Decomposition of $S^*\alpha$-continuity

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Keywords: semi$^\ast\alpha$-open, semi$^\ast\alpha_I$-set, semi$^\ast\alpha_I$-continuity.

Abstract. The main purpose of this paper is to introduce the concepts of $S^*\alpha_I$-sets, $S^*\alpha_I$-continuity and obtain decomposition of semi$^\ast\alpha$-continuity in topological spaces.

Introduction

Levine \cite{3}, Mashhour \cite{5} et al. and Njastad \cite{6} introduced the topological notions of semi-open sets, pre-open sets and $\alpha$-open sets respectively. The concept of $g$-closed sets was introduced and studied by levine \cite{4}. M.K.R.S.Veera Kumar \cite{10} introduced the notion of $g^*$-closed sets and Govindappa Navalagi defined the concept of semi $\alpha$-open sets in topological spaces. Jafari et. al.\cite{12} introduced and studied the notions of $g_\alpha$-closed sets and $g_\alpha$-closed sets in topological spaces. Recently, Robert and Pious Missier \cite{8} introduce a new class of sets namely semi$^\ast\alpha$-open sets in topological spaces.

In recent Years, the decomposition of continuity is one of the main interest for general topologists. In 1961, Levine obtained a decomposition of continuity which was later improved by Rose \cite{9}. Tong decomposed continuity into $\alpha$-continuity and $A$-continuity finally showed that his decomposition is independent of Levine’s. Ganster and Reilly \cite{2} have improved Tong’s decomposition result and provided a decomposition of $A$ -continuity. Ravi \cite{11} et. al. obtained a decomposition of $g$-continuity. In this paper we obtain a decomposition of $S^*\alpha$-continuity in topological spaces using $S^*p$-continuity and $S^*\alpha_I$-continuity.

Preliminaries

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ (simply, $X$ and $Y$) denote topological space on which no separation axioms are assumed. Let $A$ be a subset of a space $X$. The closure of $A$ and the interior of $A$ are denoted by $Cl(A)$ and $Int(A)$ respectively.

Definition 1 A subset $A$ of a space $(X, \tau)$ is called:

(1) a semi-open set \cite{3} if $A \subseteq Cl(Int(A))$ and a semi-closed set if $Int(Cl(A)) \subseteq A$,

(2) a pre-open set \cite{5} if $A \subseteq Int(Cl(A))$ and a pre-closed set if $Cl(Int(A)) \subseteq A$,

(3) an $\alpha$-open set \cite{6} if $A \subseteq Int(Cl(Int(A)))$ and an $\alpha$-closed set if $Cl(Int(Cl(A))) \subseteq A$,

(4) generalized closed \cite{4} (briefly $g$-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open,

(5) a $t$-set \cite{13} if $Int(Cl(A)) = Int(A)$,
(6) a B-set [13] if \( A = M \cap N \) where \( M \) is open and \( N \) is a t-set,

(7) an \( \alpha B \)-set [1] if \( A = M \cap N \) where \( M \) is \( \alpha \)-open and \( N \) is a t-set,

(8) an \( \eta \)-set [7] if \( A = M \cap N \) where \( M \) is open and \( N \) is an \( \alpha \)-closed set,

(9) semi\( \star \alpha \)-open [8] if there is an \( \alpha \)-open set \( U \) in \( X \) such that \( U \subseteq A \subseteq Cl^*(U) \) or equivalently if \( A \subseteq Cl^*(\alpha Int(A)) \).

(10) semi\( \star \)-preopen [8] if \( A \subseteq Cl^*(pInt(A)) \).

For a subset \( A \) of a space \( X \), the \( \alpha \)-closure (resp. semi-closure, pre-closure) of \( A \), denoted by \( \text{Cl}^\alpha(A) \) (resp. \( s\text{Cl}^\alpha(A) \), \( p\text{Cl}^\alpha(A) \)) is the intersection of all \( \alpha \)-closed (resp. semi-closed, pre-closed) sub-
sets of \( A \) containing \( A \). Dually, the \( \alpha \)-interior (resp. semi-interior, pre-interior) of \( A \), denoted by \( \text{Int}^\alpha(A) \) (resp. \( s\text{Int}^\alpha(A) \), \( p\text{Int}^\alpha(A) \)), is the union of all \( \alpha \)-open (resp. semi-open, pre-open) subsets
of \( X \) contained in \( A \), the generalized closure of \( A \) is defined as the intersection of all \( g \)-closed sets
containing \( A \) and is denoted by \( Cl^\alpha(A) \):

