The Forgotten Topological Index of Four Operations on Some Special Graphs

Sirus Ghobadi 1,a *, Mobina Ghorbaninejad 2,b

1,2 Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran.

a GhobadiMath46@gmail.com, b Ghorbani325@gmail.com

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Abstract. For a graph, the forgotten topological index (F–index) is defined as the sum of cubes of degrees of vertices. In 2009, Eliasi and Taeri [M. Eliasi, B. Taeri, Four new sums of graphs and their wiener indices, Discrete Appl. Math. 157 (2009) 794-803] introduced four new sums (F–sums) of graphs. In this paper we study the F–index for the F–sums of some special well-known graphs.

1. Introduction

For a graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, the degree of a vertex $v$ in $G$ is the number of edges incident to $v$ and denoted by $d_G(v)$. In chemical graph theory, a topological index is a number related to a graph which is structurally invariant. One of the oldest most popular and extremely studied topological indices are well–known Zagreb indices first introduced in 1972 by Gutman and Trinajestic [6] as follows:

For a graph $G$ with a vertex set $V(G)$ and an edge set $E(G)$, the first and second Zagreb indices are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G^3(v) = \sum_{u \in V(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{u \in E(G)} d_G(u) d_G(v)$$

respectively.

In [6], beside the first Zagreb index, another topological index defined as

$$F(G) = \sum_{v \in V(G)} d_G^4(v) = \sum_{u \in E(G)} [d_G^2(u) + d_G^2(v)].$$

However this index, except (implicitly) in a few works about the general first Zagreb index [9,10] and the Zeroth–Order general Randic index [8], was not further studied till then, except in a recent article by Furtula and Gutman [5], where they reinvestigated this index and studied some basic properties of this index. They proposed that $F(G)$ be named the forgotten topological index, or shortly the F–index.

The extremal trees that maximize or minimize the F–index is obtained by Abdo et. al. in [1]. De N. et. al. studied behavior of F–index under several operations and applied their results to find the F–index of different chemically interesting molecular graphs and nano–Structures [3]. In this work we will study the F–index of four operations on Paths, Cycles, Stars and Complete graphs. For this purpose we recall four related graphs as follows:

(a) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of $G$. Equivalently, each edge of $G$ is replaced by a Path of length 2.

(b) $R(G)$ is obtained from $G$ by adding a new vertex corresponding to each edge of $G$, then joining each new vertex to the end vertices of the corresponding edge.

(c) $Q(G)$ is obtained from $G$ by inserting a new vertex into each edge of $G$, then joining with edges those pairs of new vertices on adjacent edges of $G$.
(d) $T(G)$ has as its vertices the edges and vertices of $G$. Adjacency in $T(G)$ is defined as adjacency or incidence for corresponding elements of $G$.

The graph $P_5$ has the following representation:

![Graph P5 Diagram]

The graph $S(P_5)$ is shown below:

![Graph S(P5) Diagram]

The graph $R(P_5)$ is shown below:

![Graph R(P5) Diagram]

The graph $Q(P_5)$ is shown below:

![Graph Q(P5) Diagram]

The graph $T(P_5)$ is shown below:

![Graph T(P5) Diagram]

Fig 1. $P_5$, $S(P_5)$, $R(P_5)$, $Q(P_5)$, $T(P_5)$

The graph $P_5 + S P_2$ is shown below:

![Graph P5 + S P2 Diagram]

The graph $P_5 + Q P_2$ is shown below:

![Graph P5 + Q P2 Diagram]

The graph $P_5 + R P_2$ is shown below:

![Graph P5 + R P2 Diagram]

The graph $P_5 + T P_2$ is shown below:

![Graph P5 + T P2 Diagram]

Fig 2. Graphs $P_5 + F P_2$

The graph $S(G)$ and $T(G)$ are called the subdivision and total graph of $G$, respectively. For more details on these operations we refer the reader to [2].

If $G$ is $P_5$, then $S(P_5)$, $R(P_5)$, $Q(P_5)$ and $T(P_5)$ are shown in Fig 1.

Suppose that $G_1$ and $G_2$ are two connected graphs. Based on these operations above, *Eliasi and Taeri* [4] introduced four new operations on these graphs in the following:

Let $F \in \{S,R,Q,T\}$. The $F$-sum of $G_1$ and $G_2$, denoted by $G_1 +_F G_2$ is a graph with the set of vertices $V(G_1 +_F G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ of
$G_1 + F G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1) \text{ and } u_2 v_2 \in E(G_2)] \text{ or } [u_2 = v_2 \in V(G_2) \text{ and } u_1 v_1 \in E(F(G_1))].$

$P_5 + S P_2, P_3 + R P_2, P_5 + Q P_2$ and $P_5 + T P_2$ are shown in Fig 2.

