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The Forgotten Topological Index of Four Operations on Some Special Graphs

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Abstract. For a graph, the forgotten topological index (F–index) is defined as the sum of cubes of degrees of vertices. In 2009, *Eliasi and Taeri* [M. Eliasi, B. Taeri, Four new sums of graphs and their wiener indices, Discrete Appl. Math. 157 (2009) 794-803] introduced four new sums (F–sums) of graphs. In this paper we study the F–index for the F–sums of some special well-known graphs.

1. Introduction

For a graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, the degree of a vertex v in G is the number of edges incident to v and denoted by $d_G(v)$. In chemical graph theory, a topological index is a number related to a graph which is structurally invariant. One of the oldest most popular and extremely studied topological indices are well–known Zagreb indices first introduced in 1972 by *Gutman and Trinajestic* [6] as follows:

For a graph G with a vertex set $V(G)$ and an edge set $E(G)$, the first and second Zagreb indices are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G^2(v) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)\, d_G(v) \quad \text{respectively.}$$

In [6], beside the first Zagreb index, another topological index defined as

$$F(G) = \sum_{v \in V(G)} d_G^3(v) = \sum_{uv \in E(G)} [d_G^2(u) + d_G^2(v)].$$

However this index, except (implicitly) in a few works about the general first Zagreb index [9,10] and the Zeroth–Order general *Randic index* [8], was not further studied till then, except in a recent article by Furtula and Gutman [5], where they reinvestigated this index and studied some basic properties of this index. They proposed that $F(G)$ be named the forgotten topological index, or shortly the F–index.

The extremal trees that maximize or minimize the F–index is obtained by *Abdo et. al.* in [1]. *De N. et. al.* studied behavior of F–index under several operations and applied their results to find the F–index of different chemically interesting molecular graphs and nano–Structures [3].

In this work we will study the F–index of four operations on Paths, Cycles, Stars and Complete graphs. For this purpose we recall four related graphs as follows:

(a) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a Path of length 2.

(b) $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.

(c) $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

(d) $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for corresponding elements of G.

