

## Common Fixed Point Theorem in Intuitionistic Fuzzy Metric Spaces Satisfying Integral Type Inequality

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**Abstract:** We prove common fixed point theorem for weakly compatible maps in Intuitionistic fuzzy metric space satisfying integral type inequality but without using the completeness of space or continuity of the mappings involved. We prove by using the concept of E A property.

### 1. Introduction

In the study of common fixed points of compatible mappings we often require assumption on completeness of the space or continuity of mappings involved besides some contractive condition but the study of fixed points of non compatible mappings can be extend to the class of non expansive or Lipschitz type mapping pairs even without assuming the continuity of the mappings involved or completeness of the space. Aamri and El Moutawakil [6] generalized the concepts of non compatibility by defining the notion of (E.A) property and proved common fixed point theorems under strict contractive condition.

We prove common fixed point theorems for weakly compatible maps in Intuitionistic fuzzy metric space by using the concept of (E.A) property, however, without assuming either the completeness of the space or continuity of the mappings involved.

### 2. Preliminaries

**Definition:-** A binary operation  $*$ :  $[0,1] \times [0,1] \rightarrow [0,1]$  is continuous t-norm if is satisfying the following condition:

- (i)  $*$  is commutative and associative ;
- (ii)  $*$  is continuous ;
- (iii) ;  $a * 1 = a$  for some  $a \in [0,1]$ ;
- (iv)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$  and  $a, b, c, d \in [0,1]$

**Definition:-** A binary operation  $^{\circ}$ :  $[0,1] \times [0,1] \rightarrow [0,1]$  is continuous t-conorm if  $^{\circ}$  is satisfying the following condition:

- (i)  $^{\circ}$  is commutative and associative ;
- (ii)  $^{\circ}$  is continuous ;
- (iii)  $a ^{\circ} 0 = a$  for some  $a \in [0,1]$
- (iv)  $a ^{\circ} b \leq c ^{\circ} d$  whenever  $a \leq c$  and  $b \leq d$  and  $a, b, c, d \in [0,1]$

**Definition:-**The 5- tuple  $(X, M, N, *, ^{\circ})$  is said to be an Intuitionistic Fuzzy Metric Space if  $X$  is an arbitrary non empty set,  $*$  is a continuous t-norm,  $^{\circ}$  is a continuous t-conorm and  $M, N$  are a Fuzzy set on  $X^2 \times [0,1]$  satisfying the following conditions:

For all  $x, y, z, \in X, s, t > 0$

- (i)  $M(x, y, t) + N(x, y, t) \leq 1$  ;
- (ii)  $M(x, y, 0) > 0$ ;
- (iii)  $M(x, y, t) = 1$  for all  $t > 0$  iff  $x = y$ ;
- (iv)  $M(x, y, t) = M(y, x, t)$ ;
- (v)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ ;
- (vi)  $M(x, y, \cdot): (0, \infty) \rightarrow (0, 1]$  is continuous;
- (vii)  $N(x, y, 0) > 0$ ;
- (viii)  $N(x, y, t) = 0$  for all  $t > 0$  iff  $x = y$ ;
- (ix)  $N(x, y, t) = N(y, x, t)$ ;
- (x)  $N(x, y, t) \diamond N(y, z, s) \leq N(x, z, t + s)$ ;
- (xi)  $N(x, y, \cdot): (0, \infty) \rightarrow (0, 1]$  is continuous;

**Example:** - Let  $(X, d)$  be a metric space. Define  $a * b = ab$  and  $a \diamond b = \min\{1, a + b\}$  for all  $a, b \in [0, 1]$  and let  $M$  and  $N$  be fuzzy sets on  $X^2 \times [0, 1]$  defined as follows :

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

$$N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

Then  $(X, M, N, *, \diamond)$  is an intuitionistic fuzzy metric space.

**Definition:** - Let  $U$  and  $V$  be two self maps of an intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$ .  $U$  and  $V$  are said to be compatible if  $M(UVx_n, VUx_n, t) \rightarrow 1$  and  $N(UVx_n, VUx_n, t) \rightarrow 0$  as  $n \rightarrow \infty$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $UVx_n, VUx_n \rightarrow z$  as  $n \rightarrow \infty$ , for some  $z \in X$ .

**Definition :-** Two self maps  $U$  and  $V$  of intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  are said to be weakly compatible if they commute at their coincidence point, i.e.  $UVx = VUx$  whenever  $Ux = Vx$   $x \in X$ .

*Clearly each pair of compatible self maps is weakly compatible but the converse is not true always.*

**Definition:-** Let  $A$  and  $B$  be two self-maps of a intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$ . We say that  $A$  and  $B$  satisfy the property (E.A) if there exists a sequence  $\{x_n\}$  such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = z$  for some  $z \in X$

*Note that weakly compatible and property (E.A) are independent to each other.*

**Definition:** - Let  $f$  and  $g$  be two self maps of a metric space  $(X, d)$  and  $f$  and  $g$  to be weakly commuting if

$$d(fgx, gfx) \leq d(gx, fx) \text{ for all } x \in X$$

It can be seen that commuting maps  $(fgx = gfx \forall x \in X)$  are weakly compatible, but converse is false.

