Forgotten topological index and hyper-Zagreb index of generalized transformation graphs

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Abstract. In this paper, we obtain the expressions for forgotten topological index, hyper-Zagreb index and coindex for generalized transformation graphs $G^{ab}$ and their complements $\overline{G}^{ab}$.

1 Introduction

Let $G$ be a simple, undirected graph with $n$ vertices and $m$ edges. Let $V(G)$ and $E(G)$ be the vertex set and edge set of $G$ respectively. If $u$ and $v$ are adjacent vertices of $G$, then the edge connecting them will be denoted by $uv$. The degree of a vertex $u$ in $G$ is the number of edges incident to it and is denoted by $d_G(u)$. The complement of $G$, denoted by $\overline{G}$, is a graph having the same vertex set as $G$, in which two vertices are adjacent if and only if they are not adjacent in $G$. Thus, the size of $\overline{G}$ is $\binom{n}{2} - m$ and $d_{\overline{G}}(v) = n - 1 - d_G(v)$ holds for all $v \in V(G)$. We refer to [9] for unexplained terminology and notation.

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modeled by means of molecular-graph-based structure-descriptors, which are also referred to as topological indices [8, 12]. The first and second Zagreb indices, respectively, defined $M_1(G) = \sum_{u \in V(G)} d_G(u)^2$ and $M_2(G) = \sum_{u \in V(G)} d_G(u)d_G(v)$ are widely studied degree-based topological indices, that were introduced by Gutman and Trinajstić’ [7] in 1972.

Noticing that the contribution of non-adjacent vertex pairs to be taken into account when computing the weighted Wiener polynomials of certain composite graphs first and second Zagreb coindex were defined (see [1, 5])

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v)$$

respectively.

The vertex-degree-based graph invariant

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2]$$

was encountered in [7]. Recently there has been some interest to $F$, called “forgotten topological index” [6].

Shirdel et al.[11] introduced a new Zagreb index of a graph $G$ named “hyper-Zagreb index” and is defined as:

$$HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2.$$\

In [3], Basavanagoud et al. corrects some errors of [11] and gave the correct expressions for hyper-Zagreb index of some graph operations.

Recently, Veylaki et al.[13] defined the hyper-Zagreb coindices as

$$HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2.$$
In [4], Basavanagoud et al. corrects some errors of [13] and gave the correct expressions for hyper-Zagreb coindex of some graph operations.

The following results will be needed for the present considerations.

**Proposition 1.1** [4] Let $G$ be a simple graph on $n$ vertices and $m$ edges. Then $F(G) = n(n - 1)^3 - F(G) - 6(n - 1)^2m + 3(n - 1)M_1(G)$.

**Proposition 1.2** Let $G$ be a simple graph. Then $HM(G) = F(G) + 2M_2(G)$.

**Proof.** By definition of the hyper-Zagreb index, we have

\[ HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2 \]

\[ = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2 + 2d_G(u)d_G(v)] \]

\[ = \sum_{uv \in E(G)} d_G(u)^2 + d_G(v)^2 + 2 \sum_{uv \in E(G)} d_G(u)d_G(v) \]

\[ = F(G) + 2M_2(G). \]

**Proposition 1.3** [4] Let $G$ be a simple graph. Then $HM(G) = F(G) + 2M_2(G)$.

**Proposition 1.4** [4] Let $G$ be a simple graph on $n$ vertices and $m$ edges. Then $\overline{HM}(G) = 2\overline{M}_2(G) + (n - 1)M_1(G) - F(G)$.

2 Generalized transformation graphs $G^{ab}$

The semitotal-point graph $T_2(G)$ of a graph $G$ is a graph whose vertex set is $V(T_2(G)) = V(G) \cup E(G)$ and two vertices are adjacent in $T_2(G)$ if and only if (i) they are adjacent vertices of $G$ or (ii) one is a vertex of $G$ and other is an edge of $G$ incident with it. It was introduced by Sampathkumar and Chikkodimath [10]. Recently some new graphical transformations were defined by Basavanagoud et al. [2], which generalizes the concept of semitotal-point graph.

The *generalized transformation graph* $G^{ab}$ is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{ab})$. The vertices $\alpha$ and $\beta$ are adjacent in $G^{ab}$ if and only if $(\ast)$ and $(\ast\ast)$ holds:

$(\ast)$ $\alpha, \beta \in V(G)$, $\alpha$, $\beta$ are adjacent in $G$ if $a = +$ and $\alpha$, $\beta$ are not adjacent in $G$ if $a = -$.