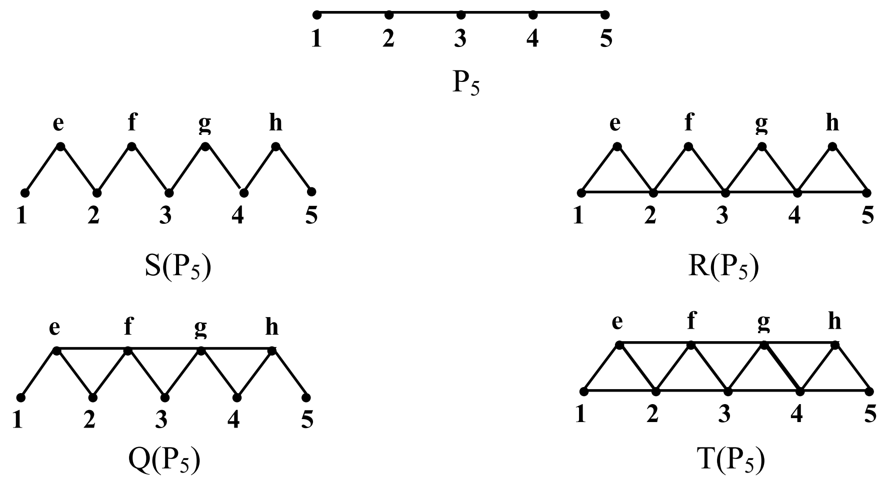

Fig1. P_5, $S(P_5)$, $R(P_5)$, $Q(P_5)$, $T(P_5)$

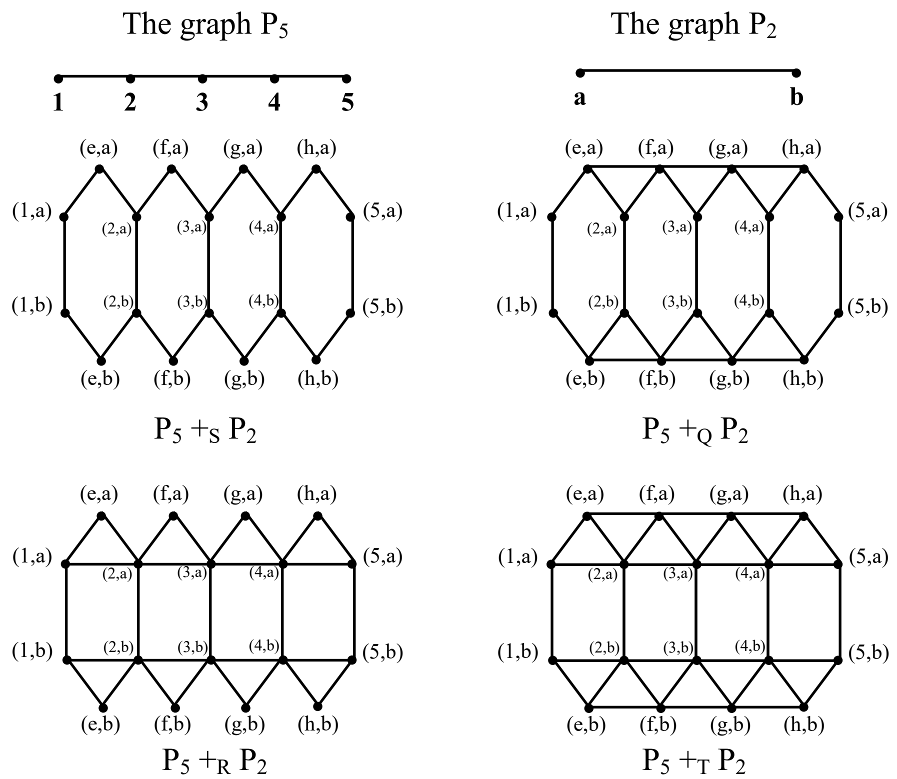

Fig 2. Graphs $P_5 +_F P_2$

The graph $S(G)$ and $T(G)$ are called the subdivision and total graph of G, respectively. For more details on these operations we refer the reader to [2].

If G is P_5, then $S(P_5)$, $R(P_5)$, $Q(P_5)$ and $T(P_5)$ are shown in Fig 1.

Suppose that G_1 and G_2 are two connected graphs. Based on these operations above, *Eliasi and Taeri* [4] introduced four new operations on these graphs in the following:

Let $F \in \{S, R, Q, T\}$. The F–sum of G_1 and G_2, denoted by $G_1 +_F G_2$ is a graph with the set of vertices $V(G_1 +_F G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of

$G_1 +_F G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)\ and\ u_2v_2 \in E(G_2)]$ or$[u_2 = v_2 \in V(G_2)\ and\ u_1v_1 \in E(F(G_1))]$.

$P_5 +_S P_2$, $P_5 +_R P_2$, $P_5 +_Q P_2$ and $P_5 +_T P_2$ are shown in Fig 2.

In [4], *Eliasi and Taeri* obtained the expression for the wiener index $W(G_1 +_F G_2)$ in terms of $W(F(G_1))$ and $W(G_2)$. In [7] *Hanyuan Deng et. al.* obtained the Zagreb indices of four operations on connected graphs. Here, we will study the F–index for the F–sums of Paths, Cycles, Stars and Complete graphs.

2. The F–index for F–sums of some special graphs

In the following four Theorems let G_1 and G_2 be two Path of order n_1 and n_2 respectively with $|G_1| = e_1$ and $|G_2| = e_2$. At first we consider the case $F = S$.

Theorem 1.

$$F(G_1 +_S G_2) = 2e_2M_1(G_1) + 6e_1M_1(G_2) + n_1F(G_2) + n_2F(G_1) + 8e_1n_2 + 16e_2n_1 - 24e_2$$

Proof. Let $d(u,v) = d_{G_1 +_S G_2}(u,v)$ be the degree of vertex (u,v) in the graph $G_1 +_S G_2$.

