

Adomian Decomposition Method for Solving Ratio-Dependent Prey-Predator System with Harvesting on Predator

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Keywords: Adomian Decomposition Method, Ratio-dependent Prey-Predator model, Predator harvesting

Abstract: In this article, Adomian Decomposition Method is used to find out the approximate solution of ratio-dependent prey-predator model with predator harvesting. The significance of ADM over other numerical discretization techniques is that it has the eminence to solve problems directly with no un-physical restrictive assumptions such as linearization, per-turbation, massive computation and any other transformation. ADM solves the problem and arrives at the approximate solution in the form of series with solution components that are easily computable. It requires very less work in comparison with other traditional methods. The graphical representations of prey and predator population contrasted with time are drawn to examine the performance and reliability of this technique.

1. Introduction

Many biological systems are based on the system of non-linear ordinary differential equations. The qualitative and quantitative analysis of these systems can be done by mathematical modeling. In order to mathematically examine ecology, it is significant to design models that involve the interaction between species. There are many discrete and continuous prey-predator models in mathematical ecology. These models are investigated for global stability, periodicity, boundedness and other features [2,3]. One of the most well-known models is the prey-predator model developed by Lotka-Volterra [7].

Adomian Decomposition Method is a powerful approach to solve non-linear differential equations by substitution of the non-linear part of the equation with Adomian Polynomials. A step by step procedure is used for finding the solution. The number of partial solutions used decides the accuracy of the system. It is noteworthy that the solution of this method has a fast convergence to the exact solution. The numerical solution of prey –predator model has been treated in several papers in order to numerically investigate reliability and efficiency. For instance, in [6 9, 10], Adomian has been tested numerically using prey-predator model. The analysis of ratio-dependent predator-prey model by Homotopy method is done by [1, 12].

The objective of this paper is to numerically examine the reliability of ADM applied to ratio-dependent prey-predator models governed by a two dimensional system of non-linear ordinary differential equation. In section (2), the mathematical model of the system is analyzed. Section (3) presents the basics of Adomian Decomposition Method. In section (4), the computational method for the considered system is discussed. Section (5) consists of the numerical simulation and the it's result. Section (6) concludes the work.

2. Mathematical Model

The general case of predator- prey system is as follows:

$$\begin{aligned} \frac{dx}{dt} &= xh(x) - yp(x, y) \\ \frac{dy}{dt} &= \beta yp(x, y) - \gamma y \end{aligned} \tag{1}$$

Where $x(t)$ and $y(t)$ represents the size of the prey and predator population respectively at time t , $h(x)$ is the per-capita growth rate in the absence of the predator, $p(x, y)$ is the functional response $\beta p(x, y)$ is the per-capita production of predator due to predation which is often called the numerical response, γ is the natural mortality of predator. The functional response plays a major role in system (1). The knowledge of this function determines the dynamics of the whole system and the transfer of bio-mass in the predation because it is proportional to numerical response. Now

by letting $h(x) = (1 - x)$, $p(x, y) = \frac{x}{x + y}$ we obtain the special case of predator-prey system called predator-prey system with constant effort harvesting on predator which formulated as follows:

$$\frac{dx}{dt} = x(1 - x) - \frac{\alpha xy}{x + y} \tag{2}$$

$$\frac{dy}{dt} = \frac{\beta xy}{x + y} - \gamma y - hy \tag{3}$$

Here $\alpha > 0$ and $h > 0$ represents capturing rate and harvesting activity respectively. Equations (2) and (3) are to be solved according to biological meaningful initial conditions $x(t) \geq 0, y(t) \geq 0$ A qualitative examination of the system described by equations (2) and (3) reveals that the long term behavior falls into two categories: predator extinction and co-existence [1]. When the predator population becomes extinct, the solution asymptotically approaches equilibrium P_1 of the form

$$P_1 = (1, 0) \tag{4}$$

The eigen values of the Jacobian matrix evaluated at equilibrium P_1 shows that the predator extinction equilibrium is a locally asymptotically stable node provided if the death rate combined with harvesting of the predator exceeds the conversion rate(i.e.) $\gamma + h > \beta$.