$(\ast\ast)$ $\alpha \in V(G)$ and $\beta \in E(G)$, $\alpha$, $\beta$ are incident in $G$ if $b = +$ and $\alpha$, $\beta$ are not incident in $G$ if $b = -$.

One can obtain the four graphical transformations of graphs as $G^{++}$, $G^{+-}$, $G^{-+}$ and $G^{--}$. The vertex $v_i$ of $G^{ab}$ corresponding to a vertex $v_i$ of $G$ is referred to as *point vertex* and vertex $e_i$ of $G^{ab}$ corresponding to an edge $e_i$ of $G$ is referred to as *line vertex*.

In [2], Basavanagoud et al. obtained the expressions for first and second Zagreb indices and coindices for generalized transformation graphs $G^{ab}$ and their complements $\overline{G^{ab}}$. Now we obtain the expressions for forgotten topological index, hyper-Zagreb index and coindex for generalized transformation graphs $G^{ab}$ and their complements $\overline{G^{ab}}$.

**Proposition 2.1** [2] Let $G$ be a $(n, m)$-graph. Then the degree of point and line vertices in $G^{ab}$ are

1. $d_G^{++}(v_i) = 2d_G(v_i)$ and $d_G^{++}(e_i) = 2$.
2. $d_G^{+-}(v_i) = m$ and $d_G^{+-}(e_i) = n - 2$.
3. $d_G^{-+}(v_i) = n - 1$ and $d_G^{-+}(e_i) = 2$.
4. $d_G^{--}(v_i) = n + m - 1 - 2d_G(v_i)$ and $d_G^{--}(e_i) = n - 2$. 
3 Results

**Theorem 3.1** Let $G$ be an $(n, m)$-graph. Then $F(G^{++}) = 8F(G) + 8m$.

*Proof.* Since $G^{++}$ has $n + m$ vertices and $3m$ edges,

$$F(G^{++}) = \sum_{u \in V(G^{++})} d_{G^{++}}(u)^3 = \sum_{u \in V(G^{++}) \cap E(G)} d_{G^{++}}(u)^3 + \sum_{u \in V(G^{++}) \cap E(G)} d_{G^{++}}(u)^3.$$

From Proposition 2.1, we have

$$F(G^{++}) = \sum_{u \in E(G)} (2d_u(u))^3 + \sum_{u \in E(G)} (2)^3 = 8F(G) + 8m.$$

**Theorem 3.2** Let $G$ be an $(n, m)$-graph. Then $F(G^{+-}) = nm^3 + m(n - 2)^3$.

*Proof.* Since $G^{+-}$ has $n + m$ vertices and $m(n - 1)$ edges,

$$F(G^{+-}) = \sum_{u \in V(G^{+-})} d_{G^{+-}}(u)^3 = \sum_{u \in V(G^{+-}) \cap E(G)} d_{G^{+-}}(u)^3 + \sum_{u \in V(G^{+-}) \cap E(G)} d_{G^{+-}}(u)^3.$$

In view of Proposition 2.1, we have

$$F(G^{+-}) = \sum_{u \in E(G)} (m)^3 + \sum_{u \in E(G)} (n - 2)^3 = nm^3 + m(n - 2)^3.$$

**Theorem 3.3** Let $G$ be an $(n, m)$-graph. Then

$$F(G^{-+}) = n(n - 1)^3 + 8m.$$

*Proof.* Since $G^{-+}$ has $n + m$ vertices and $m + \frac{1}{2}n(n - 1)$ edges,

$$F(G^{-+}) = \sum_{u \in V(G^{-+})} d_{G^{-+}}(u)^3 = \sum_{u \in V(G^{-+}) \cap E(G)} d_{G^{-+}}(u)^3 + \sum_{u \in V(G^{-+}) \cap E(G)} d_{G^{-+}}(u)^3.$$

From Proposition 2.1, we have

$$F(G^{-+}) = \sum_{u \in E(G)} (n - 1)^3 + \sum_{u \in E(G)} (2)^3 = n(n - 1)^3 + 8m.$$

**Theorem 3.4** Let $G$ be an $(n, m)$-graph. Then $F(G^{--}) = n(n + m - 1)^3 - 8F(G) - 12m(n + m - 1)^2 + 12(n + m - 1)M_1(G) + m(n - 2)^3$.

