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The Sum Degree Distance and the Product Degree Distance of Generalized Transformation Graphs G^{ab}

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Abstract. In this contribution, we consider line splitting graph $L_s(G)$ of a graph G as transformation graph G^{++} of G^{ab}. We investigate the sum degree distance $DD_+(G)$ and product degree distance $DD_*(G)$ of transformation graph G^{ab}, which are weighted version of Wiener index. The Transformation graphs of G^{ab} are G^{++}, G^{+-}, G^{-+} and G^{--}.

1. Introduction

Throughout this paper, we consider finite, un-directed, simple, connected, r-regular graphs with vertex set $V(G)=\{v_1,v_2,v_3,...,v_n\}$ and edge set $E(G)=\{e_1,e_2,e_3,...,e_m\}$. For the undefined terminologies we refer[8].

The degree of vertex in a graph G is denoted by $deg_G(v)$ or $d_G(v)$ and the distance between two vertices v_i and v_j, denoted by $dist_G(v_i,v_j)$ or $d_G(v_i,v_j)$, is the length of a shortest path between the vertices v_i and v_j in G. The shortest v_i- v_j path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic. The graphs considered in this construction are with $diam \le 2$. The degree of an edge e_i in G is the number of edges adjacent to e_i and is denoted by $deg_G(e_i)$ .The degree of edge in a graph G is

$$deg_G(e_i)=deg_G(uv)=deg_G(u)+deg_G(v)-2.$$

Topological indices and graph invariants based on distances between vertices of a graph are widely used in mathematical chemistry[2], which are due to their correlations with physical, chemical and thermodynamic parameters of chemical compounds.

One of the oldest and well studied distance based graph invariant associated with a connected graph G is the Wiener number W(G), also termed as Wiener index in chemical or mathematical chemistry literature, which is defined in [13] as the sum of distances over all unordered vertex pairs in G,

i.e.,

$$W(G)=\sum_{i\le j} d(v_i,v_j) \tag{1}$$

Which was first time introduced by Wiener. Initially, the Wiener index W(G) was considered as a molecular structure descriptor used in chemical applications, but soon it attracted the interest of pure mathematicians[1,3,5,14,15].

Eventually a number of modifications of the Wiener index were proposed, which are as follows.

$$DD_+(G)=\sum_{u,v\subseteq V(G)}[deg_G(u)+deg_G(v)]d_G(u,v) \tag{2}$$

$$DD_*(G)=\sum_{u,v\subseteq V(G)}[deg_G(u)\cdot deg_G(v)]d_G(u,v) \tag{3}$$

The graph invariants defined in (2) and (3) have all been much studied in the past. The invariant DD_+ was first time introduced by Dobrynin and Kochetova[4] and named as sum-degree distance. Later the same quantity was examined under the name "Schulz index" [7]. For mathematical research on degree distance see[9,12] and the references cited therein. A remarkable property of DD_+ is that in the case of trees of order n, the identity DD_+=4W-n(n-1) holds [10].

Gutman [7] proved that the multiplicative variant of the degree distance, namely DD_*, *i.e*., from (2) , obeys an analogous relation: DD_*= 4W-(2n-1) (n-1). This latter quantity is sometimes referred to as the "Gutman index"[6], but here we call it product-degree distance.

The open neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i

i.e, $N(e_i)=\{e_i/e_i, e_j$ are adjacent in G $\}$.

For each edge e_i of G, a new vertex e_i' is taken and the resulting set of vertices is denoted by $E'(G)$.

The line Splitting graph $L_s(G)$ of a graph G is defined as the graph having vertex set $E(G)\cup E'(G)$ with two vertices adjacent if they correspond to adjacent edges of G or one corresponds to an element e_i' of $E'(G)$ and the other to an element e_j of E(G), and e_j is in $N(e_i)$. This concept was introduced by Kulli and Biradar in[11].

2 Generalized Transformation Graphs G^{ab}

Let G = (V, E) be a graph. Let α, β and α', β' be the element of $E(G)$ and $E'(G)$ respectively. We say that the associativity of α and β is +, if they are adjacent in G otherwise is - and the associativity of α and β' or α' and β is +, if α is the neighborhood point of β or β is neighborhood point of α in G, otherwise is - .