Another long term possibility is the prey-predator co-existence equilibrium P_2 of the form

$$P_2 = \left(\frac{\beta(1 - \alpha) + \alpha(\gamma + h)}{\beta}, \frac{\beta - \gamma - h}{\gamma + h} \left(\frac{\beta(1 - \alpha) + \alpha(\gamma + h)}{\beta} \right) \right) \tag{5}$$

The eigen value λ of the Jacobian matrix evaluated at P_2 satisfy

$$\lambda^2 - (\alpha - \gamma - h - 1) + \left[\frac{(\gamma + h)^2 (\beta - \alpha)}{\beta^2} \right] \lambda + \left[\frac{(\gamma + h)(\beta - \gamma - h)(\beta - \beta\alpha + \alpha\gamma + \gamma h)}{\beta^2} \right] \tag{6}$$

Hence P_2 is locally asymptotically stable provided

$$\beta > \gamma + h \text{ and } \alpha < \beta \left(\frac{(\gamma + h + 1)\beta - (\gamma + h)^2}{\beta^2 - (\gamma + h)^2} \right) \tag{7}$$

From equation (5), we observe that wherever co-existence equilibrium P_2 occurs, the predator extinction equilibrium P_1 becomes an unstable saddle point. This is because the predator conversion rate exceeds the sum of death and harvest rate.

3. Basics Adomian Decomposition Method

Consider the standard operator

$$Lu + Ru + Nu = g$$

Where u is the unknown function, L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than the order of L , Nu represents the nonlinear terms and g is the source term. By applying the inverse operator L^{-1} to both sides of equation () we obtain

$$u = v - L^{-1}(Ru) - L^{-1}(Nu)$$

Where the function v represents the terms arising from integrating the source term g and from using the auxiliary conditions. The standard Adomian method defines the solution u by the series

$$u = \sum_{n=0}^{\infty} u_n$$

And the non-linear term by the series

$$Nu = \sum_{n=0}^{\infty} A_n$$

Where A_n are the Adomian Polynomials determined formally from the relation []

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right] \right]_{\lambda=0}, n = 0, 1, \dots,$$

The solution components u_0, u_1, u_2, \dots , are then determined recursively by using the relation

$$\left\{ \begin{array}{l} u_0 = v \\ u_{k+1} = -L^{-1}Ru_k - L^{-1}A_k, k \geq 0 \end{array} \right\}$$

Where u_0 is referred to as zeroth component.

4. Computational method

In order to explicitly construct approximate non-perturbative solution of the system described by the (2) and (3), Adomian Decomposition which is well addressed in [4,5,8,13] is employed. The advantage of this method is that it provides a direct scheme for solving the problem. This decomposition method is also an approximation but does not change the original problem. Another advantage of decomposition method is that it provides analytical approximation to a wide class of non-linear equations without any linearization as in perturbation, massive computation and any other transformation.

Now

$$\frac{dx}{dt} = x(t) - x^2(t) - \frac{ax(t)y(t)}{x(t) + y(t)} \quad (8)$$

$$\frac{dy}{dt} = \frac{\beta x(t)y(t)}{x(t) + y(t)} - (\gamma + h)y(t) \tag{9}$$

Considering f and g to be a nonlinear functions of x and (x, y) respectively. The equation (8-9) can now written in the form

$$\frac{dx}{dt} = x(t) - f(x) - \alpha g(x, y) \tag{10}$$

$$\frac{dy}{dt} = \beta g(x, y) - (\gamma + h)y(t) \tag{11}$$

With initial values $x(0) = x_0, y(0) = y_0$. To find the solution of x and y that satisfying equation (10) and (11), the decomposition method is used. It consists of approximating solution of (10) and (11) as an infinite series in the following form

$$x = \sum_{n=0}^{\infty} x_n, \quad y = \sum_{n=0}^{\infty} y_n \tag{12}$$

Decomposing f and g as

$$f(x) = \sum_{n=0}^{\infty} A_n(x_0, \dots, x_n) \tag{13}$$

$$g(x, y) = \sum_{n=0}^{\infty} B_n(x_0, \dots, x_n, y_0, \dots, y_n) \tag{14}$$

Where A_n, B_n are called Adomian polynomials that can be determined formally from the relation

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left[\left(\sum_{i=0}^{\infty} \lambda^i x_i \right) \right] \right]_{\lambda=0}, n = 0, 1, \dots, \tag{15}$$

$$B_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left[\frac{\sum_{i=0}^{\infty} \lambda^i x_i \sum_{i=0}^{\infty} \lambda^i y_i}{\sum_{i=0}^{\infty} \lambda^i x_i + \sum_{i=0}^{\infty} \lambda^i y_i} \right] \right]_{\lambda=0}, n = 0, 1, \dots, \tag{16}$$

