

The Sum Degree Distance and the Product Degree Distance of Generalized Transformation Graphs G^{ab}

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Abstract. In this contribution, we consider line splitting graph $L_s(G)$ of a graph G as transformation graph G^{++} of G^{ab} . We investigate the sum degree distance $DD_+(G)$ and product degree distance $DD_*(G)$ of transformation graph G^{ab} , which are weighted version of Wiener index. The Transformation graphs of G^{ab} are G^{++} , G^{+-} , G^{-+} and G^{--} .

1. Introduction

Throughout this paper, we consider finite, un-directed, simple, connected, r -regular graphs with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$. For the undefined terminologies we refer[8].

The degree of vertex in a graph G is denoted by $\deg_G(v)$ or $d_G(v)$ and the distance between two vertices v_i and v_j , denoted by $dist_G(v_i, v_j)$ or $d_G(v_i, v_j)$, is the length of a shortest path between the vertices v_i and v_j in G . The shortest $v_i - v_j$ path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic. The graphs considered in this construction are with $diam \leq 2$. The degree of an edge e_i in G is the number of edges adjacent to e_i and is denoted by $deg_G(e_i)$. The degree of edge in a graph G is

$$deg_G(e_i) = deg_G(uv) = deg_G(u) + deg_G(v) - 2.$$

Topological indices and graph invariants based on distances between vertices of a graph are widely used in mathematical chemistry[2], which are due to their correlations with physical, chemical and thermodynamic parameters of chemical compounds.

One of the oldest and well studied distance based graph invariant associated with a connected graph G is the Wiener number $W(G)$, also termed as Wiener index in chemical or mathematical chemistry literature, which is defined in [13] as the sum of distances over all unordered vertex pairs in G ,

i.e.,

$$W(G) = \sum_{i \leq j} d(v_i, v_j) \tag{1}$$

Which was first time introduced by Wiener. Initially, the Wiener index $W(G)$ was considered as a molecular structure descriptor used in chemical applications, but soon it attracted the interest of pure mathematicians[1,3,5,14,15].

Eventually a number of modifications of the Wiener index were proposed, which are as follows.

$$DD_+(G) = \sum_{u, v \in V(G)} [deg_G(u) + deg_G(v)] d_G(u, v) \tag{2}$$

$$DD_*(G) = \sum_{u, v \in V(G)} [deg_G(u) \cdot deg_G(v)] d_G(u, v) \tag{3}$$

The graph invariants defined in (2) and (3) have all been much studied in the past. The invariant DD_+ was first time introduced by Dobrynin and Kochetova[4] and named as sum-degree distance. Later the same quantity was examined under the name "Schulz index" [7]. For mathematical research on degree distance see[9,12] and the references cited therein. A remarkable property of DD_+ is that in the case of trees of order n , the identity $DD_+ = 4W - n(n-1)$ holds [10].

Gutman [7] proved that the multiplicative variant of the degree distance, namely DD_* , i.e., from (2), obeys an analogous relation: $DD_* = 4W - (2n-1)(n-1)$. This latter quantity is sometimes referred to as the "Gutman index"[6], but here we call it product-degree distance.

The open neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i

$$i.e., \quad N(e_i) = \{e_i/e_j, e_j \text{ are adjacent in } G \}.$$

For each edge e_i of G , a new vertex e'_i is taken and the resulting set of vertices is denoted by $E'(G)$.

The line Splitting graph $L_s(G)$ of a graph G is defined as the graph having vertex set $E(G) \cup E'(G)$ with two vertices adjacent if they correspond to adjacent edges of G or one corresponds to an element e'_i of $E'(G)$ and the other to an element e_j of $E(G)$, and e_j is in $N(e_i)$. This concept was introduced by Kulli and Biradar in[11].

2 Generalized Transformation Graphs G^{ab}

Let $G = (V, E)$ be a graph. Let α, β and α', β' be the element of $E(G)$ and $E'(G)$ respectively. We say that the associativity of α and β is $+$, if they are adjacent in G otherwise is $-$ and the associativity of α and β' or α' and β is $+$, if α is the neighborhood point of β or β is neighborhood point of α in G , otherwise is $-$.