*Proof.* Since $G^{--}$ has $n + m$ vertices and $\frac{1}{2}n(n - 1) + m(n - 3)$ edges,

$$F(G^{--}) = \sum_{u \in V(G^{--})} d_{G^{--}}(u)^3 = \sum_{u \in V(G^{--}) \cap E(G)} d_{G^{--}}(u)^3 + \sum_{u \in V(G^{--}) \cap E(G)} d_{G^{--}}(u)^3.$$

By Proposition 2.1, we have

$$F(G^{--}) = \sum_{u \in E(G)} (n + m - 1 - 2d_u(u))^3 + \sum_{u \in E(G)} (n - 2)^3 = n(n + m - 1)^3 - 8F(G) - 12m(n + m - 1)^2 + 12(n + m - 1)M_1(G) + m(n - 2)^3.$$

**Lemma 3.1** [2] Let $G$ be an $(n, m)$-graph. Then

1. $M_1(G^{++}) = 4[m + M_1(G)].$
2. $M_1(G^{+-}) = nm^2 + m(n - 2)^2.$
3. $M_1(G^{-+}) = n(n - 1)^2 + 4m.$
4. $M_1(G^{--}) = 4M_1(G) + m(n - 2)^2 + (n + m - 1)[n(n + m - 1) - 8m].$

Apply Proposition 1.1, Lemma 3.1, from the results of the Theorems 3.1-3.4, we can deduce expressions for forgotten topological index of the complement of generalized transformation graphs $G^{ab}$. These are collected in the following.

**Theorem 3.5** Let $G$ be an $(n, m)$-graph. Then

1. $F(G^{++}) = (n + m - 1)[(n + m - 1)[(n + m)(n + m - 1) - 18m] + 3[4m + 4M_1(G)]] - 8F(G) - 8m.$
2. \( F(G^{++}) = (n + m - 1)((n + m - 1)(n + m(n + m - 1) - 6m(n - 1)] + 3[nm^2 + m(n - 2)^2]) - nm^2 - m(n - 2)^3. \)

3. \( F(G^{-+}) = (n + m - 1)((n + m - 1)(n + m(n + m - 1) - 6m - 3n(n - 1)] + 3[n(n - 1)^2 + 4m]) - n(n - 1)^3 - 8m. \)

4. \( F(G^{--}) = 8F(G) - m(n - 2)^3 + 3m(n - 2)^2(n + m - 1) + 3(n - n^2 - 2nm + 2m)(n + m - 1)^2 + (3n + m)(n + m - 1)^3. \)

Lemma 3.2 [2] Let \( G \) be an \((n,m)\)-graph. Then

1. \( M_2(G^{++}) = 4[M_1(G) + M_2(G)]. \)
2. \( M_2(G^{+-}) = m^3 + m^2(n - 2)^2. \)
3. \( M_2(G^{-+}) = \frac{n-1}{2}[n(n - 1)^2 - 2m(n - 1) + 8m]. \)
4. \( M_2(G^{--}) = (n + m - 1)((n + m - 1)[\left(\frac{1}{2}ight) - m] + m(n - 2)^2 - 2M_1(G)] + 4M_2(G) - 2(n - 2)[2m^2 - M_1(G)]. \)

Apply Proposition 1.2, Lemma 3.2, from the results of the Theorems 3.1-3.4, we can deduce expressions for hyper-Zagreb index of the generalized transformation graphs \( G^{ab} \). These are collected in the following.

Theorem 3.6 Let \( G \) be \((n,m)\)-graph. Then

1. \( HM(G^{++}) = 8[F(G) + m + M_1(G) + M_2(G)]. \)
2. \( HM(G^{+-}) = m^3(n + 2) + m(n - 2)^2(n + 2m - 2). \)
3. \( HM(G^{-+}) = 2n(n - 1)^3 - 2m(n - 1)^2 + 8nm. \)
4. \( HM(G^{--}) = 8m_2(G) - 8F(G) + m(n - 2)^2 - 4(n - 2)(2m^2 - M_1(G)] + [2m(n - 2)^2 - 4M_1(G) + 12M_1(G)](n + m - 1) + (n^2 - n - 14m)(n + m - 1)^2 + n(n + m - 1)^3. \)

