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GRACEFUL LABELING FOR DISCONNECTED GRID RELATED GRAPHS

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Abstract: In this paper we have proved that union of three grid graphs, $U_{l=1}^{3}(P_{n_l} \times P_{m_l})$ and union of finite copies of a grid graph $(P_n \times P_m)$ are graceful. We have also given two graceful labeling functions to the grid graph$(P_n \times P_m)$.

1 Introduction

We begin with a simple undirected finite graph $G = (V, E)$, with $|V| = p$ vertices and $|E| = q$ edges. For all terminology and basic we follows Harary [1]. First we shall recall some definitions which are used in this paper.

Definition−1.1 **:** A function f is called *graceful labeling* of a graph $G = (V, E)$ if $f : V \longrightarrow \{0,1,...,q\}$ is injective and the induce function $f^* : E \longrightarrow \{1,2,...,q\}$ defined as $f^*(e) = |f(u) - f(v)|$ is every edge $e = (u, v) \in E$. A graph G is called *graceful graph* if it admits a graceful bijective for labeling.

Definition−1.2: The *Cartesian product* of graphs G and H denoted as $G \times H$, is the graph withvertex set $V(G)\times V(H) = \{(u, v)/u \in V(G) \text{ and } v \in V(H)\}$ and vertex (u, v) adjacent to another vertex (u', v') if and only if either $u = u'$ and edge $(v, v') \in E(H)$ or $v = v'$ and edge $(u, u') \in E(G)$.

The Cartesian product of two paths P_n and P_m denoted as $(P_n \times P_m)$ is known as *grid graph* on mn vertices.

The graceful labeling was introduced by A Rosa [2] in 1967. Acharya and Gill [3] have investigated graceful labeling for the grid graph $P_n \times P_m$, whose vertex graceful labeling we define in *Theorem*−2.1, where $V(P_n \times P_m) = \{u_{i,j} \,/\, i = 1,2,...,n, \;\; j = 1,2,...,m\}$. Kaneria and Makadia [4] discussed gracefulness of union of two grid graphs, for this the vertex graceful labeling function we define in *Theorem*−2.3. In [5] Kaneria et al. investigate graceful labeling for path union of a grid graph, star of a grid graph and cycle of even copies of a grid graph. For detail survey of various graph labelings and bibliographic references one can refer Gallian [6]. Labelled have many diversified applications.

In this paper we get graceful labeling pattern for union of finite copies of a grid graph and union of three grid graphs. Following labeling pattern for a grid graph $(P_n \times P_m)$ is graceful, but it may not help to prove union of grid graphs is also graceful.

$f : V(P_n \times P_m) \longrightarrow \{0,1,...,q\}$, where $q = 2mn - (m + n)$ define by

$f(u_{i,1}) = q - \left(\frac{i-1}{2}\right)$ when i is odd

$= \left(\frac{i-2}{2}\right)$ when i is even, $\forall\, i = 1,2,...,n$;

$f(u_{i,2}) = (n-1) + \left(\frac{i-1}{2}\right)$,when i is odd

$= (q - n + 1) - \left(\frac{i}{2}\right)$ when i is even, $\forall\, i = 1,2,...,n$;

$f(u_{i,j}) = f(u_{i,j-2}) - (2n-1)$,when $f(u_{i,j-2}) > \frac{q}{2}$,

$= f(u_{i,j-2}) + (2n-1)$, when $f(u_{i,j-2}) < \frac{q}{2}$,
$\forall\, i = 1,2,\ldots,n$; $\forall\, j = 3,4,\ldots m$;

2 Main Results:

Theorem−2.1 **:**Union of finite copies of a grid graph is graceful.