In [4], Eliasi and Taeri obtained the expression for the wiener index $W(G_1 + F G_2)$ in terms of $W(F(G_1))$ and $W(G_2)$. In [7] Hanyuan Deng et. al. obtained the Zagreb indices of four operations on connected graphs. Here, we will study the $F$–index for the $F$–sums of Paths, Cycles, Stars and Complete graphs.

2. The $F$–index for $F$–sums of some special graphs

In the following four Theorems let $G_1$ and $G_2$ be two Path of order $n_1$ and $n_2$ respectively with $|G_1| = e_1$ and $|G_2| = e_2$. At first we consider the case $F = S$.

**Theorem 1.**

$$F(G_1 + S G_2) = 2e_2 M_1(G_1) + 6e_1 M_1(G_2) + n_1 F(G_2) + n_2 F(G_1) + 8e_1 n_2 + 16e_2 n_1 - 24 e_2$$

**Proof.** Let $d(u, v) = d_G(S G_2(u, v))$ be the degree of vertex $(u, v)$ in the graph $G_1 + S G_2$.

\[
F(G_1 + S G_2) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1 + S G_2)} [d^2(u_1, v_1) + d^2(u_2, v_2)]
\]

\[
= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)]
\]

\[
+ \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} [d^2(u, v) + d^2(u, v)]
\]

\[
= I_1 + I_2
\]

Then

\[
I_1 = \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)]
\]

\[
= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [2d_G^2(u) + 2d_G(u)(d_G(v_1) + d_G(v_2)) + (d_G^2(v_1) + d_G^2(v_2))]
\]

\[
= \sum_{u \in V(G_1)} [2e_2 d_G^2(u) + 2d_G(u)M_1(G_2) + F(G_2)]
\]

\[
= 2e_2 M_1(G_1) + 4e_1 M_1(G_2) + n_1 F(G_2)
\]

and

\[
I_2 = \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} [d^2(u_1, v) + d^2(u_2, v)]
\]

\[
= \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} \left[ (d_{S(G_1)}(u_1) + d_{G_2}(v))^2 + d_{S(G_1)}^2(u_2) \right]
\]
\[
\begin{align*}
&= \sum_{v \in V(G_2)} \sum_{u_1u_2 \in E(S(G_1))} \left( \left[ (d_{S(G_1)}^2(u_1) + d_{S(G_1)}^2(u_2)) + d_{G_2}^2(v) + 2d_{S(G_1)}(u_1)d_{G_2}(v) \right] \right) \\
&= \sum_{v \in V(G_2)} \left[ F(S(G_1)) + 2e_1 d_{G_2}^2(v) + 2(4n_1 - 6)d_{G_2}(v) \right] \\
&= n_2 F(S(G_1)) + 2e_1 M_1(G_2) + 4e_2(4n_1 - 6)
\end{align*}
\]

Hence
\[
F(G_1 +_S G_2) = 2e_2 M_1(G_1) + 4e_1 M_1(G_2) + n_1 F(G_2) + n_2 F(S(G_1)) + 2e_1 M_1(G_2) + 4e_2(4n_1 - 6)
\]

Note that \(F(S(G_1)) = F(G_1) + 8e_1\). We then have the proof. \(\square\)

In the next three Theorems let \(X\) and \(Y\) be the sets of endvertices of Paths \(G_1\) and \(G_2\) respectively. Then \(|X| = |Y| = 2\) and \(d_{G_1}(x) = d_{G_2}(y) = 1\) for \(x \in X\) and \(y \in Y\).

**Theorem 2.**
\[
F(G_1 +_R G_2) = 10e_1 M_1(G_2) + 8e_2 M_1(G_1) + 4(e_2 + 1)F(G_1) + (e_1 + 1)F(G_2) + 112e_1 e_2 + 36e_1 - 56e_2 - 24
\]

**Proof.**
\[
F(G_1 +_R G_2) = \sum_{(u_1,v_1)(u_2,v_2) \in E(G_1 +_R G_2)} [d^2(u_1,v_1) + d^2(u_2,v_2)]
\]

\[= \sum_{u \in X} \sum_{v_1v_2 \in E(G_2)} [d^2(u,v_1) + d^2(u,v_2)]
\]

\[+ \sum_{v \in V(G_1) - X} \sum_{v_1v_2 \in E(G_2)} [d^2(u,v_1) + d^2(u,v_2)]
\]

\[+ \sum_{v \in V(G_2)} \sum_{u_1u_2 \in E(R(G_1))} [d^2(u_1,v) + d^2(u_2,v)]
\]

\[+ \sum_{v \in V(Y)} \sum_{u_1u_2 \in E(R(G_1))} [d^2(u_1,v) + d^2(u_2,v)]
\]

\[= I_1 + I_2 + I_3 + I_4 + I_5.
\]

