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The Sum Degree Distance and the Product Degree Distance of Generalized Transformation Graphs G^{ab}

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Abstract. In this contribution, we consider line splitting graph $L_s(G)$ of a graph G as transformation graph G^{++} of G^{ab}. We investigate the sum degree distance $DD_+(G)$ and product degree distance $DD_*(G)$ of transformation graph G^{ab}, which are weighted version of Wiener index. The Transformation graphs of G^{ab} are G^{++}, G^{+-}, G^{-+} and G^{--}.

1. Introduction

Throughout this paper, we consider finite, un-directed, simple, connected, r-regular graphs with vertex set $V(G)=\{v_1,v_2,v_3,...,v_n\}$ and edge set $E(G)=\{e_1,e_2,e_3,...,e_m\}$. For the undefined terminologies we refer[8].

The degree of vertex in a graph G is denoted by $deg_G(v)$ or $d_G(v)$ and the distance between two vertices v_i and v_j, denoted by $dist_G(v_i,v_j)$ or $d_G(v_i,v_j)$, is the length of a shortest path between the vertices v_i and v_j in G. The shortest v_i- v_j path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic. The graphs considered in this construction are with $diam \leq 2$. The degree of an edge e_i in G is the number of edges adjacent to e_i and is denoted by $deg_G(e_i)$ .The degree of edge in a graph G is

$$deg_G(e_i) = deg_G(uv) = deg_G(u) + deg_G(v) - 2.$$

Topological indices and graph invariants based on distances between vertices of a graph are widely used in mathematical chemistry[2], which are due to their correlations with physical, chemical and thermodynamic parameters of chemical compounds.

One of the oldest and well studied distance based graph invariant associated with a connected graph G is the Wiener number W(G), also termed as Wiener index in chemical or mathematical chemistry literature, which is defined in [13] as the sum of distances over all unordered vertex pairs in G,

i.e.,

$$W(G) = \sum_{i \leq j} d(v_i, v_j) \tag{1}$$

Which was first time introduced by Wiener. Initially, the Wiener index W(G) was considered as a molecular structure descriptor used in chemical applications, but soon it attracted the interest of pure mathematicians[1,3,5,14,15].

Eventually a number of modifications of the Wiener index were proposed, which are as follows.

$$DD_+(G) = \sum_{u,v \subseteq V(G)} [deg_G(u) + deg_G(v)] d_G(u,v) \tag{2}$$

$$DD_*(G) = \sum_{u,v \subseteq V(G)} [deg_G(u) \cdot deg_G(v)] d_G(u,v) \tag{3}$$

The graph invariants defined in (2) and (3) have all been much studied in the past. The invariant DD_+ was first time introduced by Dobrynin and Kochetova[4] and named as sum-degree distance. Later the same quantity was examined under the name "Schulz index" [7]. For mathematical research on degree distance see[9,12] and the references cited therein. A remarkable property of DD_+ is that in the case of trees of order n, the identity DD_+=4W-n(n-1) holds [10].

Gutman [7] proved that the multiplicative variant of the degree distance, namely DD_*, *i.e*., from (2) , obeys an analogous relation: DD_*= 4W-(2n-1) (n-1). This latter quantity is sometimes referred to as the "Gutman index"[6], but here we call it product-degree distance.

The open neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i

$$i.e, \qquad N(e_i) = \{e_i/e_i, e_j \text{ are adjacent in G }\}.$$

For each edge e_i of G, a new vertex e_i' is taken and the resulting set of vertices is denoted by $E'(G)$.

The line Splitting graph $L_s(G)$ of a graph G is defined as the graph having vertex set $E(G) \cup E'(G)$ with two vertices adjacent if they correspond to adjacent edges of G or one corresponds to an element e_i' of $E'(G)$ and the other to an element e_j of E(G), and e_j is in $N(e_i)$. This concept was introduced by Kulli and Biradar in[11].

2 Generalized Transformation Graphs G^{ab}

Let G = (V, E) be a graph. Let α, β and α', β' be the element of $E(G)$ and $E'(G)$ respectively. We say that the associativity of α and β is +, if they are adjacent in G otherwise is - and the associativity of α and β' or α' and β is +, if α is the neighborhood point of β or β is neighborhood point of α in G, otherwise is - .

Let ab be a 2-permutation of the set $\{+,-\}$. We say that α and β corresponds to the first term a of ab, and α , β $\in$ E(G). Whereas α and β' or β and α' corresponds to the both first and second term of ab and $\alpha', \beta' \in E'(G)$.