$$F(G_1 +_S G_2) = \sum_{(u_1,v_1)(u_2,v_2)\in E(G_1 +_S G_2)} [d^2(u_1,v_1) + d^2(u_2,v_2)]$$

$$= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d^2(u,v_1) + d^2(u,v_2)]$$

$$+ \sum_{v\in V(G_2)} \sum_{u_1u_2\in E(S(G_1))} [d^2(u_1,v) + d^2(u_2,v)]$$

$$= I_1 + I_2$$

Then

$$I_1 = \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} [d^2(u,v_1) + d^2(u,v_2)]$$

$$= \sum_{u\in V(G_1)} \sum_{v_1v_2\in E(G_2)} \left[2d^2_{G_1}(u) + 2d_{G_1}(u)\big(d_{G_2}(v_1) + d_{G_2}(v_2)\big) + \left(d^2_{G_2}(v_1) + d^2_{G_2}(v_2)\right)\right]$$

$$= \sum_{u\in V(G_1)} \left[2e_2d^2_{G_1}(u) + 2d_{G_1}(u)M_1(G_2) + F(G_2)\right]$$

$$= 2e_2M_1(G_1) + 4e_1M_1(G_2) + n_1F(G_2)$$

and

$$I_2 = \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(S(G_1))\\ u_1\in V(G_1),u_2\in V(S(G_1))-V(G_1)}} [d^2(u_1,v) + d^2(u_2,v)]$$

$$= \sum_{v\in V(G_2)} \sum_{\substack{u_1u_2\in E(S(G_1))\\ u_1\in V(G_1),u_2\in V(S(G_1))-V(G_1)}} \left[\left(d_{S(G_1)}(u_1) + d_{G_2}(v)\right)^2 + d^2_{S(G_1)}(u_2)\right]$$

$$= \sum_{v \in V(G_2)} \sum_{\substack{u_1u_2 \in E(S(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S(G_1)) - V(G_1)}} \left[\left(d^2_{S(G_1)}(u_1) + d^2_{S(G_1)}(u_2)\right) + d^2_{G_2}(v) + 2d_{S(G_1)}(u_1)d_{G_2}(v)\right]$$

$$= \sum_{v \in V(G_2)} \left[F\big(S(G_1)\big) + 2e_1 d^2_{G_2}(v) + 2(4n_1 - 6)d_{G_2}(v)\right]$$

$$= n_2 F\big(S(G_1)\big) + 2e_1 M_1(G_2) + 4e_2(4n_1 - 6)$$

Hence

$F(G_1 +_S G_2) = 2e_2M_1(G_1) + 4e_1M_1(G_2) + n_1F(G_2) + n_2F\big(S(G_1)\big) + 2e_1M_1(G_2) + 4e_2(4n_1 - 6)$

Note that $F\big(S(G_1)\big) = F(G_1) + 8e_1$. We then have the proof. □

In the next three Theorems let X and Y be the sets of endvertices of Paths G_1 and G_2 respectively. Then $|X| = |Y| = 2$ and $d_{G_1}(x) = d_{G_2}(y) = 1$ for $x \in X$ and $y \in Y$.

Theorem 2.

$$F(G_1 +_R G_2) = 10e_1M_1(G_2) + 8e_2M_1(G_1) + 4(e_2 + 1)F(G_1) + (e_1 + 1)F(G_2) + 112e_1e_2 + 36e_1 - 56e_2 - 24$$

Proof.

$$F(G_1 +_R G_2) = \sum_{(u_1,v_1)(u_2,v_2) \in E(G_1 +_R G_2)} [d^2(u_1, v_1) + d^2(u_2, v_2)]$$

$$= \sum_{u \in X} \sum_{v_1v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)]$$

$$+ \sum_{u \in V(G_1) - X} \sum_{v_1v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)]$$

$$+ \sum_{v \in V(G_2)} \sum_{\substack{u_1u_2 \in E(R(G_1)) \\ u_1, u_2 \in V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)]$$

$$+ \sum_{v \in Y} \sum_{\substack{u_1u_2 \in E(R(G_1)) \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)]$$

$$+ \sum_{v \in V(G_2) - Y} \sum_{\substack{u_1u_2 \in E(R(G_1)) \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)]$$

$$= I_1 + I_2 + I_3 + I_4 + I_5.$$

Then

$$I_1 = \sum_{u \in X} \sum_{v_1v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)]$$

$$= \sum_{u \in X} \sum_{v_1 v_2 \in E(G_2)} \left[8d_{G_1}^2(u) + \left(d_{G_2}^2(v_1) + d_{G_2}^2(v_2)\right) + 4d_{G_1}(u)\left(d_{G_2}(v_1) + d_{G_2}(v_2)\right)\right]$$

$$= \sum_{u \in X} \left[8e_2 d_{G_1}^2(u) + F(G_2) + 4d_{G_1}(u) M_1(G_2)\right]$$
$$= 16e_2 + 2F(G_2) + 8M_1(G_2).$$