Applying decomposition method, the system (10) and (11) can be written as

$$Lx = x(t) - f(x) - \alpha g(x, y) \tag{17}$$

$$Ly = \beta g(x, y) - (\gamma + h)y(t) \tag{18}$$

Where $L = \frac{d}{dt}$, the linear differential operator. Applying the integration inverse operator

$L^{-1} = \int_0^t (\cdot) dt$ to both sides of equation (17) and (18) we get

$$x(t) = x(0) + L^{-1}x(t) - L^{-1}f(x(t)) - \alpha L^{-1}g(x(t), y(t)) \tag{19}$$

$$y(t) = y(0) + \beta L^{-1}g(x(t), y(t)) - (\gamma + h)L^{-1}y(t) \tag{20}$$

Using equation (12), (13) and (14)

$$\sum_{n=0}^{\infty} x_n = x_0 + L^{-1} \sum_{n=0}^{\infty} x_n - L^{-1} \sum_{n=0}^{\infty} A_n - \alpha L^{-1} \sum_{n=0}^{\infty} B_n \quad (21)$$

$$\sum_{n=0}^{\infty} y_n = y_0 + \beta L^{-1} \sum_{n=0}^{\infty} B_n - (\gamma + h) L^{-1} \sum_{n=0}^{\infty} y_n \quad (22)$$

Now, the iterates are determined in the following recursive way

$$\begin{aligned} x(0) &= x_0 \\ x_{n+1} &= L^{-1} x_n - L^{-1} A_n - \alpha L^{-1} B_n, n = 0, 1, 2 \\ y(0) &= y_0 \\ y_{n+1} &= \beta L^{-1} B_n - (\gamma + h) L^{-1} y_n, n = 0, 1, 2 \end{aligned} \quad (23)$$

We define a solution of initial value problem as

$$(x, y) = \left(\lim_{n \rightarrow \infty} \sum_{n=0}^n x_n, \lim_{n \rightarrow \infty} \sum_{n=0}^n y_n \right) \quad (24)$$

Computing Adomian polynomials by using relation the algorithm presented in[4] yields to

$$\begin{aligned} A_0(x_0) &= x_0^2 \\ A_1(x_0, x_1) &= 2x_0x_1 \\ A_2(x_0, x_1, x_2) &= 2x_0x_2 + x_1^2 \\ A_3(x_0, x_1, x_2, x_3) &= 2x_0x_3 + 2x_2x_1 \end{aligned} \quad (25)$$

and

$$\begin{aligned} B_0(x_0, y_0) &= \frac{x_0 y_0}{x_0 + y_0} \\ B_1(x_0, x_1, y_0, y_1) &= \frac{x_1 y_0^2 + x_0^2 y_1}{(x_0 + y_0)^2} \\ B_2(x_0, x_1, x_2, y_0, y_1, y_2) &= \frac{x_2 y_0^3 + x_0 x_2 y_0^2 + 2x_1 x_0 y_0 y_1 - x_1^2 y_0^2 + x_0 y_2 y_0 + x_0^3 y_2 - x_0^2 y_1^2}{(x_0 + y_0)^3} \end{aligned} \quad (26)$$

Four terms approximations to the solutions are considered

$$\begin{cases} x(t) = x_0 + x_1 + x_2 + x_3 \\ y(t) = y_0 + y_1 + y_2 + y_3 \end{cases} \quad (27)$$

5. Numerical simulation and results

In this section we present some graphical results based on the decomposition method together with numerical validation of the qualitative results obtained in equation (2) and (3). According to the values introduced in Table 1, the following decomposition series with partial sums are derived.

Case1:

$$\begin{aligned} x(t) &\approx 0.5 + 0.175t - 0.027046t^2 + 0.0205461313t^3 \\ y(t) &\approx 0.3 - 0.3525t + 0.22791414t^2 - 0.0993106909t^3 \end{aligned}$$

Case 2:

$$x(t) \approx 0.4 + 0.17152857t + 0.09798022t^2 - 0.001987925t^3$$

$$y(t) \approx 0.3 - 0.1328571t + 0.01300744t^2 - 0.00248976t^3$$

Case 3:

$$x(t) \approx 0.3 + 0.15857t + 0.02536t^2 - 0.00514075t^3$$

$$y(t) \approx 0.4 - 0.051432t + 0.01618104t^2 - 0.0014306125t^3$$

Case4:

$$x(t) \approx 0.3 + 0.17t + 0.028t^2 - 0.004138683t^3$$

$$y(t) \approx 0.6 - 0.14t + 0.04t^2 - 0.009736627t^3$$

Case	x_0	y_0	α	β	γ	h	Comments
1	0.5	0.3	0.4	0.2	0.5	0.8	Stable- predator extinction
2	0.4	0.3	0.4	0.1	0.2	0.3	Stable predator extinction
3	0.3	0.4	0.3	0.4	0.2	0.1	Stable-coexistence
4	0.3	0.6	0.2	0.5	0.2	0.2	Stable- co-existence

Fig 1-4 shows the relation between prey and predator population contrasted with time.