Let ab be a 2-permutation of the set $\{+, -\}$. We say that α and β corresponds to the first term a of ab , and $\alpha, \beta \in E(G)$. Whereas α and β' or β and α' corresponds to the both first and second term of ab and $\alpha', \beta' \in E'(G)$.

The transformation graph G^{ab} of a graph G is the graph with vertex set $E(G) \cup E'(G)$. α and β or α and β' or β and α' are adjacent if and only if the following conditions holds;

* $\alpha, \beta \in E(G)$, α and β are adjacent in G if $a = +$ otherwise $a = -$.

** $\alpha, \beta \in E(G)$ and $\alpha', \beta' \in E'(G)$, if α neighborhood points of β or β is neighborhood point of α in G then $b = +$ otherwise $b = -$.

Since there are four distinct 2-permutations of $\{+, -\}$, we obtain 4-graphical transformations of G . Here we consider G^{++} , which is nothing but line splitting graph of G and the other generalized transformation graphs are G^{+-} , G^{-+} and G^{--} .

Note that, in this paper we consider graphs with $n \geq 5$ for G^{++} and G^{-+} and in particular for G^{+-} and G^{--} we consider graphs with $n > 5$ and having atleast three edges e_i, e_j and $e_w \in E(G)$; $i, j, w = 1, 2, 3, \dots, m$ and $i \neq j \neq w$ such that e_i and e_j are non adjacent edges and e_w is non adjacent to e_i and e_j .

The aim of present work is to obtain the expression for the sum degree distance and product degree distance of the generalized transformation graphs G^{ab} .

3. Results

In this section we obtain the sum degree distance and product degree distance of the transformation graphs G^{ab} , which is line splitting graph i.e., G^{++} , and its generalized transformation graphs G^{+-} , G^{-+} , G^{--} .

We start by stating the following propositions and observations, needed for proving our main results.

Proposition 3.1 Let G be an (n,m) graph. Then by the definition order of G^{ab} is $2m$ and

$$(i) \quad \text{The size of } G^{++} \text{ is } -m + \frac{1}{2}nr^2 + 2m(r-1).$$

$$(ii) \quad \text{The size of } G^{+-} \text{ is } -m^2 + \frac{1}{2}(nr^2) - 2mr.$$

$$(iii) \quad \text{The size of } G^{-+} \text{ is } \frac{m}{2}[m + 2r - 3].$$

$$(iv) \quad \text{The size of } G^{--} \text{ is } \frac{3}{2}m[m - 2r + 1].$$

Proof. Let G be a (n,m) -graph with regular degree r , then

$$(i) \quad \begin{aligned} E(G^{++}) &= E(L(G)) + \sum_{i=1}^m \deg_G(e_i) \\ &= -m + \frac{1}{2} \sum_{i=1}^n d_i^2 + \sum_{i=1}^m \deg_G(e_i) \\ &= -m + \frac{1}{2}nr^2 + m(2r - 2) \quad [\because G \text{ is } r\text{-regular graph}] \end{aligned}$$

$$E(G^{++}) = -m + \frac{1}{2}nr^2 + 2m(r - 1).$$

$$(ii) \quad E(G^{+-}) = E(L(G)) + \sum_{(uv) \in E(G)} \text{the edges which are not incident to } u \text{ and } v \text{ in } G$$

$$= -m + \frac{1}{2}nr^2 + m(m - 2r + 1) \quad [\because G \text{ is } r\text{-regular graph}]$$

$$= -m + \frac{1}{2}nr^2 + m^2 - 2mr + m$$

$$E(G^{+-}) = m^2 + \frac{1}{2}nr^2 - 2mr.$$

$$(iii) \quad E(G^{-+}) = \frac{1}{2} \sum_{(uv) \in E(G)} \text{the edges which are not incident to } u \text{ and } v \text{ in } G + \sum_{i=1}^m \deg_G(e_i)$$

$$= \frac{1}{2}m(m - 2r + 1) + m(\deg_G(u) + \deg_G(v) - 2)$$

$$[\because G \text{ is } r\text{-regular graph}]$$

$$= \frac{1}{2}m^2 - mr + \frac{1}{2}m + m(2r - 2)$$

$$E(G^{-+}) = \frac{m}{2}[m + 2mr - 3].$$

$$(iv) \quad E(G^{--}) = \frac{1}{2} \sum_{(uv) \in E(G)} \text{the edges which are not incident to } u \text{ and } v \text{ in } G +$$

$$\sum_{(uv) \in E(G)} \text{the edges which are not incident to } u \text{ and } v \text{ in } G$$

$$= \frac{1}{2}m(m - 2r + 1) + m(m - 2r + 1) \quad [\because G \text{ is } r\text{-regular graph}]$$

$$E(G^{--}) = \frac{3}{2}m[m - 2r + 1].$$

Proposition 3.2 Let G be an (n,m) graph. Then the degree of vertices e_i and e'_i of G^{ab} are,