The transformation graph G^{ab} of a graph G is the graph with vertex set $E(G) \cup E'(G)$. $\alpha\, and\, \beta$ or $\alpha\, and\, \beta'$ or $\beta\, and\, \alpha'$ are adjacent if and only if the following conditions holds;

* $\alpha, \beta \in E(G)$, α and β are adjacent in G if a = + otherwise a= -.

** $\alpha, \beta \in E(G)$ and $\alpha', \beta' \in E'(G)$, if α neighborhood points of β or β is neighborhood point of α in G then b = + otherwise b= -.

Since there are four distinct 2-permutations of $\{+,-\}$, we obtain 4-graphical transformations of G. Here we consider G^{++}, which is nothing but line splitting graph of G and the other generalized transformation graphs are G^{+-}, G^{-+} and G^{--}.

Note that, in this paper we consider graphs with n ≥ 5 for G^{++} and G^{-+} and in particular for G^{+-} and G^{--} we consider graphs with n > 5 and having atleast three edges e_i , e_j and e_w ∈E(G); i,j,w=1,2,3,...,m and i≠j≠ w such that e_i and e_j are non adjacent edges and e_w is non adjacent to e_i and e_j.

The aim of present work is to obtain the expression for the sum degree distance and product degree distance of the generalized transformation graphs G^{ab}.

3. Results

In this section we obtain the sum degree distance and product degree distance of the transformation graphs G^{ab}, which is line splitting graph i.e., G^{++}, and its generalized transformation graphs G^{+-}, G^{-+}, G^{--}.

We start by stating the following propositions and observations, needed for proving our main results.

Proposition 3.1 Let G be an (n,m) graph. Then by the definition order of G^{ab} is 2m and

(i) The size of G^{++} is $-m+\frac{1}{2}nr^2+2m(r-1)$.

(ii) The size of G^{+-} is $-m^2+\frac{1}{2}(nr^2)-2mr$.

(iii) The size of G^{-+} is $\frac{m}{2}[m+2r-3]$.

(iv) The size of G^{--} is $\frac{3}{2}m[m-2r+1]$.

Proof. Let G be a (n,m)-graph with regular degree r,then

(i) $E(G^{++}) = E(L(G)) + \sum_{i=1}^{m}\deg_G(e_i)$

$= -m+\frac{1}{2}\sum_{i=1}^{n}d_i^2+\sum_{i=1}^{m}\deg_G(e_i)$

$= -m+\frac{1}{2}nr^2+m(2r-2)$ $[\because G \text{ is } r-\text{regular graph}]$

$E(G^{++}) = -m+\frac{1}{2}nr^2+2m(r-1)$.

(ii) $E(G^{+-}) = E(L(G)) + \sum_{(uv)\in E(G)} \text{the edges which are not incident to u and v in G}$

$= -m+\frac{1}{2}nr^2+m(m-2r+1)$ $[\because G \text{ is } r-\text{regular graph}]$

$= -m+\frac{1}{2}nr^2+m^2-2mr+m$

$E(G^{+-}) = m^2+\frac{1}{2}nr^2-2mr$.

(iii) $E(G^{-+}) = \frac{1}{2}\sum_{(uv)\in E(G)} \text{the edges which are not incident to u and v in G} + \sum_{i=1}^{m}\deg_G(e_i)$

$= \frac{1}{2}m(m-2r+1)+m(\deg_G(u)+\deg_G(v)-2)$

$[\because G \text{ is } r-\text{regular graph}]$

$= \frac{1}{2}m^2-mr+\frac{1}{2}m+m(2r-2)$

$E(G^{-+}) = \frac{m}{2}[m+2mr-3]$.

(iv) $E(G^{--}) = \frac{1}{2}\sum_{(uv)\in E(G)} \text{the edges which are not incident to u and v in G} +$

$\sum_{(uv)\in E(G)} \text{the edges which are not incident to u and v in G}$

$= \frac{1}{2}m(m-2r+1)+m(m-2r+1)$ $[\because G \text{ is } r-\text{regular graph}]$

$E(G^{--}) = \frac{3}{2}m[m-2r+1]$.

Proposition 3.2 Let G be an (n,m) graph. Then the degree of vertices e_i *and* e_i' of G^{ab} are,

(i) $d_{G^{++}}(e_i) = 4(r-1)$ and $d_{G^{++}}(e_i') = 2(r-1)$.

(ii) $d_{G^{+-}}(e_i) = (m-1)$ and $d_{G^{+-}}(e_i') = m-2r+1$.

(iii) $d_{G^{-+}}(e_i) = (m-1)$ and $d_{G^{-+}}(e_i') = 2(r-1)$.