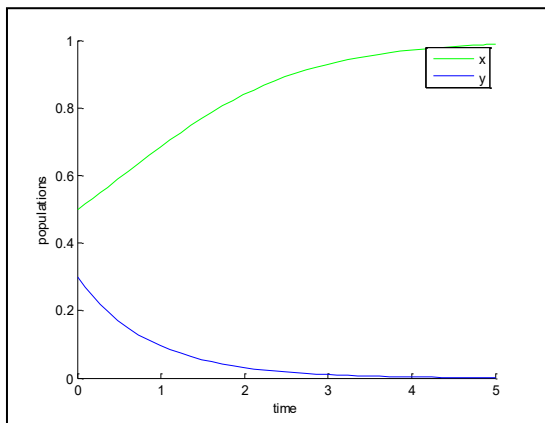


Fig 1. Case 1: For h=0.8

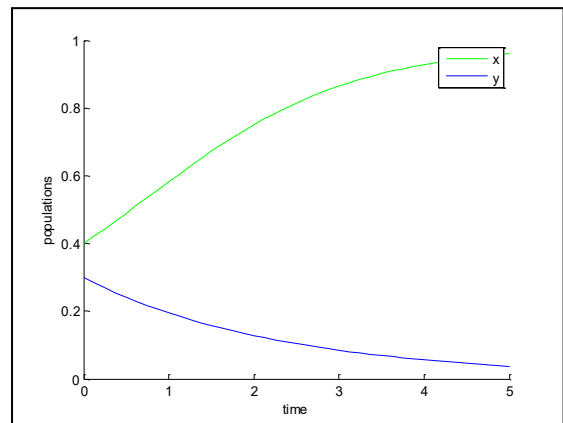


Fig 2. Case 2: For h=0.3

A noteworthy observation from Fig 1 and 2 is that predator population gradually decreases and becomes extinct despite the availability of increasing prey population. It can be observed that if the sum of death rate and harvesting rate of the predator exceeds its conversion rate, then the predator will go extinct.

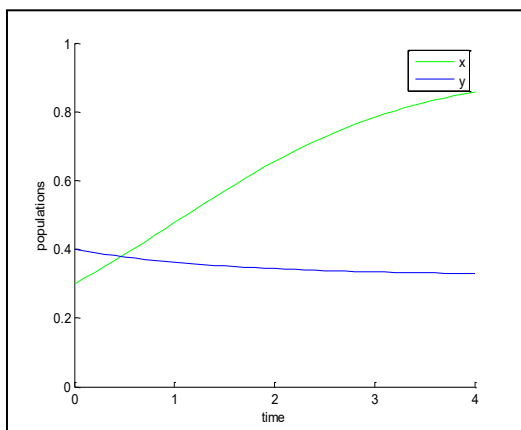


Fig 3. Case 3: For h=0.1

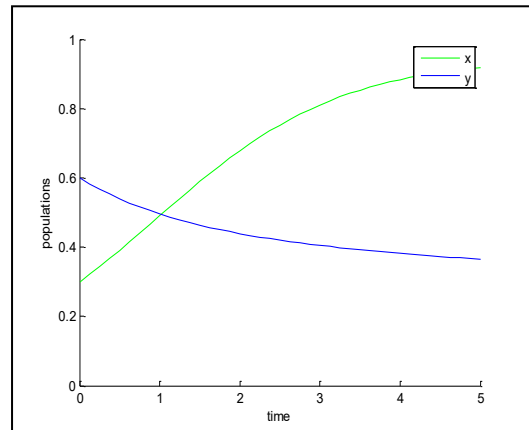


Fig 4. Case 4: For h=0.2

Fig 3 and 4 illustrate the possibility of long time co-existence of predator and prey. The predator conversion rate must exceed the sum of death rate and harvest rate for co-existence to occur among the other necessary conditions

6. Conclusion

In this paper, the approximate solution of ratio-dependent prey-predator system is simulated by Adomian Decomposition Method in a direct way without using any restrictive assumption. Therefore, the computation of work and consumption of time is reduced. The performance and reliability of this method is analyzed from the graphical representation of prey and predator population plotted against time.

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