Similar to the case I_1 we have

$$I_2 = \sum_{u \in V(G_1) - X} \left[8e_2 d_{G_1}^2(u) + F(G_2) + 4d_{G_1}(u) M_1(G_2)\right]$$

Since for $u \in V(G_1) - X$ there are $n_1 - 2$ vertices of order 2 then
$I_2 = (n_1 - 2)\left(32e_2 + F(G_2) + 8M_1(G_2)\right)$.
Now we have

$$I_3 = \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1, u_2 \in V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)]$$

$$= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1, u_2 \in V(G_1)}} \left[d_{R(G_1)}^2(u_1) + d_{R(G_1)}^2(u_2) + 2d_{G_2}^2(v) + 2d_{G_2}(v)\left(d_{R(G_1)}(u_1) + d_{R(G_1)}(u_2)\right)\right]$$

Note that $u_1, u_2 \in V(G_1)$ and $u_1 u_2 \in E\left(R(G_1)\right)$ iff $u_1 u_2 \in E(G_1)$ and $d_{R(G_1)}(u_i) = 2d_{G_1}(u_i)$, $i = 1,2$. Then

$$I_3 = \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1, u_2 \in V(G_1)}} \left[4\left(d_{G_1}^2(u_1) + d_{G_1}^2(u_2)\right) + 2d_{G_2}^2(v) + 4d_{G_2}(v)\left(d_{G_1}(u_1) + d_{G_1}(u_2)\right)\right]$$

$$= \sum_{v \in v(G_2)} \left[4F(G_1) + 2e_1 d_{G_2}^2(v) + 4d_{G_2}(v) M_1(G_1)\right]$$
$$= 4n_2 F(G_1) + 2e_1 M_1(G_2) + 8e_2 M_1(G_1).$$

$$I_4 = \sum_{v \in Y} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - v(G_1)}} \left[d_{R(G_1)}^2(u_1) + d_{R(G_1)}^2(u_2) + d_{G_2}^2(v) + 2d_{R(G_1)}(u_1) d_{G_2}(v)\right]$$

Since $d_{R(G_1)}(u) = \begin{cases} 2 & if \quad u \in X \\ 4 & if \quad u \in V(G_1) - X \end{cases}$ then

$$\sum_{u_1 u_2 \in E(R(G_1))} d_{R(G_1)}(u_1) = 2 + 4 + \cdots + 4 + 2 = 2 + (2n_1 - 4).4 + 2 = 8n_1 - 12$$

and

$$\sum_{u_1 u_2 \in E(R(G_1))} d_{R(G_1)}^2(u_1) = 32n_1 - 56$$

then

$$I_4 = \sum_{v \in Y} \left[(32n_1 - 56) + 8e_1 + 2e_1 d_{G_2}^2(v) + 2(8n_1 - 12) d_{G_2}(v)\right]$$

$= 2(32n_1 - 56) + 16e_1 + 4e_1 + 4(8n_1 - 12)$
$= 96n_1 + 20e_1 - 160$

Similar to the case I_4 , we have

$$I_5 = \sum_{v \in V(G_2)-Y} \left[(32n_1 - 56) + 8e_1 + 2e_1 d^2_{G_2}(v) + 2(8n_1 - 12)d_{G_2}(v)\right]$$

$$= (n_2 - 2)[(32n_1 - 56) + 8e_1 + 8e_1 + 4(8n_1 - 12)]$$

$$= (n_2 - 2)\,(64n_1 + 16e_1 - 104).$$

Hence the proof.□

Theorem 3.

$$F(G_1 +_Q G_2) = (e_1 + 1)F(G_2) + (e_2 + 1)\big(M_1(G_1) + 2M_2(G_1)\big) + 108e_1e_2 - 82e_2 + 48e_1 - 70$$

Proof.

$$F(G_1 +_Q G_2) = \sum_{(u_1,v_1)(u_2,v_2) \in E(G_1 +_Q G_2)} [d^2(u_1, v_1) + d^2(u_2, v_2)]$$

$$= \sum_{u \in V(G_1)} \sum_{v_1v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)]$$

$$+ \sum_{v \in Y} \sum_{\substack{u_1u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)]$$

$$+ \sum_{v \in V(G_2)-Y} \sum_{\substack{u_1u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)]$$

$$+ \sum_{v \in V(G_2)} \sum_{\substack{u_1u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)] \quad = I_1 + I_2 + I_3 + I_4$$