(iv) $d_{G^{--}}(e_i) = 2(m-2r+1)$ and $d_{G^{--}}(e_i') = (m-2r+1)$.

Proof. Let G be a (n,m)-graph with regular degree r, then

(i) $d_{G^{++}}(e_i) = 2\deg_G(e_i) = 2(2r-2) = 4(r-1)$ and

$d_{G^{++}}(e_i') = \deg_G(e_i) = 2r-2 = 2(r-1).$

(ii) $d_{G^{+-}}(e_i) = \deg_G(e_i) + (m-2r+1) = 2r-2+m-2r+1 = (m-1)$ and

$d_{G^{+-}}(e_i') =$ The total number of edges which are not incident to u and v in G and $uv = e_i$.

$= (m-2r+1).$

(iii) $d_{G^{-+}}(e_i) = (m-2r+1) + \deg_G(e_i) = m-2r+1+2r-2 = (m-1)$ and

$d_{G^{-+}}(e_i') = \deg_G(e_i) = (2r-2) = 2(r-1).$

(iv) $d_{G^{--}}(e_i) = 2($The total number of edges which are not incident to u and v in G and $uv = e_i)$

$= 2(m-2r+1)$ and

$d_{G^{--}}(e_i') =$ The total number of edges which are not incident to u and v in G and $uv = e_i$.

$= (m-2r+1).$

We use Proposition 3.2 for the following observations.

Observation A.

1. G be any (n, m) graph.

If $d_{G^{++}}(e_i, e_j) = 1$, then

$\sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i) + deg_{G^{++}}(e_j)]d_{G^{++}}(e_i, e_j)$ in $G^{++} = 8m(r-1)^2$.

If $d_{G^{++}}(e_i, e_j) = 2$, then

$\sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i) + deg_{G^{++}}(e_j)]d_{G^{++}}(e_i, e_j)$ in $G^{++} = 16(r-1)(m^2+m-2mr)$.

2. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i, e_j') = 1$, then

$\sum_{(e_i,e_j')\subseteq V(G^{++})}[deg_{G^{++}}(e_i) + deg_{G^{++}}(e_j')]d_{G^{++}}(e_i, e_j')$ in $G^{++} = 12m(r-1)^2$.

If $d_{G^{++}}(e_i, e_j') = 2$, then

$\sum_{(e_i,e_j')\subseteq V(G^{++})}[deg_{G^{++}}(e_i) + deg_{G^{++}}(e_j')]d_{G^{++}}(e_i, e_j')$ in $G^{++} = 12m(r-1)(m-2r+2)$.

3. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i', e_j') = 2$, then

$\sum_{(e_i',e_j')\subseteq V(G^{++})}[deg_{G^{++}}(e_i') + deg_{G^{++}}(e_j')]d(e_i', e_j')$ in $G^{++} = 8(r-1)\sum_{k=1}^{m}(k-1)$.

and

If $d_{G^{++}}(e_i', e_j') = 3$ when r = 2, then

$\sum_{(e_i',e_j')\subseteq V(G^{++})}[deg_{G^{++}}(e_i') + deg_{G^{++}}(e_j')]d(e_i', e_j')$ in G^{++} =12m(r-1).

Theorem 3.3. For any (n, m) graph G with r ≥ 2,
if r = 2 then

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r)+3m+nr^2+2\sum_{k=1}^{m}(k-1)] \qquad (*)$$

And if r > 2 then

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r)+nr^2+2\sum_{k=1}^{m}(k-1)] \qquad (**)$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{++} contains 2m vertices and $(-m+\frac{1}{2}nr^2+2m(r-1))$ edges.

From (2), we have

$$DD_+(G) = \sum_{u,v\subseteq V(G)}[deg_G(u)+deg_G(v)]d_G(u,v)$$

Therefore,

$$DD_+(G^{++}) = \sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)+deg_{G^{++}}(e_j)]d_{G^{++}}(e_i,e_j)+$$
$$\sum_{(e_i,e'_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)+deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i,e'_j)+$$
$$\sum_{(e'_i,e'_j)\subseteq V(G^{++})}[deg_{G^{++}}(e'_i)+deg_{G^{++}}(e'_j)]d_{G^{++}}(e'_i,e'_j).$$

