \sum_{u_1\in V(G_1)} \sum_{u_2v_2\in E(G_2)} [2n_2 d_{G_1}(u_1) + (d_{G_2}(u_2) + d_{G_2}(v_2))] = 4n_1m_1m_2 + n_1M_1(G_2).$$

$$\prod_2^* (G_1[G_2]) \le [\tfrac{1}{C_1}[n_2^4 HM(G_1) + 2m_1n_2M_1(G_2) + 8m_1m_2^2 + 8n_2^2m_2M_1(G_1)]]^{C_1} \times [\tfrac{1}{C_2}[4n_2^2m_2M_1(G_1) + n_1HM(G_2) + 8n_2m_1M_1(G_2)]]^{C_2} \tag{9}$$

which gives the required result in (7) by substituting C_1 and C_2 value in (9). The equality holds in (8) and (9) if and only if $d_{G_1}(u_1) = d_{G_1}(v_1)$ for any $u_1, v_1 \in V(G_1)$ and $d_{G_2}(u_2) = d_{G_2}(v_2)$ for any $u_2, v_2 \in V(G_2)$ by Lemma 1.1. Hence the equality holds in (7) if and only if both G_1 and G_2 are regular graphs.

2.5. Corona product

The corona product $G_1 \circ G_2$ is defined as the graph obtained from G_1 and G_2 by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and then joining by an edge each vertex of the i^{th} copy of G_2 is named (G_2, i) with the i^{th} vertex of G_1. If u is a vertex of $G_1 \circ G_2$ then

$$d_{G_1\circ G_2}(u) = \begin{cases} d_{G_1}(u) + n_2 & if\ \ u \in V(G_1) \\ d_{G_2}(u) + 1 & if\ \ u \in V(G_2, i). \end{cases}$$

Theorem 2.5. *The modified second multiplicative Zagreb index of $G_1 \circ G_2$ satisfies the following inequality:*

$$\prod_2^* (G_1 \circ G_2) \le [\tfrac{HM(G_1)+4n_2^2m_1+4n_2M_1(G_1)}{M_1(G_1)+2m_1n_2}]^{M_1(G_1)+2m_1n_2} [\tfrac{n_1[HM(G_2)+4m_2+4M_1(G_2)]}{n_1M_1(G_2)+2m_2n_1}]^{n_1M_1(G_2)+2m_2n_1} \times [\tfrac{n_2M_1(G_1)+n_1M_1(G_2)+8m_1m_2+n_1n_2(n_2+1)^2+4(n_2+1)(n_2m_1+m_2n_1)}{2n_2m_1+2m_2n_1+n_1n_2(n_2+1)}]^{2n_2m_1+2m_2n_1+n_1n_2(n_2+1)} \tag{10}$$

the equality holds in (10) if and only if G_1 and G_2 are regular graphs.

Proof. We have,

$$\prod\nolimits_2^* (G_1 \circ G_2) = \prod\nolimits_{uv\in E(G_1\circ G_2)} [d_{G_1\circ G_2}(u) + d_{G_1\circ G_2}(v)]^{[d_{G_1\circ G_2}(u)+d_{G_1\circ G_2}(v)]}$$
$$= \prod\nolimits_{uv\in E(G_1)} [d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2]^{[d_{G_1}(u)+n_2+d_{G_1}(v)+n_2]}$$
$$\times \prod\nolimits_{w\in V(G_1)} \prod\nolimits_{uv\in E(G_2)} [d_{G_2}(u) + 1 + d_{G_2}(v) + 1]^{[d_{G_2}(u)+1+d_{G_2}(v)+1]}$$
$$\times \prod\nolimits_{u\in V(G_1)} \prod\nolimits_{v\in V(G_2)} [d_{G_1}(u) + n_2 + d_{G_2}(v) + 1]^{[d_{G_1}(u)+n_2+d_{G_2}(v)+1]}.$$

By Lemma 1.1, we have

$$\prod\nolimits_2^* (G_1 \circ G_2) \le [\tfrac{1}{D_1} \sum\nolimits_{uv\in E(G_1)} [d_{G_1}(u) + d_{G_1}(v) + 2n_2]^2]^{D_1}$$
$$\times [\tfrac{1}{D_2} \sum\nolimits_{w\in V(G_1)} \sum\nolimits_{uv\in E(G_2)} [d_{G_2}(u) + d_{G_2}(v) + 2]^2]^{D_2}$$
$$\times [\tfrac{1}{D_3} \sum\nolimits_{u\in V(G_1)} \sum\nolimits_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v) + n_2 + 1]^2]^{D_3} \quad (11)$$

where

$$D_1 = \sum\nolimits_{uv\in E(G_1)} [d_{G_1}(u) + d_{G_1}(v) + 2n_2] = M_1(G_1) + 2m_1n_2$$
$$D_2 = \sum\nolimits_{w\in V(G_1)} \prod\nolimits_{uv\in E(G_2)} [d_{G_2}(u) + d_{G_2}(v) + 2] = n_1M_1(G_2) + 2m_2n_1$$
$$D_3 = \sum\nolimits_{u\in V(G_1)} \sum\nolimits_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v) + n_2 + 1] = 2n_2m_1 + 2m_2n_1 + n_1n_2(n_2 + 1).$$

$$\prod\nolimits_2^* (G_1 \circ G_2) \le [\tfrac{1}{D_1}[HM(G_1) + 4n_2^2m_1 + 4n_2M_1(G_1)]]^{D_1} \times [\tfrac{1}{D_2}[n_1[HM(G_2) + 4m_2 + 4M_1(G_2)]]]^{D_2} \times [\tfrac{1}{D_3}[n_2M_1(G_1) + n_1M_1(G_2) + 8m_1m_2 + n_1n_2(n_2 + 1)^2 + 4(n_2 + 1)(n_2m_1 + m_2n_1)]]^{D_3}. \quad (12)$$

Substituting D_1, D_2 and D_3 value in (12), we get the inequality (10). The equality holds in (10) if and only if $d_{G_1}(u_i) = d_{G_1}(u_j)$, $u_i, u_j \in V(G_1)$ and $d_{G_2}(v_k) = d_{G_2}(v_l)$, $v_k, v_l \in V(G_2)$, that is, both G_1 and G_2 are regular graphs.

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