Let ab be a 2-permutation of the set $\{+,-\}$. We say that α and β corresponds to the first term a of ab, and α , β $\in$ E(G). Whereas α and β' or β and α' corresponds to the both first and second term of ab and $\alpha', \beta' \in E'(G)$.

The transformation graph G^{ab} of a graph G is the graph with vertex set $E(G)\cup E'(G)$. α *and* β or α *and* β' or β *and* α' are adjacent if and only if the following conditions holds;

* $\alpha, \beta \in E(G)$, α and β are adjacent in G if a = + otherwise a= -.

** $\alpha, \beta \in E(G)$ and $\alpha', \beta' \in E'(G)$, if α neighborhood points of β or β is neighborhood point of α in G then b = + otherwise b= -.

Since there are four distinct 2-permutations of $\{+,-\}$, we obtain 4-graphical transformations of G. Here we consider G^{++}, which is nothing but line splitting graph of G and the other generalized transformation graphs are G^{+-}, G^{-+} and G^{--}.

Note that, in this paper we consider graphs with n $\geq$ 5 for G^{++} and G^{-+} and in particular for G^{+-} and G^{--} we consider graphs with n > 5 and having atleast three edges e_i , e_j and e_w ∈E(G); i,j,w=1,2,3,...,m and i≠j≠ w such that e_i and e_j are non adjacent edges and e_w is non adjacent to e_i and e_j.

The aim of present work is to obtain the expression for the sum degree distance and product degree distance of the generalized transformation graphs G^{ab}.

3. Results

In this section we obtain the sum degree distance and product degree distance of the transformation graphs G^{ab}, which is line splitting graph i.e., G^{++}, and its generalized transformation graphs G^{+-}, G^{-+}, G^{--}.

We start by stating the following propositions and observations, needed for proving our main results.

Proposition 3.1 Let G be an (n,m) graph. Then by the definition order of G^{ab} is 2m and

(i) The size of G^{++} is $-m+\frac{1}{2}nr^2+2m(r-1)$.

(ii) The size of G^{+-} is $-m^2+\frac{1}{2}(nr^2)-2mr$.

(iii) The size of G^{-+} is $\frac{m}{2}[m+2r-3]$.

(iv) The size of G^{--} is $\frac{3}{2}m[m-2r+1]$.

Proof. Let G be a (n,m)-graph with regular degree r,then

(i) $E(G^{++}) = E(L(G))+\sum_{i=1}^{m}\deg_G(e_i)$

$= -m+\frac{1}{2}\sum_{i=1}^{n}d_i^2+\sum_{i=1}^{m}\deg_G(e_i)$

$= -m+\frac{1}{2}nr^2+m(2r-2)$ $[\because G \textit{ is } r-\textit{regular graph}]$

$E(G^{++}) = -m+\frac{1}{2}nr^2+2m(r-1)$.

(ii) $E(G^{+-}) = E(L(G))+\sum_{(uv)\in E(G)}$ the edges which are not incident to u and v in G

$= -m+\frac{1}{2}nr^2+m(m-2r+1)$ $[\because G \textit{ is } r-\textit{regular graph}]$

$= -m+\frac{1}{2}nr^2+m^2-2mr+m$

$E(G^{+-}) = m^2+\frac{1}{2}nr^2-2mr$.

(iii) $E(G^{-+}) = \frac{1}{2}\sum_{(uv)\in E(G)}$ the edges which are not incident to u and v in G + $\sum_{i=1}^{m}\deg_G(e_i)$

$= \frac{1}{2}m(m-2r+1)+m(\deg_G(u)+\deg_G(v)-2)$

$[\because G \textit{ is } r-\textit{regular graph}]$

$= \frac{1}{2}m^2-mr+\frac{1}{2}m+m(2r-2)$

$E(G^{-+}) = \frac{m}{2}[m+2mr-3]$.

(iv) $E(G^{--}) = \frac{1}{2}\sum_{(uv)\in E(G)}$ the edges which are not incident to u and v in G +

$\sum_{(uv)\in E(G)}$ the edges which are not incident to u and v in G

$= \frac{1}{2}m(m-2r+1)+m(m-2r+1)$ $[\because G \textit{ is } r-\textit{regular graph}]$

$E(G^{--}) = \frac{3}{2}m[m-2r+1]$.