Applying observation A to the above equation,
when r = 2,

$$DD_+(G^{++}) = 8\text{m}(r-1)^2(-m+\frac{1}{2}nr^2)+16\text{m(r-1)}(m-2r+1)+12\text{m}(r-1)^2+12\text{m(r-1)}$$
$$\text{(m-2r+2)+12m(r-1)+8(r-1)}\sum_{k=1}^{m}(k-1).$$

and r >2,

$$DD_+(G^{++}) = 8\text{m}(r-1)^2(-m+\frac{1}{2}nr^2)+16\text{m(r-1)}(m-2r+1)+12\text{m}(r-1)^2+12\text{m(r-1)}$$
$$\text{(m-2r+2)+8(r-1)}\sum_{k=1}^{m}(k-1).$$

On simplification, we get $(*)$ *and* $(**)$
i.e.,

$$DD_+(G^{++}) = 4(r-1)[m(7m+5-11r)+3m+nr^2+2\sum_{k=1}^{m}(k-1)]$$

and

$$DD_+(G^{++}) = 4\text{(r-1)}[m(7m+5-11r)+nr^2+2\sum_{k=2}^{m}(k-1)].$$

Observation B.

1. Let G be any (n,m) graph.

   If $d_{G^{++}}(e_i,e_j)=1$, then

   $$\sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)\cdot deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i,e_j) \text{ in } G^{++} = 16m\,(r-1)^2(-m+\frac{1}{2}nr^2).$$

   If $d_{G^{++}}(e_i,e_j)=2$, then

   $$\sum_{(e_i,e_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)\cdot deg_{G^{++}}(e_j)]d_{G^{++}}(e_i,e_j) \text{ in } G^{++} = 32\,(r-1)^2(m^2+m-2mr).$$

2. Let G be any (n,m) graph.

   If $d_{G^{++}}(e_i,e'_j)=1$, then

   $$\sum_{(e_i,e'_j)\subseteq V(G^{++})}[deg_{G^{++}}(e_i)\cdot deg_{G^{++}}(e'_j)]d_{G^{++}}(e_i,e'_j) \text{ in } G^{++} = 16m\,(r-1)^3m.$$

If $d_{G^{++}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) \text{ in } G^{++} = 16m(r-1)^2(m-2r+2).$$

3. Let G be any (n,m) graph.

If $d_{G^{++}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j) \text{ in } G^{++} = 8(r-1)^2 \sum_{k=2}^{m} (k-1).$$

If $d_{G^{++}}(e'_i, e'_j) = 3$, when r = 2, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j) \text{ in } G^{++} = 12m(r-1)^2.$$

Theorem 3.4. For any (n,m) graph G with $r \geq 2$,

when r = 2

$$DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^{m}(k-1)] \qquad (*')$$

and when r > 2

$$DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^{m}(k-1)] \qquad (**')$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{++} contains 2m vertices and $(-m + \frac{1}{2}nr^2 + 2m(r-1))$ edges.

From (3), we have

$$DD_*(G) = \sum_{u,v \subseteq V(G)} [deg_G(u) \cdot deg_G(v)] d_G(u,v)$$

Therefore,

$$DD_*(G^{++}) = \sum_{(e_i, e_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e_j)] d_{G^{++}}(e_i, e_j) + \sum_{(e_i, e'_j) \subseteq V(G^{++})} [deg_{G^{++}}(e_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e_i, e'_j) + \sum_{(e'_i, e'_j) + \subseteq V(G^{++})} [deg_{G^{++}}(e'_i) \cdot deg_{G^{++}}(e'_j)] d_{G^{++}}(e'_i, e'_j).$$

Applying observation B to the above equation,

when r = 2,

$$DD_*(G^{++}) = 16(r-1)^2(-m + \frac{1}{2}) + 32(r-1)^2(m^2 + m - 2mr) + 16m(r-1)^3 + 16m(r-1)^2 (m-2r+2) + 12m(r-1)^2 + 8(r-1)^2 \sum_{k=2}^{m}(k-1).$$

When r > 2

$$DD_*(G^{++}) = 16(r-1)^2(-m + \frac{1}{2}) + 32(r-1)^2(m^2 + m - 2mr) + 16m(r-1)^3 + 16m(r-1)^2 (m-2r+2) + 8(r-1)^2 \sum_{k=2}^{m}(k-1).$$

On simplification, we get $(*')$ *and* $(**')$

$$DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + 3m + \sum_{k=2}^{m}(k-1)]$$

and

$$DD_*(G^{++}) = 8(r-1)^2[2m(3m-5r+2) + nr^2 + \sum_{k=2}^{m}(k-1)].$$

Observation C.

1. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i,e_j)=1$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j) \text{ in } G^{+-} = 2(m-1)(-m+\frac{1}{2}nr^2).$$

If $d_{G^{+-}}(e_i,e_j)=2$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j) \text{ in } G^{+-} = 4(m-1)(m^2+m-2mr).$$

2. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i,e'_j)=1$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) \text{ in } G^{+-} = 2m(m-r)(m-2r+1).$$

If $d_{G^{+-}}(e_i,e'_j)=2$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) \text{ in } G^{+-} = 8m(m-r)(r-1).$$

If $d_{G^{+-}}(e_i,e'_j)=3$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) \text{ in } G^{+-} = 6m(m-r).$$

3. Let G be any (n,m) graph.

If $d_G(e'_i,e'_j)=2$, then

$$\sum_{(e'_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e'_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e'_i,e'_j) \text{ in } G^{+-} = 4(m-2r+1)\sum_{k=2}^{m}(k-1).$$

Theorem 3.5. For any (n,m) graph G,

$$DD_+(G^{+-}) = 2m[m(3m-5)-1]+mr[nr-6m-4r+8]-nr^2+$$
$$4(m-2r+1)\sum_{k=1}^{r}(k-1).$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{+-} contains 2m vertices and $(-m^2+\frac{1}{2}(nr^2)-2mr)$ edges.

From (2), we have

$$DD_+(G)=\sum_{u,v\subseteq V(G)}[deg_G(u)+deg_G(v)]d_G(u,v)$$

Therefore,

$$DD_+(G^{+-}) = \sum_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j)+$$
$$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j)+$$
$$\sum_{(e'_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e'_i)+deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e'_i,e'_j).$$

Applying observation C to the above equation, we get

$$DD_+(G^{+-}) = 2m(m-1)(r-1)+4(m-1)(m^2+m-2mr)+2(m-r)(m-2r+1)+$$
$$8m(m-r)(r-1)+6m(m-r)+2m(m-2r+1)\sum_{k=1}^{m}(k-2).$$

On simplification,

$$DD_+(G^{+-}) = 2m[m(3m-5)-1]+mr[nr-6m-4r+8]-nr^2+$$
$$4(m-2r+1)\sum_{k=1}^{r}(k-1).$$

Observation D.

1. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i, e_j) = 1$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j) \text{ in } G^{+-} = (m-1)^2(-m+\frac{1}{2}nr^2).$$

If $d_{G^{+-}}(e_i, e_j) = 2$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j) \text{ in } G^{+-} = 2(m-1)^2(m^2+m-2mr).$$

2. Let G be any (n,m) graph.

If $d_{G^{+-}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) \text{ in } G^{+-} = (m^2-2mr+2r-1)$$

$(m^2-2rm+m)$.

If $d_{G^{+-}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) \text{ in } G^{+-} = 4(m^2-2mr+2r-1)(r-1)m.$$

If $d_{G^{+-}}(e_i, e'_j) = 3$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) \text{ in } G^{+-} = 3m(m^2-2m+2r-1).$$

3. Let G be any (n,m) graph.

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e'_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e'_i,e'_j) \text{ in } G^{+-} = 2(m-2r+2)^2\sum_{k=2}^{m}(k-1). .$$

Theorem 3.6. For any (n,m) graph G,

$$DD_*(G^{+-}) = m[m^2(3m-3)-7(m+1)] + \frac{1}{2}nr^2(m^2-m+1) - 2mr[2m(3-r)+2r-3] +$$
$$2(m-2r+2)^2\sum_{k=2}^{m}(k-1)^2.$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{+-} contains 2m vertices and $-m^2+\frac{1}{2}(nr^2)-2mr$ edges

From (3), we have

$$DD_*(G) = \sum_{u,v\subseteq V(G)}[deg_G(u)\cdot deg_G(v)]d_G(u,v)$$

Therefore,

$$DD_*(G^{+-}) = \sum_{(e_i,e_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e_j)]d_{G^{+-}}(e_i,e_j) +$$
$$\sum_{(e_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e_i,e'_j) +$$
$$\sum_{(e'_i,e'_j)\subseteq V(G^{+-})}[deg_{G^{+-}}(e'_i)\cdot deg_{G^{+-}}(e'_j)]d_{G^{+-}}(e'_i,e'_j).$$

Applying observation D to the above equation, we get

$$DD_*(G^{+-}) = (m-1)^2(-m+\frac{1}{2}nr^2) + 2(m-1)^2(m^2+m-2mr) +$$
$$(m^2-2mr+2r-1)(m-2r+1) + 4(m^2-2mr+2r-1)(r-1)m +$$
$$3m(m^2-2m+2r-1) + 2(m-2r+1)^2\sum_{k=1}^{r}(k-1).$$

On simplification,

$$DD_*(G^{+-}) = m[m^2(3m-3)-7(m+1)] + \frac{1}{2}nr^2(m^2-m+1) - 2mr[2m(3-r)+2r-3] + 2(m-2r+2)^2\sum_{k=2}^{m}(k-1)^2.$$

Observation E.