**Definition 2** A function \( f : X \rightarrow Y \) is said to be

1. B-continuous [13] if \( f^{-1}(V) \) is a B-set in \( X \) for every open set \( V \) of \( Y \),
2. \( \alpha B \)-continuous [1] if \( f^{-1}(V) \) is an \( \alpha B \)-set in \( X \) for every open set \( V \) of \( Y \),
3. \( \eta \)-continuous [7] if \( f^{-1}(V) \) is an \( \eta \)-set in \( X \) for every open set \( V \) of \( Y \),
4. semi\( \star \alpha \)-continuous [8] if \( f^{-1}(V) \) is semi\( \star \alpha \)-open in \( X \) for every open set \( V \) in \( Y \),
5. semi\( \star \)-pre continuous [8] if \( f^{-1}(V) \) is semi\( \star \)-preopen in \( X \) for every open set \( V \) in \( Y \).

**Remark 1** In any space \( (X, \tau) \),

1. Every semi\( \star \alpha \)-open set is semi\( \star \)-preopen set but not conversely.[8]
2. Every semi\( \star \alpha \)-continuous function is semi\( \star \)-pre continuous but not
   conversely.[8]

**Semi\( \star \alpha_t \)-sets**

**Definition 3** A subset \( A \) of a topological space \( (X, \tau) \) is said to be semi\( \star \alpha_t \)-set if \( A = M \cap N \) where \( M \) is a semi\( \star \alpha \)-open in \( X \) and \( N \) is a t-set in \( X \). The family of all semi\( \star \alpha_t \)-sets in a space \( (X, \tau) \) is denoted by \( S^*\alpha_t(X, \tau) \).

**Remark 2** The following implications are hold:

\[ \eta(X) \]
\[ B(X) \longrightarrow \alpha B(X) \longrightarrow S^*\alpha_t(X) \]
\[ S^*\alpha O(X) \longrightarrow S^*pO(X) \]
where none of these implications are reversible as shown in the following examples.

**Example 1** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\} \). In \( (X, \tau) \), the set \( \{a, c\} \) is semi-\( \alpha \)-set but not semi-\( \alpha \)-open.

**Example 2** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\} \). In \( (X, \tau) \), the set \( \{b, c, d\} \) is semi-\( \alpha \)-set but not \( B \)-set.

**Example 3** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\} \). In \( (X, \tau) \), the set \( \{b, d\} \) is semi-\( \alpha \)-set but not \( \alpha \)-set.

**Example 4** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a, c\}, \{a, c, d\}, X\} \). In \( (X, \tau) \), the set \( \{a, c\} \) is semi-\( \alpha \)-set but not \( \eta \)-set.

**Remark 3** The notions of semi-\( \alpha \)-sets and semi-\( \eta \)-preopen sets are independent are shown in the following example.

**Example 5** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a, b\}, X\} \). In \( (X, \tau) \), the set \( \{c\} \) is semi-\( \alpha \)-set but not semi-\( \eta \)-preopen and also the set \( \{a, c\} \) is semi-\( \eta \)-preopen but not semi-\( \alpha \)-set.

**Theorem 1** For a subset \( A \) of a space \( X \), the following are equivalent:

1. \( A \) is semi-\( \alpha \)-open
2. \( A \) is semi-\( \alpha \)-set and semi-\( \eta \)-preopen.

**Proof.**

(1) \( \Rightarrow \) (2): The proof is obvious.