Then
\[
I_1 = \sum_{u \in X} \sum_{v_1v_2 \in E(G_2)} [d^2(u,v_1) + d^2(u,v_2)]
\]
\[ \sum_{x \in X} \sum_{v_1v_2 \in E(G_2)} \left[ 8d_{G_1}^2(u) + \left( d_{G_2}^2(v_1) + d_{G_2}^2(v_2) \right) + 4d_{G_1}(u) \left( d_{G_2}(v_1) + d_{G_2}(v_2) \right) \right] \]

\[ = \sum_{u \in x} \sum_{v \in X} \left[ 8e_2d_{G_1}^2(u) + F(G_2) + 4d_{G_1}(u)M_1(G_2) \right] \]

\[ = 16e_2 + 2F(G_2) + 8M_1(G_2). \]

Similar to the case \( I_1 \) we have

\[ I_2 = \sum_{u \in V(G_1) - X} \left[ 8e_2d_{G_1}^2(u) + F(G_2) + 4d_{G_1}(u)M_1(G_2) \right] \]

Since for \( u \in V(G_1) - X \) there are \( n_1 - 2 \) vertices of order 2 then

\[ I_2 = (n_1 - 2)(32e_2 + F(G_2) + 8M_1(G_2)). \]

Now we have

\[ I_3 = \sum_{v \in V(G_2)} \sum_{u_1u_2 \in E(R(G_1))} \left[ d^2(u_1, v) + d^2(u_2, v) \right] \]

\[ = \sum_{v \in V(G_2)} \sum_{u_1u_2 \in E(R(G_1))} \left[ d_{R(G_1)}^2(u_1) + d_{R(G_1)}^2(u_2) + 2d_{G_2}(v) + 2d_{G_2}(v) \left( d_{R(G_1)}(u_1) + d_{R(G_1)}(u_2) \right) \right] \]

Note that \( u_1, u_2 \in V(G_1) \) and \( u_1u_2 \in E(R(G_1)) \) if \( u_1u_2 \in E(G_1) \) and \( d_{R(G_1)}(u_i) = 2d_{G_1}(u_i), \ i = 1, 2. \) Then

\[ I_3 = \sum_{v \in V(G_2)} \sum_{u_1u_2 \in E(R(G_1))} \left[ 4 \left( d_{G_1}^2(u_1) + d_{G_1}^2(u_2) \right) + 2d_{G_2}(v) + 4d_{G_2}(v) \left( d_{G_1}(u_1) + d_{G_1}(u_2) \right) \right] \]

\[ = \sum_{v \in V(G_2)} \left[ 4F(G_1) + 2e_1d_{G_2}^2(v) + 4d_{G_2}(v)M_1(G_1) \right] \]

\[ = 4n_2F(G_1) + 2e_1M_1(G_2) + 8e_2M_1(G_1). \]

\[ I_4 = \sum_{v \in Y} \sum_{u_1u_2 \in E(R(G_1))} \left[ d_{R(G_1)}^2(u_1) + d_{R(G_1)}^2(u_2) + d_{G_2}^2(v) + 2d_{R(G_1)}(u_1)d_{G_2}(v) \right] \]

Since \( d_{R(G_1)}(u) = \begin{cases} 2 & \text{if } u \in X \\ 4 & \text{if } u \in V(G_1) - X \end{cases} \) then

\[ \sum_{u_1u_2 \in E(R(G_1))} d_{R(G_1)}(u_1) = 2 + 4 + \cdots + 4 + 2 = 2 + (2n_1 - 4).4 + 2 = 8n_1 - 12 \]

and

\[ \sum_{u_1u_2 \in E(R(G_1))} d_{R(G_1)}^2(u_1) = 32n_1 - 56 \]

then

\[ I_4 = \sum_{v \in Y} \left[ (32n_1 - 56) + 8e_1 + 2e_1d_{G_2}^2(v) + 2(8n_1 - 12)d_{G_2}(v) \right] \]
\( I_5 = \sum_{v \in V(G_2) - Y} [(32n_1 - 56) + 8e_1 + 2e_1 d_{G_2}^2(v) + 2(8n_1 - 12)d_{G_2}(v)] \)

\( = (n_2 - 2)[(32n_1 - 56) + 8e_1 + 8e_1 + 4(8n_1 - 12)] \)

\( = (n_2 - 2)(64n_1 + 16e_1 - 104). \)

Hence the proof. \( \square \)

**Theorem 3.**

\( F(G_1 + Q G_2) = (e_1 + 1)F(G_2) + (e_2 + 1)(M_1(G_1) + 2M_2(G_1)) + 108e_1e_2 - 82e_2 + 48e_1 - 70 \)