1. Let G be any (n,m) graph.
   If $d_{G^{-+}}(e_i,e_j)=1$, then
   $\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)+deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j)$ in G^{-+} = $(m-2r+1)(m-1)m$.
   If $d_{G^{-+}}(e_i,e_j)=2$, then
   $\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)+deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j)$ in G^{-+} = $(m-1)(r-1)m$.
2. Let G be any (n,m) graph.
   If $d_{G^{-+}}(e_i,e'_j)=1$, then
   $\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)+deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e_i,e'_j)$ $=2m(m+2r-3)(r-1)$.
   If $d_{G^{-+}}(e_i,e'_j)=2$, then
   $\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)+deg_{G^{-+}})]d_{G^{-+}}(e_i,e'_j)=$ $2m(m+2r-3)(m-2r+2)$.
   If $d_{G^{-+}}(e_i,e'_j)=3$, then
   $\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)+deg_{G^{-+}})]d_{G^{-+}}(e_i,e'_j)=$ $3m(m+2r-3)$.
3. Let G be any (n,m) graph.
   If $d_{G^{-+}}(e'_i,e'_j)=2$, then
   $\sum_{(e'_i,e'_j)\subseteq V(G^{-+})}[deg_G(e'_i)+deg_G(e'_j)]d(e'_i,e'_j)$ in G^{-+} $=8(r-1)\sum_{k=2}^{m}(k-1)$.

Theorem 3.7. For any (n,m) graph G,

$$DD_+(G^{-+}) = m[m(3m-2)-15] - mr[m-4r-17] + 8(r-1)\sum_{k=2}^{m}(k-1).$$

Proof. Let G be any (n,m)-graph, from the Proposition 3.1, G^{-+} contains 2m vertices and $\frac{m}{2}[m+2r-3]$ edges.

From (2), we have

$$DD_+(G) = \sum_{u,v\subseteq V(G)}[deg_G(u)+deg_G(v)]d_G(u,v)$$

Therefore,

$$\begin{aligned}DD_+(G^{-+}) = &\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)+deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j) + \\ &\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)+deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e_i,e'_j) + \\ &\sum_{(e'_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e'_i)+deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e'_i,e'_j).\end{aligned}$$

Applying observation E to the above equation, we get

$$\begin{aligned}DD_+(G^{-+}) = &(m-2r+1)(m-1)m + (m-1)(r-1)m + 2m(m+2r-3)(r-1) + \\ &2m(m+2r-3)(m-2r+2) + 3m(m+2r-3) + 8(r-1)\sum_{k=2}^{m}(k-1).\end{aligned}$$

On simplification,

$$DD_{+}(G^{-+}) = m[m(3m-2)-15] - mr[m-4r-17] + 8(r-1)\sum_{k=2}^{m}(k-1).$$

Observation F.

1. Let G be any (n,m) graph.

   If $d_{G^{-+}}(e_i,e_j)=1$, then

   $\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j)$ in $G^{-+} = \frac{1}{2}(m-1)^2(m^2-2mr+m)$.

   If $d_{G^{-+}}(e_i,e_j)=2$, then

   $\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j)$ in $G^{-+} = 2(m-1)^2(r-1)m$.

2. Let G be any (n,m) graph.

   If $d_{G^{-+}}(e_i,e'_j)=1$, then

   $\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e_i,e'_j)$ in $G^{-+} = 4m(mr-m-r+1)(r-1)$.

   If $d_{G^{-+}}(e_i,e'_j)=2$, then

   $\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e_i,e'_j)$ in $G^{-+} = 4m(mr-m-r-1)(m-2r+2)$.

   If $d_{G^{-+}}(e'_i,e'_j)=3$, then

   $\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e_i,e'_j)$ in $G^{-+} = 6m(mr-m-r+1)$.

3. Let G be any (n,m) graph.

   If $d_{G^{-+}}(e'_i,e'_j)=2$, then

   $\sum_{(e'_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e'_i)\cdot deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e'_i,e'_j)$ in $G^{-+} = 8(r-1)^2\sum_{k=2}^{m}(k-1)$.