Proposition 3.2 Let G be an (n,m) graph. Then the degree of vertices e_i *and* e_i' of G^{ab} are,

(i) $d_{G^{++}}(e_i)=4(r-1)$ and $d_{G^{++}}(e_i')=2(r-1)$.

$(ii) d_{G^{+-}}(e_i)=(m-1)$ and $d_{G^{+-}}(e_i')=m-2r+1$.

$(iii) d_{G^{-+}}(e_i)=(m-1)$ and $d_{G^{-+}}(e_i')=2(r-1)$.

$(iv) d_{G^{--}}(e_i)=2(m-2r+1)$ and $d_{G^{--}}(e_i')=(m-2r+1)$.

Proof. Let G be a (n,m)-graph with regular degree r, then

(i) $d_{G^{++}}(e_i)=2\deg_G(e_i)=2(2r-2)=4(r-1)$ and

$d_{G^{++}}(e_i')=\deg_G(e_i)=2r-2=2(r-1).$

(ii) $d_{G^{+-}}(e_i)=\deg_G(e_i)+(m-2r+1)=2r-2+m-2r+1=(m-1)$ and

$d_{G^{+-}}(e_i')=$ The total number of edges which are not incident to u and v in G and $uv=e_i$.

$=(m-2r+1).$

(iii) $d_{G^{-+}}(e_i)=(m-2r+1)+\deg_G(e_i)=m-2r+1+2r-2=(m-1)$ and

$d_{G^{-+}}(e_i')=\deg_G(e_i)=(2r-2)=2(r-1).$

(iv) $d_{G^{--}}(e_i)=2($The total number of edges which are not incident to u and v in G and $uv=e_i)$

$=2(m-2r+1)$ and

$d_{G^{--}}(e_i')=$ The total number of edges which are not incident to u and v in G and $uv=e_i$.

$=(m-2r+1).$

We use Proposition 3.2 for the following observations.

Observation A.

1. G be any (n, m) graph.

If $d_{G^{++}}(e_i,e_j)=1$, then

$\sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)+deg_{G^{++}}(e_j)]d_{G^{++}}(e_i,e_j)$ in G^{++} $=8m(r-1)^2$.

If $d_{G^{++}}(e_i,e_j)=2$, then

$\sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)+deg_{G^{++}}(e_j)]d_{G^{++}}(e_i,e_j)$ in G^{++} $=16(r-1)(m^2+m-2mr)$.

2. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i,e_j')=1$, then

$\sum_{(e_i,e_j')\subseteq V(G^{++})}[deg_{G^{++}}(e_i)+deg_{G^{++}}(e_j')]d_{G^{++}}(e_i,e_j')$ in G^{++} $=12m(r-1)^2$.

If $d_{G^{++}}(e_i,e_j')=2$, then

$\sum_{(e_i,e_j')\subseteq V(G^{++})}[deg_{G^{++}}(e_i)+deg_{G^{++}}(e_j')]d_{G^{++}}(e_i,e_j')$ in G^{++} $=12m\,(r-1)(m-2r+2)$.

3. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i',e_j')=2$, then

$\sum_{(e_i',e_j')\subseteq V(G^{++})}[deg_{G^{++}}(e_i')+deg_{G^{++}}(e_j')]d(e_i',e_j')$ in G^{++} $=8(r-1)\sum_{k=1}^{m}(k-1)$.

and

If $d_{G^{++}}(e_i',e_j')=3$ when r = 2, then

$\sum_{(e_i',e_j')\subseteq V(G^{++})}[deg_{G^{++}}(e_i')+deg_{G^{++}}(e_j')]d(e_i',e_j')$ in G^{++} =12m(r-1).

Theorem 3.3. For any (n, m) graph G with r ≥ 2,
if r = 2 then

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r)+3m+nr^2+2\sum_{k=1}^{m}(k-1)] \qquad (*)$$

And if r > 2 then

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r)+nr^2+2\sum_{k=1}^{m}(k-1)] \qquad (**)$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{++} contains 2m vertices and $(-m+\frac{1}{2}nr^2+2m(r-1))$ edges.