Proof : Let $u_{j,k}$ $(1 \le j \le n,\ 1 \le k \le m)$ be vertices of a grid graph $P_n \times P_m$. Let G be finite union of $t$copies of the grid graph $P_n \times P_m$. Let $u_{i,j,k}$ $(1 \le j \le n,\ 1 \le k \le m)$ be vertices of i^{th} copy $(P_n \times P_m)^{(i)}$ of the grid graph in G, $\forall\, i = 1,2,\ldots,t$

We know that the grid graph $(P_n \times P_m)^{(1)}$ (where $n \le m$) is a graceful graph on $p = mn$ vertices and $q = 2mn - (m + n)$ edges with following vertex labeling function. $f : V((P_n \times P_m)^{(1)}) \longrightarrow \{0, 1, \ldots, q\}$ define by

$f(u_{1,j,1}) = q - \frac{(j-1)^2}{2}$ when j is odd

$= \frac{j(j-2)}{2}$ when j is even, $\forall\, j = 1,2,\ldots,n$;

$f(u_{1,j,m}) = \frac{q}{2} - \frac{1}{4} + (-1)^{m+j}\left[\frac{(n-j)^2}{2} + \frac{1}{4}\right]$, $\forall\, j = n, n\text{-}1, \ldots., 1$;

$f(u_{1,j,k}) = f(u_{1,j-1,k+1}) + (-1)^{j+k}$, $\forall\, k = m\text{-}1, m\text{-}2, \ldots.., m+1\text{-}n$, $\forall\, j = n, n\text{-}1, \ldots.., m\text{-}i+1$;

$f(u_{1,n,k}) = f(u_{1,n,1}) + (-1)^n\left[\frac{(2n-1)(k-1)}{2}\right]$, when k is odd ,

$= f(u_{1,n,1}) - (-1)^n\left[\frac{(2n-1)k}{2} - 1\right]$, when k is even, $\forall\, k = 2,3, \ldots., m\text{-}n$;

$f(u_{1,j,k}) = f(u_{1,j+1,k-1}) - (-1)^{j+k}$, $\forall\, k = 2, 3, \ldots., m\text{-}1$, $\forall\, j = 1, 2, \ldots., \min\{n, m\text{-}k\}$.

In graph G it is obvious that the number of vertices $|V(G)| = P = tmn$ and the edges $|E(G)| = Q = tq$. To define a labeling function $g : V(G) \longrightarrow \{0,1,\ldots,Q\}$, we shall have following two cases.

Case−I: q is even

$g(u_{1,j,k}) = f(u_{1,j,k}) + (Q-q)$, when $f(u_{1,j,k}) \ge \frac{q}{2}$

$= f(u_{1,j,k})$, when $f(u_{1,j,k}) < \frac{q}{2}$

$\forall\, j = 1,2, \ldots, n$, $\forall\, k = 1,2, \ldots, m$;

$g(u_{i,j,k}) = g(u_{i-1,j,k}) - \frac{q}{2} + 1$, when $g(u_{i-1,j,k}) > \frac{Q}{2}$

$= g(u_{i-1,j,k}) + \frac{q}{2} + 1$, when $g(u_{i-1,j,k}) < \frac{Q}{2}$,

$\forall\, i = 2,3,\ldots,t$, $\forall\, j = 1,2,\ldots,n$, $\forall\, k = 1,2,\ldots,m$.

Case−II : q is odd

$g(u_{1,j,k}) = f(u_{1,j,k}) + (Q-q)$, when $f(u_{1,j,k}) > \frac{q}{2}$

$= f(u_{1,j,k})$, when $f(u_{1,j,k}) < \frac{q}{2}$,

$\forall\, j = 1,2, \ldots, n$, $\forall\, k = 1,2, \ldots, m$;

$$g(u_{2,j,k}) = Q + \tfrac{q}{2} - \left[g(u_{1,j,k}) + \tfrac{3}{2}\right], \quad \text{when } g(u_{1,j,k}) > \tfrac{Q}{2}$$
$$= Q - \left[g(u_{1,j,k}) + \tfrac{q}{2} + \tfrac{3}{2}\right], \quad \text{when } g(u_{1,j,k}) < \tfrac{Q}{2},$$
$$\forall\, j = 1,2,\dots, n, \quad \forall\, k = 1,2,\dots, m\,;$$
$$g(u_{i,j,k}) = g(u_{i-2,j,k}) - q, \quad \text{when } g(u_{i-2,j,k}) > \tfrac{Q}{2}$$
$$= g(u_{i-2,j,k}) + q, \quad \text{when } g(u_{i-2,j,k}) < \tfrac{Q}{2},$$
$$\forall\, i = 3,4,\dots, t, \quad \forall\, j = 1,2,\dots, n, \quad \forall\, k = 1,2,\dots, m.$$