- (i) $d_{G^{++}}(e_i) = 4(r-1)$ and $d_{G^{++}}(e'_i) = 2(r-1)$.
- (ii) $d_{G^{+-}}(e_i) = (m-1)$ and $d_{G^{+-}}(e'_i) = m-2r+1$.
- (iii) $d_{G^{-+}}(e_i) = (m-1)$ and $d_{G^{-+}}(e'_i) = 2(r-1)$.
- (iv) $d_{G^{--}}(e_i) = 2(m-2r+1)$ and $d_{G^{--}}(e'_i) = (m-2r+1)$.

Proof. Let G be a (n,m) -graph with regular degree r , then

- (i) $d_{G^{++}}(e_i) = 2 \deg_G(e_i) = 2(2r-2) = 4(r-1)$ and
 $d_{G^{++}}(e'_i) = \deg_G(e_i) = 2r-2 = 2(r-1)$.
- (ii) $d_{G^{+-}}(e_i) = \deg_G(e_i) + (m-2r+1) = 2r-2 + m-2r+1 = (m-1)$ and
 $d_{G^{+-}}(e'_i) =$ The total number of edges which are not incident to u and v in G and $uv = e_i$.
 $= (m-2r+1)$.
- (iii) $d_{G^{-+}}(e_i) = (m-2r+1) + \deg_G(e_i) = m-2r+1 + 2r-2 = (m-1)$ and
 $d_{G^{-+}}(e'_i) = \deg_G(e_i) = (2r-2) = 2(r-1)$.
- (iv) $d_{G^{--}}(e_i) = 2$ (The total number of edges which are not incident to u and v in G and $uv = e_i$)
 $= 2(m-2r+1)$ and
 $d_{G^{--}}(e'_i) =$ The total number of edges which are not incident to u and v in G and $uv = e_i$.
 $= (m-2r+1)$.

We use Proposition 3.2 for the following observations.

Observation A.

1. G be any (n, m) graph.

If $d_{G^{++}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{++})} [\deg_{G^{++}}(e_i) + \deg_{G^{++}}(e_j)] d_{G^{++}}(e_i, e_j) \text{ in } G^{++} = 8m(r-1)^2.$$

If $d_{G^{++}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{++})} [\deg_{G^{++}}(e_i) + \deg_{G^{++}}(e_j)] d_{G^{++}}(e_i, e_j) \text{ in } G^{++} = 16(r-1)(m^2 + m - 2mr).$$

2. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{++})} [\deg_{G^{++}}(e_i) + \deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) \text{ in } G^{++} = 12m(r-1)^2.$$

If $d_{G^{++}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{++})} [\deg_{G^{++}}(e_i) + \deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) \text{ in } G^{++} = 12m(r-1)(m-2r+2).$$

3. Let G be any (n, m) graph.

If $d_{G^{++}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [\deg_{G^{++}}(e'_i) + \deg_{G^{++}}(e'_j)] d(e'_i, e'_j) \text{ in } G^{++} = 8(r-1) \sum_{k=1}^m (k-1).$$

and

If $d_{G^{++}}(e'_i, e'_j) = 3$ when $r = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [\deg_{G^{++}}(e'_i) + \deg_{G^{++}}(e'_j)] d(e'_i, e'_j) \text{ in } G^{++} = 12m(r-1).$$

Theorem 3.3. For any (n, m) graph G with $r \geq 2$,
if $r = 2$ then

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r) + 3m + nr^2 + 2\sum_{k=1}^m (k-1)] \quad (*)$$

And if $r > 2$ then

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r) + nr^2 + 2\sum_{k=1}^m (k-1)] \quad (**)$$

Proof. Let G be any (n, m) -graph. From Proposition 3.1, G^{++} contains $2m$ vertices and $(-m + \frac{1}{2}nr^2 + 2m(r-1))$ edges.