From (2), we have

$$DD_+(G) = \sum_{u,v\subseteq V(G)}[deg_G(u)+deg_G(v)]d_G(u,v)$$

Therefore,

$$\begin{aligned} DD_+(G^{++}) = &\sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)+deg_{G^{++}}(e_j)]d_{G^{++}}(e_i,e_j)+\\ &\sum_{(e_i,e'_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)+deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i,e'_j)+\\ &\sum_{(e'_i,e'_j)\subseteq V(G^{++})}[deg_{G^{++}}(e'_i)+deg_{G^{++}}(e'_j)]d_{G^{++}}(e'_i,e'_j). \end{aligned}$$

Applying observation A to the above equation,
when r = 2,

$$\begin{aligned} DD_+(G^{++}) =\ &8\mathrm{m}(r-1)^2(-m+\frac{1}{2}nr^2)+16\mathrm{m(r\text{-}1)}(m-2r+1)+12\mathrm{m}(r-1)^2+12\mathrm{m(r\text{-}1)}\\ &\mathrm{(m\text{-}2r\text{+}2)+12m(r\text{-}1)+8(r\text{-}1)}\sum_{k=1}^{m}(k-1). \end{aligned}$$

and r >2,

$$\begin{aligned} DD_+(G^{++}) =\ &8\mathrm{m}(r-1)^2(-m+\frac{1}{2}nr^2)+16\mathrm{m(r\text{-}1)}(m-2r+1)+12\mathrm{m}(r-1)^2+12\mathrm{m(r\text{-}1)}\\ &\mathrm{(m\text{-}2r\text{+}2)+8(r\text{-}1)}\sum_{k=1}^{m}(k-1). \end{aligned}$$

On simplification, we get $(*)$ *and* $(**)$
i.e.,

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r)+3m+nr^2+2\sum_{k=1}^{m}(k-1)]$$

and

$$DD_+(G^{++}) = 4\mathrm{(r\text{-}1)}[m(7m+5-11r)+nr^2+2\sum_{k=2}^{m}(k-1)].$$

Observation B.

1. Let G be any (n,m) graph.
   If $d_{G^{++}}(e_i,e_j)=1$, then
   $$\sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)\cdot deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i,e_j) \text{ in } G^{++} = 16m\,(r-1)^2(-m+\frac{1}{2}nr^2).$$
   If $d_{G^{++}}(e_i,e_j)=2$, then
   $$\sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)\cdot deg_{G^{++}}(e_j)]d_{G^{++}}(e_i,e_j) \text{ in } G^{++} = 32\,(r-1)^2(m^2+m-2mr).$$
2. Let G be any (n,m) graph.
   If $d_{G^{++}}(e_i,e'_j)=1$, then
   $$\sum_{(e_i,\,e'_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)\cdot deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i,e'_j) \text{ in } G^{++} = 16m\,(r-1)^3 m.$$

If $d_{G^{++}}(e_i, e'_j) = 2$, then

$\sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j)$ in $G^{++} = 16m\,(r-1)^2(m-2r+2)$.

3. Let G be any (n,m) graph.

If $d_{G^{++}}(e'_i, e'_j) = 2$, then

$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j)$ in $G^{++} = 8(r-1)^2 \sum_{k=2}^{m}(k-1)$.

If $d_{G^{++}}(e'_i, e'_j) = 3$, when r = 2, then

$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j)$ in G^{++} =12m(r-1)^2.

Theorem 3.4. For any (n,m) graph G with r ≥ 2,

when r = 2

$$DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^{m}(k-1)] \qquad (*')$$

and when r > 2

$$DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^{m}(k-1)] \qquad (**')$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{++} contains 2m vertices and $(-m + \frac{1}{2}nr^2 + 2m(r-1))$ edges.

