

The Forgotten Topological Index of Four Operations on Some Special Graphs

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Abstract. For a graph, the forgotten topological index (F–index) is defined as the sum of cubes of degrees of vertices. In 2009, *Eliasi and Taeri* [M. Eliasi, B. Taeri, Four new sums of graphs and their wiener indices, Discrete Appl. Math. 157 (2009) 794-803] introduced four new sums (F–sums) of graphs. In this paper we study the F–index for the F–sums of some special well-known graphs.

1. Introduction

For a graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, the degree of a vertex v in G is the number of edges incident to v and denoted by $d_G(v)$. In chemical graph theory, a topological index is a number related to a graph which is structurally invariant. One of the oldest most popular and extremely studied topological indices are well-known Zagreb indices first introduced in 1972 by *Gutman and Trinajestic* [6] as follows:

For a graph G with a vertex set $V(G)$ and an edge set $E(G)$, the first and second Zagreb indices are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G^2(v) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v) \text{ respectively.}$$

In [6], beside the first Zagreb index, another topological index defined as

$$F(G) = \sum_{v \in V(G)} d_G^3(v) = \sum_{uv \in E(G)} [d_G^2(u) + d_G^2(v)].$$

However this index, except (implicitly) in a few works about the general first Zagreb index [9,10] and the Zeroth–Order general *Randic index* [8], was not further studied till then, except in a recent article by *Furtula and Gutman* [5], where they reinvestigated this index and studied some basic properties of this index. They proposed that $F(G)$ be named the forgotten topological index, or shortly the F–index.

The extremal trees that maximize or minimize the F–index is obtained by *Abdo et. al.* in [1]. *De N. et. al.* studied behavior of F–index under several operations and applied their results to find the F–index of different chemically interesting molecular graphs and nano–Structures [3].

In this work we will study the F–index of four operations on Paths, Cycles, Stars and Complete graphs. For this purpose we recall four related graphs as follows:

- $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G . Equivalently, each edge of G is replaced by a Path of length 2.
- $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G , then joining each new vertex to the end vertices of the corresponding edge.
- $Q(G)$ is obtained from G by inserting a new vertex into each edge of G , then joining with edges those pairs of new vertices on adjacent edges of G .

(d) $T(G)$ has as its vertices the edges and vertices of G . Adjacency in $T(G)$ is defined as adjacency or incidence for corresponding elements of G .