Theorem 3.8. For any (n,m) graph G,

$$DD_{*}(G^{-+}) = \frac{1}{2}m[m^2(m-13)-5m+8] + mr[m(5m-4r+8)+4r-13] +$$
$$8(r-1)^2\sum_{k=2}^{m}(k-1).$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{-+} contains 2m vertices and $\frac{m}{2}[m+2r-3]$ edges.

From (3), we have

$$DD_{*}(G) = \sum_{u,v\subseteq V(G)}[deg_G(u)\cdot deg_G(v)]d_G(u,v)$$

Therefore,

$$\begin{aligned} DD_{*}(G^{-+}) = &\sum_{(e_i,e_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e_j)]d_{G^{-+}}(e_i,e_j) + \\ &\sum_{(e_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e_i)\cdot deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e_i,e'_j) + \\ &\sum_{(e'_i,e'_j)\subseteq V(G^{-+})}[deg_{G^{-+}}(e'_i)\cdot deg_{G^{-+}}(e'_j)]d_{G^{-+}}(e'_i,e'_j). \end{aligned}$$

Applying observation F to the above equation, we get

$$DD_{*}(G^{-+}) = \frac{1}{2}(m-1)^2(m^2-2mr+m) + 2(m-1)^2(r-1)m + 4m(mr-m-r+1)(r-1)$$

$$+4m(mr-m-r-1)(m-2r+2)+\ 6m(mr-m-r+1)+$$
$$8(r-1)^2\sum_{k=2}^{m}(k-1)\,.$$

On simplification,

$$DD_*(G^{-+})=\frac{1}{2}m[m^2(m-13)-5m+8]+mr[m(5m-4r+8)+4r-13]+$$
$$8(r-1)^2\sum_{k=2}^{m}(k-1).$$

Observation G.

1. Let G be any (n,m) graph.

   If $d_{G^{--}}(e_i,e_j)=1$, then

   $\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)$ in G^{--} = $2m\ (m-2r+1)^2$.

   If $d_{G^{--}}(e_i,e_j)=2$, then

   $\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)$ in G^{--} = $8\ (m-2r+1)(r-1)$.

2. Let G be any (n,m) graph.

   If $d_{G^{--}}(e_i,e'_j)=1$, then

   $\sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)$ in G^{--} = $3m\ (m-2r+1)^2$.

   if $d_{G^{--}}(e_i,e'_j)=2$, then

   $\sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)$ in G^{--} = $6m\ (2r-1)(m-2r+1)$.

3. Let G be any (n,m) graph.

   If $d_{G^{--}}(e'_i,e'_j)=2$, then

   $\sum_{(e'_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e'_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e'_i,e'_j)$ in G^{--} $=4(m-2r+1)\sum_{k=2}^{m}(k-1)$.

Theorem 3.9. For any (n,m) graph G,

$$DD_+(G^{--})=m[5m^2-4m-9]-4mr[5r-7]+4(m-2r+1)\sum_{k=2}^{m}(k-1).$$

Proof. Let G be any (n,m)-graph. From proposition 3.1, G^{--} contains 2m vertices and $\frac{3}{2}m[m-2r+1]$ edges.

From (2), we have

$$DD_+(G)=\sum_{u,v\subseteq V(G)}[deg_G(u)+deg_G(v)]d_G(u,v)$$

Therefore,

$$DD_+(G^{--})=\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)]+$$
$$\sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)]+$$
$$\sum_{(e'_i,e'_j)V(G^{--})}[deg_{G^{--}}(e'_i)+deg_{G^{--}}(e'_j)]d_{G^{--}}(e'_i,e'_j)].$$

Applying observation G to the above equation, we get

$$DD_+(G^{--})=2m(m-2r+1)^2+8(m-2r+1)(r-1)+3m(m-2r+1)^2+6m(2r-1)(m-2r+1)$$
$$+4(m-2r+1)\sum_{k=2}^{m}(k-1)$$

On simplification,

$$DD_+(G^{--}) = m[5m^2 - 4m - 9] - 4mr[5r - 7] + 4(m - 2r + 1)\sum_{k=2}^{m}(k-1).$$

Observation H.

1. Let G be any (n,m) graph.

   If $d_{G^{--}}(e_i, e_j) = 1$, then

   $\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)$ in $G^{--} = 2m(m-2r+1)^3$.

   If $d_{G^{--}}(e_i, e_j) = 2$, then

   $\sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j)$ in $G^{--} = 8m(m-2r+1)^2(r-1)$.

2. Let G be any (n,m) graph.

   If $d_{G^{--}}(e_i, e'_j) = 1$, then

   $\sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)$ in $G^{--} = 2m(m-2r+1)^3$.