From (2), we have

$$DD_+(G) = \sum_{u, v \subseteq V(G)} [deg_G(u) + deg_G(v)]d_G(u, v)$$

Therefore,

$$\begin{aligned} DD_+(G^{++}) &= \sum_{(e_i, e_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) + deg_{G^{++}}(e_j)]d_{G^{++}}(e_i, e_j) + \\ &\quad \sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) + deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i, e'_j) + \\ &\quad \sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) + deg_{G^{++}}(e'_j)]d_{G^{++}}(e'_i, e'_j). \end{aligned}$$

Applying observation A to the above equation,
when $r = 2$,

$$\begin{aligned} DD_+(G^{++}) &= 8m(r-1)^2(-m + \frac{1}{2}nr^2) + 16m(r-1)(m-2r+1) + 12m(r-1)^2 + 12m(r-1) \\ &\quad (m-2r+2) + 12m(r-1) + 8(r-1)\sum_{k=1}^m (k-1). \end{aligned}$$

and $r > 2$,

$$\begin{aligned} DD_+(G^{++}) &= 8m(r-1)^2(-m + \frac{1}{2}nr^2) + 16m(r-1)(m-2r+1) + 12m(r-1)^2 + 12m(r-1) \\ &\quad (m-2r+2) + 8(r-1)\sum_{k=1}^m (k-1). \end{aligned}$$

On simplification, we get (*) and (**)

i.e.,

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r) + 3m + nr^2 + 2\sum_{k=1}^m (k-1)]$$

and

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r) + nr^2 + 2\sum_{k=2}^m (k-1)].$$

Observation B.

1. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e_j)]d_{G^{++}}(e_i, e_j) \text{ in } G^{++} = 16m(r-1)^2(-m + \frac{1}{2}nr^2).$$

If $d_{G^{++}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e_j)]d_{G^{++}}(e_i, e_j) \text{ in } G^{++} = 32(r-1)^2(m^2 + m - 2mr).$$

2. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i, e'_j) \text{ in } G^{++} = 16m(r-1)^3m.$$

If $d_{G^{++}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) \text{ in } G^{++} = 16m(r-1)^2(m-2r+2).$$

3. Let G be any (n,m) graph.

If $d_{G^{++}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j) \text{ in } G^{++} = 8(r-1)^2 \sum_{k=2}^m (k-1).$$

If $d_{G^{++}}(e'_i, e'_j) = 3$, when $r = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j) \text{ in } G^{++} = 12m(r-1)^2.$$

Theorem 3.4. For any (n,m) graph G with $r \geq 2$,
when $r = 2$

$$DD_*(G^{++}) = 8(r-1)^2 [2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^m (k-1)] \quad (*)$$

and when $r > 2$

$$DD_*(G^{++}) = 8(r-1)^2 [2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^m (k-1)] \quad (**)$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{++} contains $2m$ vertices and

$$(-m + \frac{1}{2}nr^2 + 2m(r-1)) \text{ edges.}$$

From (3), we have

$$DD_*(G) = \sum_{u,v \subseteq V(G)} [deg_G(u) \cdot deg_G(v)] d_G(u,v)$$

Therefore,

$$\begin{aligned} DD_*(G^{++}) &= \sum_{(e_i, e_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e_j)] d_{G^{++}}(e_i, e_j) + \\ &\quad \sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) + \\ &\quad \sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j). \end{aligned}$$

Applying observation B to the above equation,

when $r = 2$,

$$\begin{aligned} DD_*(G^{++}) &= 16(r-1)^2(-m + \frac{1}{2}) + 32(r-1)^2(m^2 + m - 2mr) + 16m(r-1)^3 + 16m(r-1)^2 \\ &\quad (m-2r+2) + 12m(r-1)^2 + 8(r-1)^2 \sum_{k=2}^m (k-1). \end{aligned}$$

When $r > 2$

$$\begin{aligned} DD_*(G^{++}) &= 16(r-1)^2(-m + \frac{1}{2}) + 32(r-1)^2(m^2 + m - 2mr) + 16m(r-1)^3 + 16m(r-1)^2 \\ &\quad (m-2r+2) + 8(r-1)^2 \sum_{k=2}^m (k-1). \end{aligned}$$

On simplification, we get (*) and (**)

$$DD_*(G^{++}) = 8(r-1)^2 [2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^m (k-1)]$$

and

$$DD_*(G^{++}) = 8(r-1)^2 [2m(3m-5r+2) + nr^2 + \sum_{k=2}^m (k-1)].$$

Observation C.

1. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e_j)] d_{G^{+-}}(e_i, e_j) \text{ in } G^{+-} = 2(m-1)(-m + \frac{1}{2}nr^2).$$

If $d_{G^{+-}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e_j)] d_{G^{+-}}(e_i, e_j) \text{ in } G^{+-} = 4(m-1)(m^2 + m - 2mr).$$

2. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) \text{ in } G^{+-} = 2m(m-r)(m-2r+1).$$

If $d_{G^{+-}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) \text{ in } G^{+-} = 8m(m-r)(r-1).$$

If $d_{G^{+-}}(e_i, e'_j) = 3$, then

$$\sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) \text{ in } G^{+-} = 6m(m-r).$$

3. Let G be any (n,m) graph.

If $d_G(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e'_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e'_i, e'_j) \text{ in } G^{+-} = 4(m-2r+1) \sum_{k=2}^m (k-1).$$

Theorem 3.5. For any (n,m) graph G ,

$$DD_+(G^{+-}) = 2m[m(3m-5)-1] + mr[nr-6m-4r+8] - nr^2 + 4(m-2r+1) \sum_{k=1}^r (k-1).$$

Proof. Let G be any (n,m) -graph. From Proposition 3.1, G^{+-} contains $2m$ vertices and

$$(-m^2 + \frac{1}{2}(nr^2) - 2mr) \text{ edges.}$$

From (2), we have

$$DD_+(G) = \sum_{u, v \in V(G)} [deg_G(u) + deg_G(v)] d_G(u, v)$$

Therefore,

$$DD_+(G^{+-}) = \sum_{(e_i, e_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e_j)] d_{G^{+-}}(e_i, e_j) + \sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) + \sum_{(e'_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e'_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e'_i, e'_j).$$

Applying observation C to the above equation, we get

$$DD_+(G^{+-}) = 2m(m-1)(r-1) + 4(m-1)(m^2 + m - 2mr) + 2(m-r)(m-2r+1) + 8m(m-r)(r-1) + 6m(m-r) + 2m(m-2r+1) \sum_{k=1}^m (k-2).$$

On simplification,

$$DD_+(G^{+-}) = 2m[m(3m-5)-1] + mr[nr-6m-4r+8] - nr^2 + 4(m-2r+1) \sum_{k=1}^r (k-1).$$

Observation D.

1. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) \cdot deg_{G^{+-}}(e_j)] d_{G^{+-}}(e_i, e_j) \text{ in } G^{+-} = (m-1)^2(-m + \frac{1}{2}nr^2).$$

If $d_{G^{+-}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) \cdot deg_{G^{+-}}(e_j)] d_{G^{+-}}(e_i, e_j) \text{ in } G^{+-} = 2(m-1)^2(m^2 + m - 2mr).$$

2. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) \cdot deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) \text{ in } G^{+-} = (m^2 - 2mr + 2r - 1)(m^2 - 2rm + m).$$

If $d_{G^{+-}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) \cdot deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) \text{ in } G^{+-} = 4(m^2 - 2mr + 2r - 1)(r - 1)m.$$

If $d_{G^{+-}}(e_i, e'_j) = 3$, then

$$\sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) \cdot deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) \text{ in } G^{+-} = 3m(m^2 - 2m + 2r - 1).$$

3. Let G be any (n,m) graph.

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e'_i) \cdot deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e'_i, e'_j) \text{ in } G^{+-} = 2(m - 2r + 2)^2 \sum_{k=2}^m (k - 1).$$

Theorem 3.6. For any (n,m) graph G ,

$$DD_*(G^{+-}) = m[m^2(3m - 3) - 7(m + 1)] + \frac{1}{2}nr^2(m^2 - m + 1) - 2mr[2m(3 - r) + 2r - 3] + 2(m - 2r + 2)^2 \sum_{k=2}^m (k - 1)^2.$$