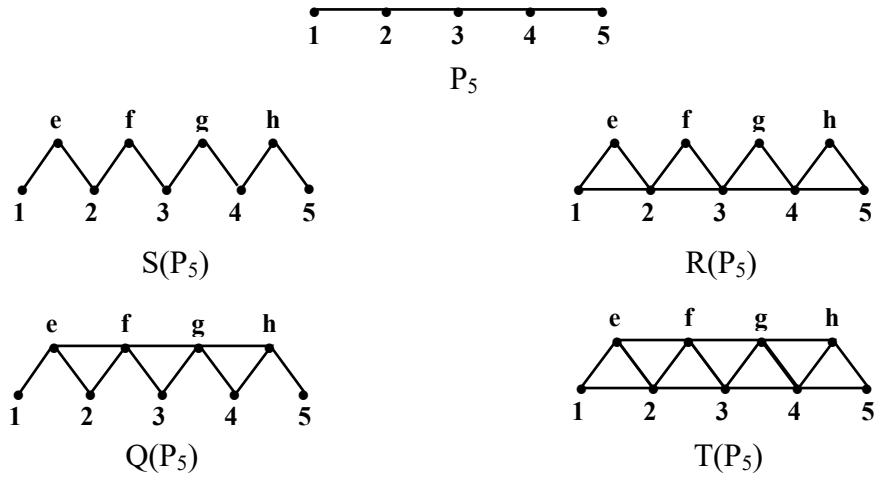


Fig1. $P_5, S(P_5), R(P_5), Q(P_5), T(P_5)$

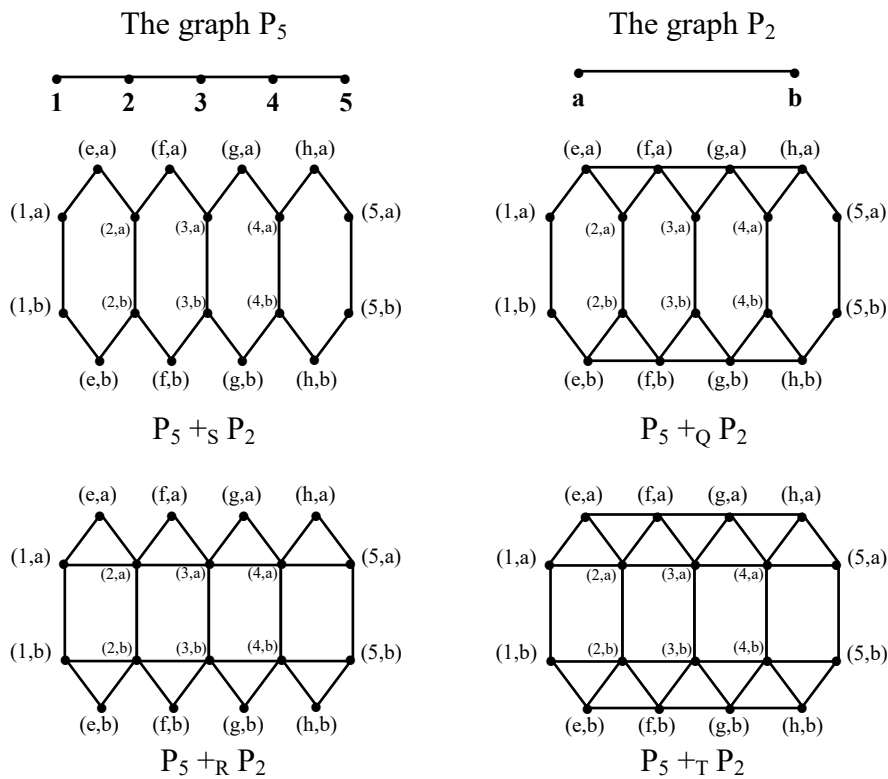


Fig 2. Graphs $P_5 +_F P_2$

The graph $S(G)$ and $T(G)$ are called the subdivision and total graph of G , respectively. For more details on these operations we refer the reader to [2].

If G is P_5 , then $S(P_5), R(P_5), Q(P_5)$ and $T(P_5)$ are shown in Fig 1.

Suppose that G_1 and G_2 are two connected graphs. Based on these operations above, *Eliasi and Taeri* [4] introduced four new operations on these graphs in the following:

Let $F \in \{S, R, Q, T\}$. The F-sum of G_1 and G_2 , denoted by $G_1 +_F G_2$ is a graph with the set of vertices $V(G_1 +_F G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of

$G_1 +_F G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1) \text{ and } u_2 v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2) \text{ and } u_1 v_1 \in E(F(G_1))]$.

$P_5 +_S P_2$, $P_5 +_R P_2$, $P_5 +_Q P_2$ and $P_5 +_T P_2$ are shown in Fig 2.

In [4], *Eliasi and Taeri* obtained the expression for the wiener index $W(G_1 +_F G_2)$ in terms of $W(F(G_1))$ and $W(G_2)$. In [7] *Hanyuan Deng et. al.* obtained the Zagreb indices of four operations on connected graphs. Here, we will study the F-index for the F-sums of Paths, Cycles, Stars and Complete graphs.

2. The F-index for F-sums of some special graphs

In the following four Theorems let G_1 and G_2 be two Path of order n_1 and n_2 respectively with $|G_1| = e_1$ and $|G_2| = e_2$. At first we consider the case $F = S$.

Theorem 1.

$$F(G_1 +_S G_2) = 2e_2 M_1(G_1) + 6e_1 M_1(G_2) + n_1 F(G_2) + n_2 F(G_1) + 8e_1 n_2 + 16e_2 n_1 - 24e_2$$

Proof. Let $d(u, v) = d_{G_1 +_S G_2}(u, v)$ be the degree of vertex (u, v) in the graph $G_1 +_S G_2$.

$$\begin{aligned} F(G_1 +_S G_2) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1 +_S G_2)} [d^2(u_1, v_1) + d^2(u_2, v_2)] \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)] \\ &+ \sum_{v \in V(G_2)} \sum_{u_1 u_2 \in E(S(G_1))} [d^2(u_1, v) + d^2(u_2, v)] \\ &= I_1 + I_2 \end{aligned}$$