With respect to the above Theorems its easy to see that

$$I_1 = 24e_1e_2 - 4e_2 - 8e_1 + n_1F(G_2)$$

and

$$I_4 = n_2M_1(G_1) + 2n_2M_2(G_1) + 20n_1n_2 - 56n_2$$

For the case I_2 and I_3 we have

$$I_2 = \sum_{v \in Y} \sum_{\substack{u_1u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} \left[d^2_{Q(G_1)}(u_1) + d^2_{Q(G_1)}(u_2) + d^2_{G_2}(v) + 2d_{G_1}(u_1)d_{G_2}(v)\right]$$

$= 2(8n_1 - 14) + 2(32n_1 - 60) + 4e_1 + 4(4n_1 - 6)$
$= 96n_1 + 4e_1 - 172$

and $I_3 = 56n_1n_2 - 98n_2 - 112n_1 + 8e_1n_2 - 16e_1 + 196.$

Note that $n_1 = e_1 + 1$ and $n_2 = e_2 + 1$ we then have

$$\begin{aligned} F\left(G_1 +_Q G_2\right) &= I_1 + I_2 + I_3 + I_4 \\ &= (e_1+1)F(G_2) + (e_2+1)\big(M_1(G_1) + 2M_2(G_1)\big) \\ &+108e_1e_2 - 82e_2 + 48e_1 - 70. \ \square \end{aligned}$$

Since $d_{G_1 +_T G_2}(u,v) = \begin{cases} d_{G_1 +_R G_2}(u,v) \text{ for } u \in V(G_1) \text{ and } v \in V(G_2) \\ d_{G_1 +_Q G_2}(u,v) \text{ for } u \in V\big(T(G_1)\big) - V(G_1) \text{ and } v \in V(G_2) \end{cases}$

we can get the following Theorem by the proofs of Theorems 2 and 3.

Theorem 4.

$$\begin{aligned} F(G_1 +_T G_2) &= n_1F(G_2) + 4n_2F(G_1) + M_1(G_1)\,(n_2 + 8e_2) + 2M_1(G_2)\,(e_1 + 4n_1 - 4) \\ &+2n_2M_2(G_1) + 32n_1e_2 + 8n_2e_1 + 100n_1n_2 - 196n_2 - 12e_1 - 48e_2 \end{aligned}$$

Applying the above four Theorems we have the following Theorems.

Theorem 5. For $n \geq 3$ and $m \geq 2$,

(a) $F(C_n +_S P_m) = n(72m - 74)$

(b) $F(C_n +_R P_m) = n(224m - 182)$

(c) $F\big(C_n +_Q P_m\big) = n(128m - 74)$

(d) $F(C_n +_T P_m) = n(280m - 182)$

Theorem 6. For $n \geq 4$ and $m \geq 2$

(a) $F\big(K_{1,n-1} +_S P_m\big) = m(n^3 + 3n^2 + 38n - 34) - 2(3n^2 + 22n - 18)$

(b) $F\big(K_{1,n-1} +_R P_m\big) = 8m(n^3 + 9n - 9) - 2(12n^2 + 31n - 36)$

(c) $F\big(K_{1,n-1} +_Q P_m\big) = m(n^4 + 3n^2 + 30n - 26) - 2(3n^2 + 22n - 18)$

(d) $F\big(K_{1,n-1} +_T P_m\big) = m(n^4 + 7n^3 + 64n - 64) - 2(12n^2 + 31n - 36)$

Theorem 7. For $n \geq 3$ and $m \geq 2$,

(a) $F(K_n +_S P_m) = 2n^4 + 4mn(n-1) + n(n+1)^3(m-2)$

(b) $F(K_n +_R P_m) = 2n(2n-1)^3 + 4mn(n-1) + 8n^4(m-2)$

(c) $F\big(K_n +_Q P_m\big) = 2n^4 + 4mn(n-1)^4 + n(n+1)^3(m-2)$

(d) $F(K_n +_T P_m) = \frac{1}{2}mn(n-1)(2n-2)^3 + 2n(2n-1)^3 + 8n^4(m-2)$

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