Proof. Let G be any (n,m) -graph. From Proposition 3.1, G^{+-} contains $2m$ vertices and

$$-m^2 + \frac{1}{2}(nr^2) - 2mr \text{ edges}$$

From (3), we have

$$DD_*(G) = \sum_{u, v \in V(G)} [deg_G(u) \cdot deg_G(v)] d_G(u, v)$$

Therefore,

$$DD_*(G^{+-}) = \sum_{(e_i, e_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) \cdot deg_{G^{+-}}(e_j)] d_{G^{+-}}(e_i, e_j) + \sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) \cdot deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) + \sum_{(e'_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e'_i) \cdot deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e'_i, e'_j).$$

Applying observation D to the above equation, we get

$$DD_*(G^{+-}) = (m - 1)^2(-m + \frac{1}{2}nr^2) + 2(m - 1)^2(m^2 + m - 2mr) + (m^2 - 2mr + 2r - 1)(m - 2r + 1) + 4(m^2 - 2mr + 2r - 1)(r - 1)m + 3m(m^2 - 2m + 2r - 1) + 2(m - 2r + 2)^2 \sum_{k=1}^r (k - 1).$$

On simplification,

$$DD_*(G^{+-}) = m[m^2(3m-3) - 7(m+1)] + \frac{1}{2}nr^2(m^2 - m + 1) - 2mr[2m(3-r) + 2r - 3] + 2(m-2r+2)^2 \sum_{k=2}^m (k-1)^2.$$

Observation E.

1. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e_j)] d_{G^{+-}}(e_i, e_j) \text{ in } G^{+-} = (m-2r+1)(m-1)m.$$

If $d_{G^{+-}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e_j)] d_{G^{+-}}(e_i, e_j) \text{ in } G^{+-} = (m-1)(r-1)m.$$

2. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) = 2m(m+2r-3)(r-1).$$

If $d_{G^{+-}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) = 2m(m+2r-3)(m-2r+2).$$

If $d_{G^{+-}}(e_i, e'_j) = 3$, then

$$\sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) = 3m(m+2r-3).$$

3. Let G be any (n,m) graph.

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \in V(G^{+-})} [deg_G(e'_i) + deg_G(e'_j)] d_{G^{+-}}(e'_i, e'_j) \text{ in } G^{+-} = 8(r-1) \sum_{k=2}^m (k-1).$$

Theorem 3.7. For any (n,m) graph G,

$$DD_+(G^{+-}) = m[m(3m-2) - 15] - mr[m - 4r - 17] + 8(r-1) \sum_{k=2}^m (k-1).$$

Proof. Let G be any (n,m)-graph, from the Proposition 3.1, G^{+-} contains 2m vertices and

$$\frac{m}{2}[m+2r-3] \text{ edges.}$$

From (2), we have

$$DD_+(G) = \sum_{u, v \in V(G)} [deg_G(u) + deg_G(v)] d_G(u, v)$$

Therefore,

$$\begin{aligned} DD_+(G^{+-}) &= \sum_{(e_i, e_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e_j)] d_{G^{+-}}(e_i, e_j) + \\ &\quad \sum_{(e_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e_i, e'_j) + \\ &\quad \sum_{(e'_i, e'_j) \in V(G^{+-})} [deg_{G^{+-}}(e'_i) + deg_{G^{+-}}(e'_j)] d_{G^{+-}}(e'_i, e'_j). \end{aligned}$$

Applying observation E to the above equation, we get

$$\begin{aligned} DD_+(G^{+-}) &= (m-2r+1)(m-1)m + (m-1)(r-1)m + 2m(m+2r-3)(r-1) + \\ &\quad 2m(m+2r-3)(m-2r+2) + 3m(m+2r-3) + 8(r-1) \sum_{k=2}^m (k-1). \end{aligned}$$

On simplification,

$$DD_+(G^{++}) = m[m(3m - 2) - 15] - mr[m - 4r - 17] + 8(r - 1) \sum_{k=2}^m (k - 1).$$

Observation F.