From (3), we have

$$DD_*(G) = \sum_{u,v \subseteq V(G)} [deg_G(u) \cdot deg_G(v)] d_G(u,v)$$

Therefore,

$$\begin{aligned} DD_*(G^{++}) = &\sum_{(e_i, e_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e_j)] d_{G^{++}}(e_i, e_j) + \\ &\sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) + \\ &\sum_{(e'_i, e'_j) + \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j). \end{aligned}$$

Applying observation B to the above equation,

when r = 2,

$$\begin{aligned} DD_*(G^{++}) = &16\,(r-1)^2(-m + \tfrac{1}{2}) + 32\,(r-1)^2(m^2 + m - 2mr) + 16m\,(r-1)^3 + 16m\,(r-1)^2 \\ &(m-2r+2) + 12m(r-1)^2 + 8(r-1)^2 \sum_{k=2}^{m}(k-1). \end{aligned}$$

When r > 2

$$\begin{aligned} DD_*(G^{++}) = &16\,(r-1)^2(-m + \tfrac{1}{2}) + 32\,(r-1)^2(m^2 + m - 2mr) + 16m\,(r-1)^3 + 16m\,(r-1)^2 \\ &(m-2r+2) + 8(r-1)^2 \sum_{k=2}^{m}(k-1). \end{aligned}$$

On simplification, we get $(*')$ *and* $(**')$

$$DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^{m}(k-1)]$$

and

$$DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + \sum_{k=2}^{m}(k-1)].$$

Observation C.

1. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i,e_j)=1$, then

$\sum_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j)$ in G^{+-} $= 2(m-1)(-m+\frac{1}{2}nr^2)$.

If $d_{G^{+-}}(e_i,e_j)=2$, then

$\sum_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j)$ in G^{+-} $= 4(m-1)(m^2+m-2mr)$.

2. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i,e'_j)=1$, then

$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j)$ in G^{+-} $= 2m(m-r)(m-2r+1)$.

If $d_{G^{+-}}(e_i,e'_j)=2$, then

$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j)$ in G^{+-} $= 8m(m-r)(r-1)$.

If $d_{G^{+-}}(e_i,e'_j)=3$, then

$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j)$ in G^{+-} $= 6m(m-r)$.

3. Let G be any (n,m) graph.

If $d_G(e'_i,e'_j)=2$, then

$\sum_{(e'_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e'_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e'_i,e'_j)$ in G^{+-} $=4(m-2r+1)\sum_{k=2}^{m}(k-1)$.

Theorem 3.5. For any (n,m) graph G,

$$DD_+(G^{+-}) = 2m[m(3m-5)-1]+mr[nr-6m-4r+8]-nr^2+$$
$$4(m-2r+1)\sum_{k=1}^{r}(k-1).$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{+-} contains 2m vertices and $(-m^2+\frac{1}{2}(nr^2)-2mr)$ edges.

From (2), we have

$$DD_+(G)=\sum_{u,v\subseteq V(G)}[deg_G(u)+deg_G(v)]d_G(u,v)$$

Therefore,

$$DD_+(G^{+-}) = \sum_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j)+$$
$$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j)+$$
$$\sum_{(e'_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e'_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e'_i,e'_j).$$

Applying observation C to the above equation, we get

$$DD_+(G^{+-}) = 2m(m-1)(r-1)+4(m-1)(m^2+m-2mr)+2(m-r)(m-2r+1)+$$
$$8m(m-r)(r-1)+6m(m-r)+2m(m-2r+1)\sum_{k=1}^{m}(k-2).$$

On simplification,

$$DD_+(G^{+-}) = 2m[m(3m-5)-1]+mr[nr-6m-4r+8]-nr^2+$$
$$4(m-2r+1)\sum_{k=1}^{r}(k-1).$$

Observation D.

1. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i,e_j)=1$, then

$$\sum\nolimits_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j) \text{ in } G^{+-} =(m-1)^2(-m+\frac{1}{2}nr^2).$$

If $d_{G^{+-}}(e_i,e_j)=2$, then

$$\sum\nolimits_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j) \text{ in } G^{+-} = 2(m-1)^2(m^2+m-2mr).$$

2. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i,e'_j)=1$, then

$$\sum\nolimits_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) \text{ in } G^{+-} = (m^2-2mr+2r-1)$$

$(m^2-2rm+m)$.