Lemma 3.3 [2] Let \( G \) be an \((n,m)\)-graph. Then

1. \( M_2(G^{++}) = 2m[11m + 2n - 3] + 2(2n + 2m - 5)M_1(G) - 4M_2(G) + (n + m - 1)^2[\left(\frac{n + m}{2}\right) - 9m]. \)
2. \( M_2(G^{+-}) = \frac{1}{2}[n^4 + m^4 - 3n^2 + m^3 + 4nm^2 - 2n^2m + 8nm + 3n^2 - 5m^2 - 7m - n]. \)
3. \( M_2(G^{-+}) = \frac{1}{2}[4nm^3 + 3n^2m^2 - 18nm^2 - n^2m + 6nm - 9n^3 + m^4 + 27m^2 - 9m]. \)
4. \( M_2(G^{--}) = 2(n + m - 1)\bar{M}_1(G) - 4\bar{M}_2(G) + 2M_1(G)(n + m - 1) + \frac{m}{2}[23m - 8nm - 8n^2 + 16n - 9 + m^2 + m^3]. \)

Apply Proposition 1.3, Lemma 3.3, from the results of the Theorem 3.5, we can deduce expressions for the hyper-Zagreb index of the complement of generalized transformation graphs \( G^{ab} \). These are collected in the following.

Theorem 3.7 Let \( G \) be \((n,m)\)-graph. Then

1. \( HM(G^{++}) = (n + m - 1)((n + m - 1)[2(n + m)(n + m - 1) - 36m] + 3[4m + 4M_1(G)] - 4[2F(G) + 2m - m(11m + 2n - 3) - (2n + 2m - 5)M_1(G) + 2M_2(G)]. \)
2. \( HM(G^{+-}) = (n + m - 1)((n + m)(n + m - 1) - 6m(n - 1)] + 3[nm^2 + m(n - 2)^2)] + n^4 + m^4 - 3n^3 + (1 - n)m^3 + 4nm^2 - 2n^2m + 8nm + 3n^2 - 5m^2 - m[7 + (n - 2)^3] - n. \)
3. $HM(G^{+}) = (n + m - 1)[(n + m)(n + m - 1) - 6m - 3(n - 1)] + 3[n(n - 1)^2 + 4m] - n(n - 1)^3 + m(6n - n^2 - 17) + m^2[4nm + 3n^2 - 18n - 9m + m^2 + 27].$

4. $HM(G^{-}) = 4[2F(G) - 2M_2(G) + (n + 2m - 1)M_1(G)] + m[23m - 8nm - 8n^2 + 16n + m^2 + m^3 - (n - 2)^3 - 9] + [3m(n - 2)^2 + 4M_1(G)](n + m - 1) + 3[n - n^2 - 2nm + 2m](n + m - 1)^2 + [3n + m](n + m - 1)^3.$

Lemma 3.4 [2] Let $G$ be an $(n,m)$-graph. Then
1. $M_2(G^{+}) = 2m(9m - 1) - 6M_1(G) - 4M_2(G).$
2. $M_2(G^{-}) = m^2[2m(2n^2 - 2m - 4 - n) - (n - 2)^2].$
3. $M_2(G^{-}) = 2(\binom{n}{2} + m^2 - (n - 1)[\binom{n}{2} + 5m - mn] - 2m.$
4. $M_2(G^{-}) = 2(n + m - 1)M_1(G) - 4M_2(G) - 2(n - 1)M_1(G) + \frac{1}{2}[m^2n^2 + 16m^2 - 4m^2n - 3mn^2 + 2m - 2m^2 - m(n - 2)^3 + [4M_1(G) - 8M_1(G) + m(n - 2)^2](n + m - 1) + 4m(n + m - 1)^2].$

Apply Proposition 1.4, Lemmas 3.4, 3.1, from the results of the Theorems 3.1-3.4, we can deduce expressions for hyper-Zagreb coindex of the complement of generalized transformation graphs $G^{ab}.$ These are collected in the following.