1. Let G be any (n,m) graph.

If $d_{G^{++}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e_j)] d_{G^{++}}(e_i, e_j) \text{ in } G^{++} = \frac{1}{2}(m - 1)^2(m^2 - 2mr + m).$$

If $d_{G^{++}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e_j)] d_{G^{++}}(e_i, e_j) \text{ in } G^{++} = 2(m - 1)^2(r - 1)m.$$

2. Let G be any (n,m) graph.

If $d_{G^{++}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) \text{ in } G^{++} = 4m(mr - m - r + 1)(r - 1).$$

If $d_{G^{++}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) \text{ in } G^{++} = 4m(mr - m - r - 1)(m - 2r + 2).$$

If $d_{G^{++}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j) \text{ in } G^{++} = 6m(mr - m - r + 1).$$

3. Let G be any (n,m) graph.

If $d_{G^{++}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j) \text{ in } G^{++} = 8(r - 1)^2 \sum_{k=2}^m (k - 1).$$

Theorem 3.8. For any (n,m) graph G,

$$DD_*(G^{++}) = \frac{1}{2} m[m^2(m - 13) - 5m + 8] + mr[m(5m - 4r + 8) + 4r - 13] + 8(r - 1)^2 \sum_{k=2}^m (k - 1).$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{++} contains 2m vertices and

$$\frac{m}{2}[m + 2r - 3] \text{ edges.}$$

From (3), we have

$$DD_*(G) = \sum_{u, v \subseteq V(G)} [deg_G(u) \cdot deg_G(v)] d_G(u, v)$$

Therefore,

$$DD_*(G^{++}) = \sum_{(e_i, e_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e_j)] d_{G^{++}}(e_i, e_j) + \sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) + \sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j).$$

Applying observation F to the above equation, we get

$$DD_*(G^{++}) = \frac{1}{2}(m - 1)^2(m^2 - 2mr + m) + 2(m - 1)^2(r - 1)m + 4m(mr - m - r + 1)(r - 1)$$

$$+ 4m(mr - m - r - 1)(m - 2r + 2) + 6m(mr - m - r + 1) + 8(r - 1)^2 \sum_{k=2}^m (k - 1).$$

On simplification,

$$DD_*(G^{--}) = \frac{1}{2} m[m^2(m - 13) - 5m + 8] + mr[m(5m - 4r + 8) + 4r - 13] + 8(r - 1)^2 \sum_{k=2}^m (k - 1).$$

Observation G.

1. Let G be any (n,m) graph.

If $d_{G^{--}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{--})} [deg_{G^{--}}(e_i) + deg_{G^{--}}(e_j)] d_{G^{--}}(e_i, e_j) \text{ in } G^{--} = 2m(m - 2r + 1)^2.$$

If $d_{G^{--}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{--})} [deg_{G^{--}}(e_i) + deg_{G^{--}}(e_j)] d_{G^{--}}(e_i, e_j) \text{ in } G^{--} = 8(m - 2r + 1)(r - 1).$$

2. Let G be any (n,m) graph.

If $d_{G^{--}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{--})} [deg_{G^{--}}(e_i) + deg_{G^{--}}(e'_j)] d_{G^{--}}(e_i, e'_j) \text{ in } G^{--} = 3m(m - 2r + 1)^2.$$

if $d_{G^{--}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{--})} [deg_{G^{--}}(e_i) + deg_{G^{--}}(e'_j)] d_{G^{--}}(e_i, e'_j) \text{ in } G^{--} = 6m(2r - 1)(m - 2r + 1).$$

3. Let G be any (n,m) graph.

If $d_{G^{--}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{--})} [deg_{G^{--}}(e'_i) + deg_{G^{--}}(e'_j)] d_{G^{--}}(e'_i, e'_j) \text{ in } G^{--} = 4(m - 2r + 1) \sum_{k=2}^m (k - 1).$$

Theorem 3.9. For any (n,m) graph G,

$$DD_+(G^{--}) = m[5m^2 - 4m - 9] - 4mr[5r - 7] + 4(m - 2r + 1) \sum_{k=2}^m (k - 1).$$

Proof. Let G be any (n,m)-graph. From proposition 3.1, G^{--} contains $2m$ vertices and

$$\frac{3}{2} m[m - 2r + 1] \text{ edges.}$$

From (2), we have

$$DD_+(G) = \sum_{u, v \subseteq V(G)} [deg_G(u) + deg_G(v)] d_G(u, v)$$

Therefore,

$$\begin{aligned} DD_+(G^{--}) &= \sum_{(e_i, e_j) \subseteq V(G^{--})} [deg_{G^{--}}(e_i) + deg_{G^{--}}(e_j)] d_{G^{--}}(e_i, e_j) + \\ &\quad \sum_{(e_i, e'_j) \subseteq V(G^{--})} [deg_{G^{--}}(e_i) + deg_{G^{--}}(e'_j)] d_{G^{--}}(e_i, e'_j) + \\ &\quad \sum_{(e'_i, e'_j) \subseteq V(G^{--})} [deg_{G^{--}}(e'_i) + deg_{G^{--}}(e'_j)] d_{G^{--}}(e'_i, e'_j). \end{aligned}$$