(2) \( \Rightarrow \) (1): Assume that \( A \) is semi-\( \alpha \)-set and semi-\( \eta \)-preopen in \( X \). Then \( A = M \cap N \) where \( M \) is semi-\( \alpha \)-open and \( N \) is a \( t \)-set in \( X \). By the hypothesis, \( A \) is semi-\( \eta \)-preopen and we have

\[
A \subseteq \text{Cl}^*(\text{pInt}(A)) = \text{Cl}^*(A \cap \text{Int}(\text{Cl}(A))) \\
\subseteq \text{Cl}^*(A) \cap \text{Cl}^*(\text{Int}(\text{Cl}(A))) \subseteq \text{Cl}^*(\text{Int}(\text{Cl}(A))) \\
M \cap N \subseteq \text{Cl}^*(\text{Int}(\text{Cl}(M \cap N))) \\
= \text{Cl}^*(\text{Int}(\text{Cl}(M))) \cap \text{Cl}^*(\text{Int}(\text{Cl}(N))) \\
\subseteq \text{Cl}^*(\text{Int}(\text{Cl}(N))) = \text{Cl}^*(\text{Int}(N)) \subseteq \text{Cl}^*(\alpha \text{Int}(N))
\]

Since \( M \) is semi-\( \alpha \)-open in \( X \), we have \( M \subseteq \text{Cl}^*(\alpha \text{Int}(M)) \)

\[
M \cap N \subseteq \text{Cl}^*(\alpha \text{Int}(M \cap N))
\]

Therefore \( A \) is semi-\( \alpha \)-open in \( X \).

**semi-\( \alpha \)-continuity**

**Definition 4** A function \( f : X \to Y \) is said to be semi-\( \alpha \)-continuous if \( f^{-1}(V) \) is semi-\( \alpha \)-set in \( X \) for every open set \( V \) of \( Y \).

**Remark 4** The following implications are hold:
where none of these implications are reversible as shown in the following examples.

Example 6 Let \( X = \{a, b, c\} \) and \( Y = \{p, q, r\} \), \( \tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\} \), and \( \sigma = \{\phi, \{q\}, \{r\}, \{q, r\}, X\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as follows: \( f(a) = p \), \( f(b) = q \), \( f(c) = r \). Then \( f \) is semi-\( \alpha_1 \)-continuous but not semi-\( \alpha \)-continuous.

Example 7 Let \( X = \{a, b, c\} \) and \( Y = \{p, q, r\} \), \( \tau = \{\phi, \{a\}, X\} \) and \( \sigma = \{\phi, \{p\}, \{r\}, \{p, r\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as follows: \( f(a) = p \), \( f(b) = r \), \( f(c) = q \). Then \( f \) is semi-\( \alpha_1 \)-continuous but not \( B \)-continuous.

Example 8 Let \( X = \{a, b, c, d\} \) and \( Y = \{p, q, r, s\} \), \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\} \) and \( \sigma = \{\phi, \{s\}, \{p, s\}, \{q, s\}, \{p, q, s\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as follows: \( f(a) = r \), \( f(b) = s \), \( f(c) = q \), \( f(d) = p \). Then \( f \) is semi-\( \alpha_1 \)-continuous but not semi-\( \alpha \)-continuous.

Example 9 Let \( X = \{a, b, c\} \) and \( Y = \{p, q, r\} \), \( \tau = \{\phi, \{a\}, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{p\}, \{q\}, \{p, q\}, \{p, r\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as follows: \( f(a) = q \), \( f(b) = r \), \( f(c) = p \). Then \( f \) is semi-\( \alpha_1 \)-continuous but not semi-\( \alpha \)-continuous.

Remark 5 The following examples show that the concept of semi-\( \alpha_1 \)-continuity and semi-\( \alpha \)-pre continuity are independent.

Example 10 Let \( X = \{a, b, c\} \) and \( Y = \{p, q, r\} \), \( \tau = \{\phi, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{q\}, \{r\}, \{p, q\}, \{q, r\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as follows \( f(a) = r \), \( f(b) = q \), \( f(c) = p \). Then \( f \) is semi-\( \alpha \)-pre continuous but not semi-\( \alpha_1 \)-continuous.

Example 11 Let \( X = \{a, b, c\} \) and \( Y = \{p, q, r\} \), \( \tau = \{\phi, \{a\}, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{p\}, \{p, q\}, \{p, q\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as follows \( f(a) = r \), \( f(b) = q \), \( f(c) = p \). Then \( f \) is semi-\( \alpha_1 \)-continuous but not semi-\( \alpha \)-pre continuous.

Theorem 2 For a function \( f : X \to Y \), the following are equivalent:

1. \( f \) is semi-\( \alpha \)-continuous.
2. \( f \) is semi-\( \alpha_1 \)-continuous and semi-\( \alpha \)-pre continuous.

Proof. The proof follows from Theorem 1.
References