In 2002, A.Branciari[1] analyzed the existence of fixed point for mapping  $T$  defined on a complete metric space  $(X; d)$  satisfying a general contractive condition of integral type in the following theorem.

**Theorem[1] :-** Let  $(X; d)$  be a complete metric space,  $c \in (0,1)$  and let  $T : X \rightarrow X$  be a mapping such that for each  $x, y \in X$ ,

$$\int_0^{d(Tx, Ty)} \varphi dt \leq c \int_0^{d(Tx, Ty)} \varphi dt \tag{i}$$

where  $\varphi : [0, +\infty) \rightarrow [0; +\infty)$  is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of  $[0; +\infty)$ , non-negative, and such that for each

$$\varepsilon > 0, \quad \int_0^\varepsilon \varphi dt \geq 0$$

then  $T$  has a unique fixed point  $a \in X$  such that for each  $x \in X$ ,

$$\lim_{n \rightarrow \infty} T^n x = a$$

After the paper of Branciari[1], a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades[2] extending the result of Branciari[1] by replacing the condition (i) by the following

$$\int_0^{d(Tx, Ty)} \varphi dt \leq \int_0^{\max \left\{ \begin{array}{l} d(x, y), \\ d(x, Tx), \\ d(y, Ty), \\ \frac{1}{2}[d(x, Ty), d(y, Tx)] \end{array} \right\}} \varphi dt$$

### 3. Main Results run as

**Theorem:-** Let  $f$  and  $g$  be two weak – compatible self maps of intuitionistic fuzzy metric  $(X, M, N, *, \diamond)$  space satisfying the property E.A. and

$$fX \subset gX$$

$$\int_0^{M(fx, fy, kt) + N(fx, fy, kt)} \phi(t) dt \geq \int_0^{M(gx, gy, t) + N(gx, gy, t)} \phi(t) dt, \quad k \geq 0$$

$$\int_0^{M(fx, ffx, t) + N(fx, ffx, t)} \phi(t) dt > \int_0^{\min \left\{ \begin{array}{l} M(ffx, gx, t) + N(ffx, gx, t), \\ M(ggx, fx, t) + N(ggx, fx, t), \\ M(ffx, ggx, t) + N(ffx, ggx, t) \end{array} \right\}} \phi(t) dt$$

Whenever  $fx \neq f^2x$

If the range of  $f$  or  $g$  is complete subspace of  $X$ , then  $f$  and  $g$  have a common fixed point

**Proof:-**

Since  $f$  and  $g$  are satisfy the property E.A., so there exists a sequence  $\{x_n\}$  in  $X$  such that  $fx_n, gx_n \rightarrow z$  as  $n \rightarrow \infty$  for  $z \in X$

Since  $z \in fX$  and  $z \in gX$ , there exists some point  $s$  in  $X$  such that  $z = fs$  where  $gx_n \rightarrow z$  as  $n \rightarrow \infty$

If then  $fs \neq gs$

$$\int_0^{M(fx_n,fs,kt)+N(fx_n,fs,kt)} \phi(t) dt \geq \int_0^{M(gx_n,gs,t)+N(gx_n,gs,t)} \phi(t) dt$$

Taking limit  $n \rightarrow \infty$  we get

$$\int_0^{M(gs,fs,kt)+N(gs,fs,kt)} \phi(t) dt \geq \int_0^{M(gs,gs,t)+N(gs,gs,t)} \phi(t) dt = \int_0^1 \phi(t) dt$$

Not possible

Hence  $fs = gs$

Since  $f$  and  $g$  are weakly compatible so  $fgs = ffs$ , therefore  $fgs = ffs = gfs = ggs$

If  $ffs \neq fs$  then

$$\begin{aligned} \int_0^{M(fs,ffs,t)+N(fs,ffs,t)} \phi(t) dt &> \int_0^{\min \left\{ \begin{matrix} M(ffs,gs,t)+N(ffs,gs,t), \\ M(ggs,fs,t)+N(ggs,fs,t), \\ M(ffs,ggg,t)+N(ffs,ggg,t) \end{matrix} \right\}} \phi(t) dt \\ &= \int_0^{\min \left\{ \begin{matrix} M(ffs,gs,t)+N(ffs,gs,t), \\ M(ggs,fs,t)+N(ggs,fs,t), \\ M(ffs,ffs,t)+N(ffs,ffs,t) \end{matrix} \right\}} \phi(t) dt \\ &= \int_0^{\min \left\{ \begin{matrix} M(ffs,fs,t)+N(ffs,fs,t), \\ M(ffs,fs,t)+N(ffs,fs,t), \\ 1+0 \end{matrix} \right\}} \phi(t) dt \\ &= \int_0^{M(ffs,fs,t)+N(ffs,fs,t)} \phi(t) dt \end{aligned}$$

This is contradiction and so  $ffs = fs$

$$\Rightarrow fs = ffs = fgs = gfs = ggs$$

Hence  $fs$  is a common fixed point of  $f$  and  $g$ .

The case when  $fX$  is a complete subspace  $X$  is similar to the above since  $fX \subset gX$

#### 4. Conclusion

We prove fixed point theorem for weakly compatible maps in intuitionistic fuzzy metric space satisfying integral type inequality by using E.A. property.

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