Then

$$\begin{aligned} I_1 &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)] \\ &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [2d_{G_1}^2(u) + 2d_{G_1}(u)(d_{G_2}(v_1) + d_{G_2}(v_2)) + (d_{G_2}^2(v_1) + d_{G_2}^2(v_2))] \\ &= \sum_{u \in V(G_1)} [2e_2 d_{G_1}^2(u) + 2d_{G_1}(u)M_1(G_2) + F(G_2)] \\ &= 2e_2 M_1(G_1) + 4e_1 M_1(G_2) + n_1 F(G_2) \end{aligned}$$

and

$$\begin{aligned} I_2 &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(S(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)] \\ &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(S(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S(G_1)) - V(G_1)}} \left[(d_{S(G_1)}(u_1) + d_{G_2}(v))^2 + d_{S(G_1)}^2(u_2) \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(S(G_1)) \\ u_1 \in V(G_1), u_2 \in V(S(G_1)) - V(G_1)}} \left[\left(d_{S(G_1)}^2(u_1) + d_{S(G_1)}^2(u_2) \right) + d_{G_2}^2(v) + 2d_{S(G_1)}(u_1)d_{G_2}(v) \right] \\
&= \sum_{v \in V(G_2)} \left[F(S(G_1)) + 2e_1 d_{G_2}^2(v) + 2(4n_1 - 6)d_{G_2}(v) \right] \\
&= n_2 F(S(G_1)) + 2e_1 M_1(G_2) + 4e_2(4n_1 - 6)
\end{aligned}$$

Hence

$$F(G_1 +_S G_2) = 2e_2 M_1(G_1) + 4e_1 M_1(G_2) + n_1 F(G_2) + n_2 F(S(G_1)) + 2e_1 M_1(G_2) + 4e_2(4n_1 - 6)$$

Note that $F(S(G_1)) = F(G_1) + 8e_1$. We then have the proof. \square

In the next three Theorems let X and Y be the sets of endvertices of Paths G_1 and G_2 respectively. Then $|X| = |Y| = 2$ and $d_{G_1}(x) = d_{G_2}(y) = 1$ for $x \in X$ and $y \in Y$.

Theorem 2.

$$\begin{aligned}
F(G_1 +_R G_2) &= 10e_1 M_1(G_2) + 8e_2 M_1(G_1) + 4(e_2 + 1)F(G_1) + (e_1 + 1)F(G_2) \\
&\quad + 112e_1 e_2 + 36e_1 - 56e_2 - 24
\end{aligned}$$

Proof.

$$\begin{aligned}
F(G_1 +_R G_2) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1 +_R G_2)} [d^2(u_1, v_1) + d^2(u_2, v_2)] \\
&= \sum_{u \in X} \sum_{v_1 v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)] \\
&\quad + \sum_{u \in V(G_1) - X} \sum_{v_1 v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)] \\
&\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1, u_2 \in V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)] \\
&\quad + \sum_{v \in Y} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)] \\
&\quad + \sum_{v \in V(G_2) - Y} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)] \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Then

$$I_1 = \sum_{u \in X} \sum_{v_1 v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)]$$

$$\begin{aligned}
 &= \sum_{u \in X} \sum_{v_1 v_2 \in E(G_2)} \left[8d_{G_1}^2(u) + (d_{G_2}^2(v_1) + d_{G_2}^2(v_2)) + 4d_{G_1}(u)(d_{G_2}(v_1) + d_{G_2}(v_2)) \right] \\
 &= \sum_{u \in X} [8e_2 d_{G_1}^2(u) + F(G_2) + 4d_{G_1}(u)M_1(G_2)] \\
 &= 16e_2 + 2F(G_2) + 8M_1(G_2).
 \end{aligned}$$

Similar to the case I_1 we have

$$I_2 = \sum_{u \in V(G_1) - X} [8e_2 d_{G_1}^2(u) + F(G_2) + 4d_{G_1}(u)M_1(G_2)]$$