**Proof.**

\[
F(G_1 + Q G_2) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1 + Q G_2)} [d^2(u_1, v_1) + d^2(u_2, v_2)]
\]

\[
= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)]
\]

\[
+ \sum_{v \in V(G_2) - Y} \sum_{u_1, u_2 \in E(Q(G_1))} [d^2(u_1, v) + d^2(u_2, v)]
\]

\[
+ \sum_{v \in V(G_2) - Y} \sum_{u_1, u_2 \in E(Q(G_1)) - V(G_1)} [d^2(u_1, v) + d^2(u_2, v)] = I_1 + I_2 + I_3 + I_4
\]

With respect to the above Theorems it’s easy to see that

\( I_1 = 24e_1e_2 - 4e_2 - 8e_1 + n_1F(G_2) \)

and

\( I_4 = n_2M_1(G_1) + 2n_2M_2(G_1) + 20n_1n_2 - 56n_2 \)

For the case \( I_2 \) and \( I_3 \) we have

\[
I_2 = \sum_{v \in V} \sum_{u_1, u_2 \in E(Q(G_1))} [d_{Q(G_1)}^2(u_1) + d_{Q(G_1)}^2(u_2) + d_{G_2}^2(v) + 2d_{G_1}(u_1)d_{G_2}(v)]
\]

\[
= 2(8n_1 - 14) + 2(32n_1 - 60) + 4e_1 + 4(4n_1 - 6)
\]

\[
= 96n_1 - 16e_1 - 172
\]

and \( I_3 = 56n_1n_2 - 98n_2 + 112n_1 + 8e_1n_2 - 16e_1 + 196. \)

Note that \( n_1 = e_1 + 1 \) and \( n_2 = e_2 + 1 \) we then have
\[ F(G_1 + Q G_2) = I_1 + I_2 + I_3 + I_4 \]
\[ = (e_1 + 1)F(G_2) + (e_2 + 1)(M_1(G_1) + 2M_2(G_1)) \]
\[ + 108e_1e_2 - 82e_2 + 48e_1 - 70. \]

Since \( d_{G_1 + T G_2}(u, v) = \begin{cases} 
  d_{G_1 + R G_2}(u, v) & \text{for } u \in V(G_1) \text{ and } v \in V(G_2) \\
  d_{G_1 + Q G_2}(u, v) & \text{for } u \in V(T(G_1)) - V(G_1) \text{ and } v \in V(G_2)
\end{cases} \)

we can get the following Theorem by the proofs of Theorems 2 and 3.

**Theorem 4.**
\[ F(G_1 + T G_2) = n_1F(G_2) + 4n_2F(G_1) + M_1(G_1) (n_2 + 8e_2) + 2M_2(G_2) (e_1 + 4n_1 - 4) + 2n_2M_2(G_1) + 32n_1e_2 + 8n_2e_1 + 100n_1n_2 - 196n_2 - 12e_1 - 48e_2 \]

Applying the above four Theorems we have the following Theorems.

**Theorem 5.** For \( n \geq 3 \) and \( m \geq 2 \),
(a) \( F(C_n + S P_m) = n(72m - 74) \)
(b) \( F(C_n + R P_m) = n(224m - 182) \)
(c) \( F(C_n + Q P_m) = n(128m - 74) \)
(d) \( F(C_n + T P_m) = n(280m - 182) \)

**Theorem 6.** For \( n \geq 4 \) and \( m \geq 2 \),
(a) \( F(K_{1,n-1} + S P_m) = m(n^3 + 3n^2 + 38n - 34) - 2(3n^2 + 22n - 18) \)
(b) \( F(K_{1,n-1} + R P_m) = 8m(n^3 + 9n - 9) - 2(12n^2 + 31n - 36) \)
(c) \( F(K_{1,n-1} + Q P_m) = m(n^4 + 3n^2 + 30n - 26) - 2(3n^2 + 22n - 18) \)
(d) \( F(K_{1,n-1} + T P_m) = m(n^4 + 7n^3 + 64n - 64) - 2(12n^2 + 31n - 36) \)

**Theorem 7.** For \( n \geq 3 \) and \( m \geq 2 \),
(a) \( F(K_n + S P_m) = 2n^4 + 4mn(n - 1) + n(n + 1)^3(m - 2) \)
(b) \( F(K_n + R P_m) = 2n(2n - 1)^3 + 4mn(n - 1) + 8n^4(m - 2) \)
(c) \( F(K_n + Q P_m) = 2n^4 + 4mn(n - 1)^4 + n(n + 1)^3(m - 2) \)
(d) \( F(K_n + T P_m) = \frac{1}{2}mn(n - 1)(2n - 2)^3 + 2n(2n - 1)^3 + 8n^4(m - 2) \)

**References**