If $d_{G^{+-}}(e_i,e'_j)=2$, then

$$\sum\nolimits_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) \text{ in } G^{+-} = 4(m^2-2mr+2r-1)(r-1)m.$$

If $d_{G^{+-}}(e_i,e'_j)=3$, then

$$\sum\nolimits_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) \text{ in } G^{+-} = 3m(m^2-2m+2r-1).$$

3. Let G be any (n,m) graph.

If $d_{G^{+-}}(e'_i,e'_j)=2$, then

$$\sum\nolimits_{(e'_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e'_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e'_i,e'_j) \text{ in } G^{+-} = 2(m-2r+2)^2\sum\nolimits_{k=2}^{m}(k-1)..$$

Theorem 3.6. For any (n,m) graph G,

$$DD_*(G^{+-})=m[m^2(3m-3)-7(m+1)]+\frac{1}{2}nr^2(m^2-m+1)-2mr[2m(3-r)+2r-3]+$$
$$2(m-2r+2)^2\sum_{k=2}^{m}(k-1)^2.$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{+-} contains 2m vertices and $-m^2+\frac{1}{2}(nr^2)-2mr$ edges

From (3), we have

$$DD_*(G)=\sum_{u,v\subseteq V(G)}[deg_G(u)\cdot deg_G(v)]d_G(u,v)$$

Therefore,

$$DD_*(G^{+-})=\sum\nolimits_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j)+$$
$$\sum\nolimits_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j)+$$
$$\sum\nolimits_{(e'_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e'_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e'_i,e'_j).$$

Applying observation D to the above equation, we get

$$DD_*(G^{+-})=(m-1)^2(-m+\frac{1}{2}nr^2)+2(m-1)^2(m^2+m-2mr)+$$
$$(m^2-2mr+2r-1)(m-2r+1)+4(m^2-2mr+2r-1)(r-1)m+$$
$$3m(m^2-2m+2r-1)+2(m-2r+1)^2\sum\nolimits_{k=1}^{r}(k-1).$$

On simplification,

$$DD_*(G^{+-}) = m[m^2(3m-3) - 7(m+1)] + \frac{1}{2}nr^2(m^2 - m + 1) - 2mr[2m(3-r) + 2r - 3] + 2(m-2r+2)^2 \sum_{k=2}^{m}(k-1)^2.$$

Observation E.

1. Let G be any (n,m) graph.
   If $d_{G^{-+}}(e_i, e_j) = 1$, then
   $\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i) + deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j)$ in G^{-+} = $(m-2r+1)(m-1)m$.
   If $d_{G^{-+}}(e_i, e_j) = 2$, then
   $\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i) + deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j)$ in G^{-+} = $(m-1)(r-1)m$.
2. Let G be any (n,m) graph.
   If $d_{G^{-+}}(e_i, e'_j) = 1$, then
   $\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i) + deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e_i,e'_j)$ $=2m(m+2r-3)(r-1)$.
   If $d_{G^{-+}}(e_i, e'_j) = 2$, then
   $\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i) + deg_{G^{-+}})]d_{G^{-+}}(e_i,e'_j) =$ $2m(m+2r-3)(m-2r+2)$.
   If $d_{G^{-+}}(e_i, e'_j) = 3$, then
   $\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i) + deg_{G^{-+}})]d_{G^{-+}}(e_i,e'_j) =$ $3m(m+2r-3)$.
3. Let G be any (n,m) graph.
   If $d_{G^{-+}}(e'_i, e'_j) = 2$, then
   $\sum_{(e'_i,e'_j)\subseteq V(G^{-+})}[deg_G(e'_i) + deg_G(e'_j)]d(e'_i,e'_j)$ in G^{-+} $=8(r-1)\sum_{k=2}^{m}(k-1)$.

Theorem 3.7. For any (n,m) graph G,

$$DD_+(G^{-+}) = m[m(3m-2) - 15] - mr[m - 4r - 17] + 8(r-1)\sum_{k=2}^{m}(k-1).$$

Proof. Let G be any (n,m)-graph, from the Proposition 3.1, G^{-+} contains 2m vertices and $\frac{m}{2}[m+2r-3]$ edges.