Applying observation G to the above equation, we get

$$\begin{aligned} DD_+(G^{--}) &= 2m(m - 2r + 1)^2 + 8(m - 2r + 1)(r - 1) + 3m(m - 2r + 1)^2 + 6m(2r - 1)(m - 2r + 1) \\ &\quad + 4(m - 2r + 1) \sum_{k=2}^m (k - 1) \end{aligned}$$

On simplification,

$$DD_+(G^-) = m[5m^2 - 4m - 9] - 4mr[5r - 7] + 4(m - 2r + 1) \sum_{k=2}^m (k - 1).$$

Observation H.

1. Let G be any (n,m) graph.

If $d_{G^-}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \in V(G^-)} [deg_{G^-}(e_i) \cdot deg_{G^-}(e_j)] d_{G^-}(e_i, e_j) \text{ in } G^- = 2m(m - 2r + 1)^3.$$

If $d_{G^-}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \in V(G^-)} [deg_{G^-}(e_i) \cdot deg_{G^-}(e_j)] d_{G^-}(e_i, e_j) \text{ in } G^- = 8m(m - 2r + 1)^2(r - 1).$$

2. Let G be any (n,m) graph.

If $d_{G^-}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \in V(G^-)} [deg_{G^-}(e_i) \cdot deg_{G^-}(e'_j)] d_{G^-}(e_i, e'_j) \text{ in } G^- = 2m(m - 2r + 1)^3.$$

If $d_{G^-}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \in V(G^-)} [deg_{G^-}(e_i) \cdot deg_{G^-}(e'_j)] d_{G^-}(e_i, e'_j) \text{ in } G^- = 4m(m - 2r + 2)^2(2r - 1).$$

3. Let G be any (n,m) graph.

If $d_{G^-}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \in V(G^-)} [deg_{G^-}(e'_i) \cdot deg_{G^-}(e'_j)] d_{G^-}(e'_i, e'_j) \text{ in } G^- = 2(m - 2r + 1)^2 \sum_{k=2}^m (k - 1).$$

Theorem 3.10. For any (n,m) graph G,

$$DD_*(G^-) = 2m[m^2(m - 3) - 5m - 6] - 4mr[3m^2 - 4r^2 + 10mr - 14m + 12r - 13] + 2(m - 2r + 1)^2 \sum_{k=2}^m (k - 1).$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^- contains 2m vertices and

$$\frac{3}{2}m[m - 2r + 1] \text{ edges.}$$

From (3), we have

$$DD_*(G) = \sum_{u, v \in V(G)} [deg_G(u) \cdot deg_G(v)] d_G(u, v)$$

Therefore,

$$DD_*(G^-) = \sum_{(e_i, e_j) \in V(G^-)} [deg_{G^-}(e_i) \cdot deg_{G^-}(e_j)] d_{G^-}(e_i, e_j) + \sum_{(e_i, e'_j) \in V(G^-)} [deg_{G^-}(e_i) \cdot deg_{G^-}(e'_j)] d_{G^-}(e_i, e'_j) + \sum_{(e'_i, e'_j) \in V(G^-)} [deg_{G^-}(e'_i) \cdot deg_{G^-}(e'_j)] d_{G^-}(e'_i, e'_j).$$

Applying observation H the above equation, we get

$$DD_*(G^-) = 2m(m - 2r + 1)^3 + 8m(m - 2r + 1)^2(r - 1) + 2m(m - 2r + 1)^3 + 4m(m - 2r + 2)^2(2r - 1) + 2(m - 2r + 1)^2 \sum_{k=2}^m (k - 1).$$

On simplification,

$$DD_*(G^-) = 2m[m^2(m - 3) - 5m - 6] - 4mr[3m^2 - 4r^2 + 10mr - 14m + 12r - 13] + 2(m - 2r + 1)^2 \sum_{k=2}^m (k - 1).$$

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