Above labeling pattern give rise a graceful labeling to given graph G = union of t copies of a grid graph $P_n \times P_m$ and so it is a graceful graph.

Illustration−2.2: $U_{l=1}^{4}(P_3 \times P_4)$ and its graceful labeling shown in *figure*−1

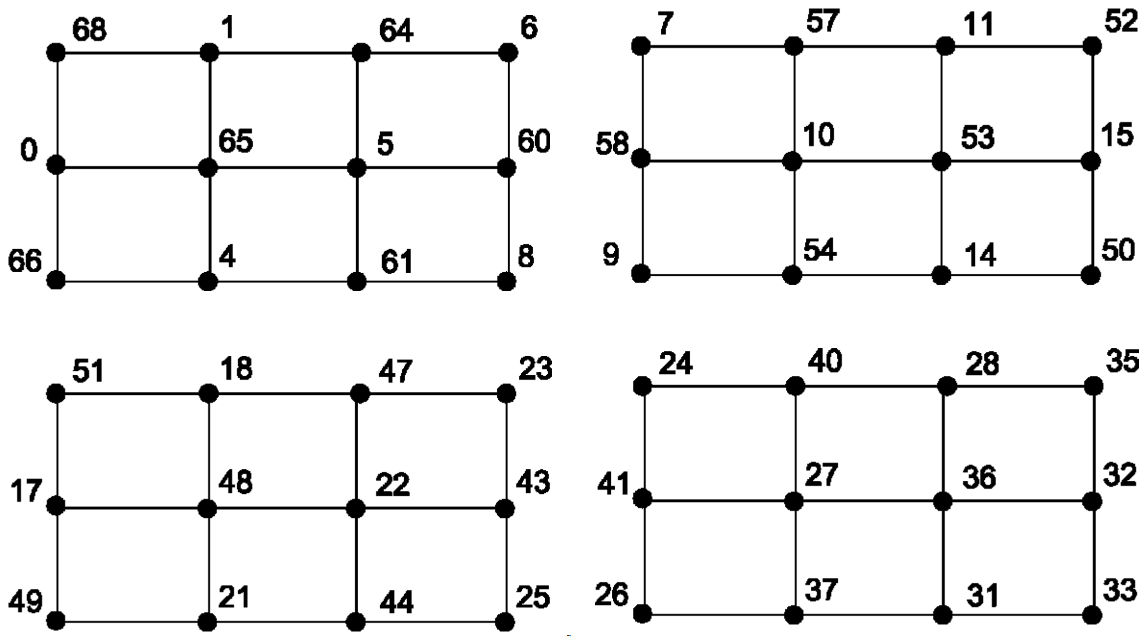

Figure 1 Disconnected graph $U_{l=1}^{4}(P_3 \times P_4)$ and its graceful labeling.

Theorem−2.3 **:** $U_{l=1}^{3}(P_{n_l} \times P_{m_l})$ is graceful.

Proof : Let $u_{i,j}^{(l)}\ (1 \le i \le n_l, 1 \le j \le m_l)$ be vertices of the grid graph $P_{nl} \times P_{ml}$, $\forall\, l = 1,2,3$.

It is obvious that the number of vertices $pl = |V(P_{nl} \times P_{ml})|$ for each grid is $nl\, ml$ and the number of edges $ql = |E(P_{nl} \times P_{ml})|$ for each grid is $2ml\,nl - (ml + nl)$, $\forall\, l = 1,2,3$.