   If $d_{G^{--}}(e_i, e'_j) = 2$, then

   $\sum_{(e'_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j)$ in $G^{--} = 4m(m-2r+2)^2(2r-1)$.

3. Let G be any (n,m) graph.

   If $d_{G^{--}}(e'_i, e'_j) = 2$, then

   $\sum_{(e'_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e'_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e'_i,e'_j)$ in $G^{--} = 2(m-2r+1)^2\sum_{k=2}^{m}(k-1)$.

Theorem 3.10. For any (n,m) graph G,

$$DD_*(G^{--}) = 2m[m^2(m-3) - 5m - 6] - 4mr[3m^2 - 4r^2 + 10mr - 14m + 12r - 13] + 2(m-2r+1)^2\sum_{k=2}^{m}(k-1).$$

Proof. Let G be any (n,m)-graph. From Proposition 3.1, G^{--} contains 2m vertices and $\frac{3}{2}m[m-2r+1]$ edges.

From (3), we have

$$DD_*(G) = \sum_{u,v\subseteq V(G)}[deg_G(u)\cdot deg_G(v)]d_G(u,v)$$

Therefore,

$$DD_*(G^{--}) = \sum_{(e_i,e_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e_j)]d_{G^{--}}(e_i,e_j) + \sum_{(e_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e_i,e'_j) + \sum_{(e'_i,e'_j)\subseteq V(G^{--})}[deg_{G^{--}}(e'_i)\cdot deg_{G^{--}}(e'_j)]d_{G^{--}}(e'_i,e'_j).$$

Applying observation H the above equation, we get

$$DD_*(G^{--}) = 2m(m-2r+1)^3 + 8m(m-2r+1)^2(r-1) + 2m(m-2r+1)^3 + 4m(m-2r+2)^2(2r-1) + 2(m-2r+1)^2\sum_{k=2}^{m}(k-1).$$

On simplification,

$$DD_*(G^{--}) = 2m[m^2(m-3) - 5m - 6] - 4mr[3m^2 - 4r^2 + 10mr - 14m + 12r - 13] + 2(m-2r+1)^2\sum_{k=2}^{m}(k-1).$$

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References

[1] G. Caporossi, M. Paiva, D. Vukičević, M. Segatto, Centrality and betweenness: vertex and edge decomposition of the Wiener index, MATCH Commun. Math. Comput. Chem., 68 (2012) 293–302.

[2] M.V. Diudea, I. Gutman, L. Jäntschi, Molecular Topology, Nova, Huntington, NY, 2001.

[3] A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications. Acta Appl Math, 66 (2001) 211-249.

[4] A.A. Dobrynin, A.A. Kochetova, Degree distance of a graph: a degree analogue of the Wiener index, J. Chem. Inf. Comput. Sci., 34 (1994) 1082–1086.

[5] R.C. Entringer, D.E. Jackson, D. A. Snyder, Distance in graphs. Czechoslov Math. J., 26 (1976) 283–296.

[6] L. Feng, B.Liu, The maximal Gutman index of bicyclic graphs, MATCH Commun. Math. Comput. Chem., 66 (2011) 699–708.

[7] I. Gutman, Selected properties of the Schultz molecular topological index, J. Chem. Inf. Comput. Sci., 34 (1994) 1087–1089.

[8] F. Harary, Graph Theory, Addison-Wesley, Reading, 1969.

[9] A. Ili c′, D. Stevanovi c′, L. Feng, G. Yu, P. Dankelmann, Degree distance of unicyclic and bicyclic graphs, Discrete Appl. Math., 159 (2011) 779–788.

[10] D.J. Kelein , Z. Mihali c′, D. Plavši c′, N.Trinajsti c′, Molecular topological index: a relation with the Wiener index, J. Chem. Inf. Comput. Sci., 32 (1992) 304–305.

[11] V. R. Kulli, M. S. Biradar, The line splitting graph of a graph, Acta Ciencia Indica., XXVII, M.No. 3 (2002) 317–322.

[12] I. Tomescu, Ordering connected graphs having small degree distance, Discrete Appl. Math., 158 (2010) 1714–1717.

[13] H. Wiener, Structural determination of paraffin boiling points, J. Am. Chem. Soc., 69 (1947) 17-20.

[14] H. Wiener, Correlation of heats of isomerization and differences in heats of vaporization of isomers among the paraffin hydrocarbons, J. Am. Chem. Soc., 69 (1947) 2636–2638.

[15] H. Wiener, Influence of interatomic forces on paraffin properties, J. Chem. Phys., 15 (1947) 766–766.