From (2), we have

$$DD_+(G) = \sum_{u,v\subseteq V(G)}[deg_G(u) + deg_G(v)]d_G(u,v)$$

Therefore,

$$\begin{aligned} DD_+(G^{-+}) = &\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i) + deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j) + \\ &\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i) + deg_{G^{-+}}(e'_j)]\, d_{G^{-+}}(e_i,e'_j) + \\ &\sum_{(e'_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e'_i) + deg_{G^{-+}}(e'_j)]\, d_{G^{-+}}(e'_i,e'_j). \end{aligned}$$

Applying observation E to the above equation, we get

$$\begin{aligned} DD_+(G^{-+}) = &(m-2r+1)(m-1)m + (m-1)(r-1)m + 2m(m+2r-3)(r-1) + \\ &2m(m+2r-3)(m-2r+2) + 3m(m+2r-3) + 8(r-1)\sum_{k=2}^{m}(k-1). \end{aligned}$$

On simplification,

$$DD_{+}(G^{-+}) = m[m(3m-2)-15] - mr[m-4r-17] + 8(r-1)\sum_{k=2}^{m}(k-1).$$

Observation F.

1. Let G be any (n,m) graph.

If $d_{G^{-+}}(e_i,e_j)=1$, then

$\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j)$ in $G^{-+} = \frac{1}{2}(m-1)^2(m^2-2mr+m)$.

If $d_{G^{-+}}(e_i,e_j)=2$, then

$\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j)$ in $G^{-+}= 2(m-1)^2(r-1)m$.

2. Let G be any (n,m) graph.

If $d_{G^{-+}}(e_i,e_j')=1$, then

$\sum_{(e_i,e_j')\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j')]d_{G^{-+}}(e_i,e_j')$ in $G^{-+}= 4m(mr-m-r+1)(r-1)$.

If $d_{G^{-+}}(e_i,e_j')=2$, then

$\sum_{(e_i,e_j')\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j')]d_{G^{-+}}(e_i,e_j')$ in $G^{-+} = 4m(mr-m-r-1)(m-2r+2)$.

If $d_{G^{-+}}(e_i',e_j')=3$, then

$\sum_{(e_i,e_j')\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j')]d_{G^{-+}}(e_i,e_j')$ in $G^{-+}= 6m(mr-m-r+1)$.

3. Let G be any (n,m) graph.

If $d_{G^{-+}}(e_i',e_j')=2$, then

$\sum_{(e_i',e_j')\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i')\cdot deg_{G^{-+}}(e_j')]d_{G^{-+}}(e_i',e_j')$ in $G^{-+}= 8(r-1)^2\sum_{k=2}^{m}(k-1)$.

Theorem 3.8. For any (n,m) graph G,

$$DD_{*}(G^{-+}) = \frac{1}{2}m[m^2(m-13)-5m+8] + mr[m(5m-4r+8)+4r-13] + 8(r-1)^2\sum_{k=2}^{m}(k-1).$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{-+} contains 2m vertices and $\frac{m}{2}[m+2r-3]$ edges.

From (3), we have

$$DD_{*}(G) = \sum_{u,v\subseteq V(G)}[deg_G(u)\cdot deg_G(v)]d_G(u,v)$$

Therefore,

$$\begin{aligned} DD_{*}(G^{-+}) = &\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j) + \\ &\sum_{(e_i,e_j')\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j')]d_{G^{-+}}(e_i,e_j') + \\ &\sum_{(e_i',e_j')\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i')\cdot deg_{G^{-+}}(e_j')]d_{G^{-+}}(e_i',e_j'). \end{aligned}$$

Applying observation F to the above equation, we get

$$DD_{*}(G^{-+}) = \frac{1}{2}(m-1)^2(m^2-2mr+m) + 2(m-1)^2(r-1)m + 4m(mr-m-r+1)(r-1)$$

$$+4m(mr-m-r-1)(m-2r+2)+6m(mr-m-r+1)+$$
$$8(r-1)^2\sum_{k=2}^{m}(k-1).$$

On simplification,

$$DD_*(G^{-+})=\frac{1}{2}m[m^2(m-13)-5m+8]+mr[m(5m-4r+8)+4r-13]+$$
$$8(r-1)^2\sum_{k=2}^{m}(k-1).$$

Observation G.