Theorem 3.8 Let $G$ be an $(n,m)$-graph. Then
1. $HM(G^{+}) = 4m(n + 10m - 4) + 4M_1(G)(n + m - 4) - 8M_2(G) + F(G).$
2. $HM(G^{-}) = m[m(2n^2 - 2m - 4 - n) - (n - 2)^2 - nm^2 - (n - 2)^3] + (n + m - 1)[nm^2 + m(n - 2)^2].$
3. $HM(G^{-}) = 4[\binom{n}{2} + m^2 - 2(n - 1)[\binom{n}{2} + 5m - mn] - 12m + (n + m - 1)[n(n - 1)^2 + 4m] - n(n - 1)^3.$
4. $HM(G^{-}) = 4[2F(G) - 2M_2(G) - (n - 1)M_1(G)] + m^2n^2 + 16m^2 - 4m^2n - 3mn^2 + 4mn - 2m + 2m^2 - m(n - 2)^3 + [4M_1(G) - 8M_1(G) + m(n - 2)^2](n + m - 1) + 4m(n + m - 1)^2.$

Lemma 3.5 [2] Let $G$ be an $(n,m)$-graph. Then
1. $M_2(G^{+}) = 4M_2(G) - 4(n + m - 2)M_1(G) + m(3m^2 - 10m + 3n^2 + 6nm - 10n + 7).$
2. $M_2(G^{-}) = m[2n^2 + 2n^2 - 3mn + 2m - 5n + 3].$
3. $M_2(G^{-}) = \frac{1}{2}[(n + m)(n + m - 1) - 2m - n(n - 1)^2] - \frac{1}{2}[4nm^3 + 3n^2m^2 - 15nm^2 - 8m^3 + m^4 + 2m^2].$
4. $M_2(G^{-}) = 4M_2(G) - 2(n + m - 1)M_1(G) + 2(n + 2m)M_1(G) + \frac{1}{2}[8m^3 + 8m^2 + 8n^2m - 16mn + 8m].$

Lemma 3.6 [2] Let $G$ be an $(n,m)$-graph. Then
1. $M_1(G^{+}) = 4[M_1(G) + m] + (n + m - 1)[(n + m)(n + m - 1) - 12m].$
2. $M_1(G^{+}) = n(n - 1)^2 + m(m + 1)^2.$
3. $M_1(G^{+}) = m^3 + 3nm^2 + n^2m - 6m^2 - 6nm + 9m.$
4. $M_1(G^{-}) = 4M_1(G) + m^3 + 2m^2 + m.$

Apply Proposition 1.4, Lemmas 3.5, 3.6, from the results of the Theorem 3.5, we can deduce expressions for hyper-Zagreb coindex of the complement of generalized transformation graphs $G^{ab}.$ These are collected in the following.
Theorem 3.9 Let $G$ be $(n,m)$-graph. Then

1. $HM(\overline{G^{++}}) = \frac{8}{3}M_2(G) - (n + m - 2)M_1(G) + F(G) + m + 2m(3m^2 - 10m + 3n^2 + 6nm - 10n + 7) + (n + m - 1)[6m(n + m - 1) - 8m - 8M_1(G)]$.

2. $HM(\overline{G^{+-}}) = 2m(n^2m + 2n^2 - 3nm + 2m - 5n + 3) + nm^3 + m(n - 2)^3 + (n + m - 1)\{n(n - 1)^2 + m(m + 1)^2 - 3[nm^2 + m(n - 2)^2] - (n + m - 1)[(n + m)(n + m - 1) - 6m(n - 1)]\}$.

3. $HM(\overline{G^{-+}}) = n(n - 1)^2(n - 2) + 6m - 4nm^3 - 3n^2m^2 + 15nm^2 + 8m^3 - m^4 - 21m^2 + (n + m - 1)\{n - 2m + m^3 + 3nm^2 + n^2m - 6m^2 - 6nm - 3n(n - 1)^2\} - (n + m - 1)^2[(n + m)(n + m - 1) - 6m - 3n(n - 1)]$.

4. $HM(\overline{G^{--}}) = 4[2M_2(G) - (n + 2m)M_1(G) - 2F(G)] + 8m[m^2 + nm + n^2 - 2n + 1] + m(n - 2)^3 + 4M_1(G) + m^3 + 2m^2 + m - 4M_1(G) - 3m(n - 2)^2](n + m - 1) - 3[n - n^2 - 2nm + 2m](n + m - 1)^2 - (3n + m)(n + m - 1)^3$.

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References