Since for $u \in V(G_1) - X$ there are $n_1 - 2$ vertices of order 2 then

$$I_2 = (n_1 - 2)(32e_2 + F(G_2) + 8M_1(G_2)).$$

Now we have

$$\begin{aligned}
 I_3 &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1, u_2 \in V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)] \\
 &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1, u_2 \in V(G_1)}} [d_{R(G_1)}^2(u_1) + d_{R(G_1)}^2(u_2) + 2d_{G_2}^2(v) + 2d_{G_2}(v)(d_{R(G_1)}(u_1) + d_{R(G_1)}(u_2))]
 \end{aligned}$$

Note that $u_1, u_2 \in V(G_1)$ and $u_1 u_2 \in E(R(G_1))$ iff $u_1 u_2 \in E(G_1)$ and $d_{R(G_1)}(u_i) = 2d_{G_1}(u_i)$, $i = 1, 2$. Then

$$\begin{aligned}
 I_3 &= \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1, u_2 \in V(G_1)}} [4(d_{G_1}^2(u_1) + d_{G_1}^2(u_2)) + 2d_{G_2}^2(v) + 4d_{G_2}(v)(d_{G_1}(u_1) + d_{G_1}(u_2))] \\
 &= \sum_{v \in V(G_2)} [4F(G_1) + 2e_1 d_{G_2}^2(v) + 4d_{G_2}(v)M_1(G_1)] \\
 &= 4n_2 F(G_1) + 2e_1 M_1(G_2) + 8e_2 M_1(G_1).
 \end{aligned}$$

$$I_4 = \sum_{v \in Y} \sum_{\substack{u_1 u_2 \in E(R(G_1)) \\ u_1 \in V(G_1), u_2 \in V(R(G_1)) - v(G_1)}} [d_{R(G_1)}^2(u_1) + d_{R(G_1)}^2(u_2) + d_{G_2}^2(v) + 2d_{R(G_1)}(u_1)d_{G_2}(v)]$$

Since $d_{R(G_1)}(u) = \begin{cases} 2 & \text{if } u \in X \\ 4 & \text{if } u \in V(G_1) - X \end{cases}$ then

$$\sum_{u_1 u_2 \in E(R(G_1))} d_{R(G_1)}(u_1) = 2 + 4 + \dots + 4 + 2 = 2 + (2n_1 - 4) \cdot 4 + 2 = 8n_1 - 12$$

and

$$\sum_{u_1 u_2 \in E(R(G_1))} d_{R(G_1)}^2(u_1) = 32n_1 - 56$$

then

$$I_4 = \sum_{v \in Y} [(32n_1 - 56) + 8e_1 + 2e_1 d_{G_2}^2(v) + 2(8n_1 - 12)d_{G_2}(v)]$$

$$\begin{aligned}
&= 2(32n_1 - 56) + 16e_1 + 4e_1 + 4(8n_1 - 12) \\
&= 96n_1 + 20e_1 - 160
\end{aligned}$$

Similar to the case I_4 , we have

$$\begin{aligned}
I_5 &= \sum_{v \in V(G_2) - Y} [(32n_1 - 56) + 8e_1 + 2e_1 d_{G_2}^2(v) + 2(8n_1 - 12)d_{G_2}(v)] \\
&= (n_2 - 2)[(32n_1 - 56) + 8e_1 + 8e_1 + 4(8n_1 - 12)] \\
&= (n_2 - 2)(64n_1 + 16e_1 - 104).
\end{aligned}$$

Hence the proof. \square

Theorem 3.

$$\begin{aligned}
F(G_1 +_Q G_2) &= (e_1 + 1)F(G_2) + (e_2 + 1)(M_1(G_1) + 2M_2(G_1)) \\
&\quad + 108e_1e_2 - 82e_2 + 48e_1 - 70
\end{aligned}$$

Proof.

$$\begin{aligned}
F(G_1 +_Q G_2) &= \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1 +_Q G_2)} [d^2(u_1, v_1) + d^2(u_2, v_2)] \\
&= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d^2(u, v_1) + d^2(u, v_2)] \\
&\quad + \sum_{v \in Y} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)] \\
&\quad + \sum_{v \in V(G_2) - Y} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)] \\
&\quad + \sum_{v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1, u_2 \in V(Q(G_1)) - V(G_1)}} [d^2(u_1, v) + d^2(u_2, v)] = I_1 + I_2 + I_3 + I_4
\end{aligned}$$