1. Let G be any (n,m) graph.
   If $d_{G^{--}}(e_i,e_j)=1$, then
   $\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)$ in G^{--} = $2m\ (m-2r+1)^2$.
   If $d_{G^{--}}(e_i,e_j)=2$, then
   $\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)$ in G^{--} = $8\ (m-2r+1)(r-1)$.
2. Let G be any (n,m) graph.
   If $d_{G^{--}}(e_i,e'_j)=1$, then
   $\sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)$ in G^{--} = $3m\ (m-2r+1)^2$.
   if $d_{G^{--}}(e_i,e'_j)=2$, then
   $\sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)$ in G^{--} = $6m\ (2r-1)(m-2r+1)$.
3. Let G be any (n,m) graph.
   If $d_{G^{--}}(e'_i,e'_j)=2$, then
   $\sum_{(e'_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e'_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e'_i,e'_j)$ in G^{--} $=4(m-2r+1)\sum_{k=2}^{m}(k-1)$.

Theorem 3.9. For any (n,m) graph G,

$$DD_+(G^{--})=m[5m^2-4m-9]-4mr[5r-7]+4(m-2r+1)\sum_{k=2}^{m}(k-1).$$

Proof. Let G be any (n,m)-graph. From proposition 3.1, G^{--} contains 2m vertices and $\frac{3}{2}m[m-2r+1]$ edges.

From (2), we have

$$DD_+(G)=\sum_{u,v\subseteq V(G)}[deg_G(u)+deg_G(v)]d_G(u,v)$$

Therefore,

$$DD_+(G^{--})=\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)]+$$
$$\sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)]+$$
$$\sum_{(e'_i,e'_j)V(G^{--})}[deg_{G^{--}}(e'_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e'_i,e'_j)].$$

Applying observation G to the above equation, we get

$$DD_+(G^{--})=2m(m-2r+1)^2+8(m-2r+1)(r-1)+3m(m-2r+1)^2+6m(2r-1)(m-2r+1)$$
$$+4(m-2r+1)\sum_{k=2}^{m}(k-1)$$

On simplification,

$$DD_{+}(G^{--}) = m[5m^2 - 4m - 9] - 4mr[5r - 7] + 4(m - 2r + 1)\sum_{k=2}^{m}(k-1).$$

Observation H.

1. Let G be any (n,m) graph.

   If $d_{G^{--}}(e_i, e_j) = 1$, then

   $\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)$ in $G^{--} = 2m(m-2r+1)^3$.

   If $d_{G^{--}}(e_i, e_j) = 2$, then

   $\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)$ in $G^{--} = 8m(m-2r+1)^2(r-1)$.

2. Let G be any (n,m) graph.

   If $d_{G^{--}}(e_i, e'_j) = 1$, then

   $\sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)$ in $G^{--} = 2m(m-2r+1)^3$.

   If $d_{G^{--}}(e_i, e'_j) = 2$, then

   $\sum_{(e'_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)$ in $G^{--} = 4m(m-2r+2)^2(2r-1)$.

3. Let G be any (n,m) graph.

   If $d_{G^{--}}(e'_i, e'_j) = 2$, then

   $\sum_{(e'_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e'_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e'_i,e'_j)$ in $G^{--} = 2(m-2r+1)^2\sum_{k=2}^{m}(k-1)$.

Theorem 3.10. For any (n,m) graph G,

$$DD_{*}(G^{--}) = 2m[m^2(m-3) - 5m - 6] - 4mr[3m^2 - 4r^2 + 10mr - 14m + 12r - 13] + 2(m-2r+1)^2\sum_{k=2}^{m}(k-1).$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{--} contains 2m vertices and $\frac{3}{2}m[m-2r+1]$ edges.

From (3), we have

$$DD_{*}(G) = \sum_{u,v\subseteq V(G)}[deg_G(u)\cdot deg_G(v)]d_G(u,v)$$

Therefore,

$$\begin{aligned}DD_{*}(G^{--}) = {} & \sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j) + \\ & \sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j) + \\ & \sum_{(e'_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e'_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e'_i,e'_j).\end{aligned}$$

Applying observation H the above equation, we get

$$DD_{*}(G^{--}) = 2m(m-2r+1)^3 + 8m(m-2r+1)^2(r-1) + 2m(m-2r+1)^3 + 4m(m-2r+2)^2(2r-1) + 2(m-2r+1)^2\sum_{k=2}^{m}(k-1).$$

On simplification,

$$DD_{*}(G^{--}) = 2m[m^2(m-3) - 5m - 6] - 4mr[3m^2 - 4r^2 + 10mr - 14m + 12r - 13] + 2(m-2r+1)^2\sum_{k=2}^{m}(k-1).$$

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