With respect to the above Theorems its easy to see that

$$I_1 = 24e_1e_2 - 4e_2 - 8e_1 + n_1F(G_2)$$

and

$$I_4 = n_2M_1(G_1) + 2n_2M_2(G_1) + 20n_1n_2 - 56n_2$$

For the case I_2 and I_3 we have

$$\begin{aligned}
I_2 &= \sum_{v \in Y} \sum_{\substack{u_1 u_2 \in E(Q(G_1)) \\ u_1 \in V(G_1), u_2 \in V(Q(G_1)) - V(G_1)}} [d_{Q(G_1)}^2(u_1) + d_{Q(G_1)}^2(u_2) + d_{G_2}^2(v) + 2d_{G_1}(u_1)d_{G_2}(v)] \\
&= 2(8n_1 - 14) + 2(32n_1 - 60) + 4e_1 + 4(4n_1 - 6) \\
&= 96n_1 + 4e_1 - 172
\end{aligned}$$

and $I_3 = 56n_1n_2 - 98n_2 - 112n_1 + 8e_1n_2 - 16e_1 + 196$.

Note that $n_1 = e_1 + 1$ and $n_2 = e_2 + 1$ we then have

$$\begin{aligned}
 F(G_1 +_Q G_2) &= I_1 + I_2 + I_3 + I_4 \\
 &= (e_1 + 1)F(G_2) + (e_2 + 1)(M_1(G_1) + 2M_2(G_1)) \\
 &\quad + 108e_1e_2 - 82e_2 + 48e_1 - 70. \quad \square
 \end{aligned}$$

$$\text{Since } d_{G_1 +_T G_2}(u, v) = \begin{cases} d_{G_1 +_R G_2}(u, v) & \text{for } u \in V(G_1) \text{ and } v \in V(G_2) \\ d_{G_1 +_Q G_2}(u, v) & \text{for } u \in V(T(G_1)) - V(G_1) \text{ and } v \in V(G_2) \end{cases}$$

we can get the following Theorem by the proofs of Theorems 2 and 3.

Theorem 4.

$$\begin{aligned}
 F(G_1 +_T G_2) &= n_1F(G_2) + 4n_2F(G_1) + M_1(G_1)(n_2 + 8e_2) + 2M_1(G_2)(e_1 + 4n_1 - 4) \\
 &\quad + 2n_2M_2(G_1) + 32n_1e_2 + 8n_2e_1 + 100n_1n_2 - 196n_2 - 12e_1 - 48e_2
 \end{aligned}$$

Applying the above four Theorems we have the following Theorems.

Theorem 5. For $n \geq 3$ and $m \geq 2$,

- (a) $F(C_n +_S P_m) = n(72m - 74)$
- (b) $F(C_n +_R P_m) = n(224m - 182)$
- (c) $F(C_n +_Q P_m) = n(128m - 74)$
- (d) $F(C_n +_T P_m) = n(280m - 182)$

Theorem 6. For $n \geq 4$ and $m \geq 2$

- (a) $F(K_{1,n-1} +_S P_m) = m(n^3 + 3n^2 + 38n - 34) - 2(3n^2 + 22n - 18)$
- (b) $F(K_{1,n-1} +_R P_m) = 8m(n^3 + 9n - 9) - 2(12n^2 + 31n - 36)$
- (c) $F(K_{1,n-1} +_Q P_m) = m(n^4 + 3n^2 + 30n - 26) - 2(3n^2 + 22n - 18)$
- (d) $F(K_{1,n-1} +_T P_m) = m(n^4 + 7n^3 + 64n - 64) - 2(12n^2 + 31n - 36)$

Theorem 7. For $n \geq 3$ and $m \geq 2$,

- (a) $F(K_n +_S P_m) = 2n^4 + 4mn(n - 1) + n(n + 1)^3(m - 2)$
- (b) $F(K_n +_R P_m) = 2n(2n - 1)^3 + 4mn(n - 1) + 8n^4(m - 2)$
- (c) $F(K_n +_Q P_m) = 2n^4 + 4mn(n - 1)^4 + n(n + 1)^3(m - 2)$
- (d) $F(K_n +_T P_m) = \frac{1}{2}mn(n - 1)(2n - 2)^3 + 2n(2n - 1)^3 + 8n^4(m - 2)$

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