We know that these grid graphs $P_{nl} \times P_{ml}$ are graceful graphs as we mentioned its graceful labeling in *Theorem*−2.1, $\forall\, l = 1,2,3$. Let $fl : V(P_{nl} \times P_{ml}) \longrightarrow \{0,1,\dots,ql\}$ be a graceful labeling for grid graph $P_{nl} \times P_{ml}$, $\forall\, l = 1,2,3$.

First we shall prove that $P_{n1} \times P_{m1} \cup P_{n2} \times P_{m2}$ is a graceful graph. We define a labeling function $f : V(P_{n1} \times P_{m1} \cup P_{n2} \times P_{m2}) \longrightarrow \{0,1,\dots, q1 + q2\}$ as follows.

$$f(u_{i,j}^{(1)}) = f_1(u_{i,j}^{(1)}), \quad \text{when } f_1(u_{i,j}^{(1)}) < \tfrac{q_1}{2}$$
$$= f_1(u_{i,j}^{(1)}) + q_2, \quad \text{when } f_1(u_{i,j}^{(1)}) \ge \tfrac{q_1}{2}$$
$\forall\, i = 1,2,\dots, n_1$, $\forall\, j = 1,2,\dots, m_1$;

$f\left(u_{i,j}^{(2)}\right) = f_2\left(u_{i,j}^{(2)}\right) + \frac{q_1+2}{2}$

when $q_1 \equiv 0 \pmod 2$

$= q_1 + q_2 - f_2\left(u_{i,j}^{(2)}\right) - \frac{q_1+3}{2}$ when $q_1 \equiv 1 \pmod 2$,

$\forall\, i = 1,2, \dots ,n_2$, $\forall\, j = 1,2, \dots ,m_2$.

Above labeling pattern give rise graceful labeling to the graph $P_{n1} \times P_{m1} \cup P_{n2} \times P_{m2}$. [Above result was proved by Kaneria and Makadia [4].]

Since $q1, q2, q3 \in N$ (one of three cases must be hold) either $q1 + q2 \equiv 0 \pmod 2$ or $q1 + q3 \equiv 0 \pmod 2$ or $q2 + q3 \equiv 0 \pmod 2$. Without loss of generality we may assume that $q1 + q2 \equiv 0 \pmod 2$. To define a labeling function $g : V(U_{l=1}^{3}(P_{n_l} \times P_{m_l})) \longrightarrow \{0,1,\dots,Q\}$, where $Q = q1 + q2 + q3$, we shall have following two cases.

Case–I : $q_1 \equiv 1 \pmod 2$ and $q_2 \equiv 1 \pmod 2$

$g\left(u_{i,j}^{(l)}\right) = f\left(u_{i,j}^{(l)}\right)$, when $f\left(u_{i,j}^{(l)}\right) < \frac{q_1+q_2-2}{2}$

$= f\left(u_{i,j}^{(l)}\right) + q_3$, when $f\left(u_{i,j}^{(l)}\right) \geq \frac{q_1+q_2-2}{2}$

$\forall\, i = 1,2, \dots, n_l$, $\forall\, j = 1,2,\dots,m_l$, $\forall\, l = 1,2$;

$g\left(u_{i,j}^{(3)}\right) = f_3\left(u_{i,j}^{(3)}\right) + \frac{q_1+q_2}{2}$, $\forall\, i = 1,2, \dots ,n_3$, $\forall\, j = 1,2, \dots ,m_3$.

Case–II : $q_1 \equiv 0 \pmod 2$ and $q_2 \equiv 0 \pmod 2$

$g\left(u_{i,j}^{(1)}\right) = f\left(u_{i,j}^{(1)}\right)$, when $f\left(u_{i,j}^{(1)}\right) < \frac{q_1}{2}$

$= f\left(u_{i,j}^{(1)}\right) + q_3$, when $f\left(u_{i,j}^{(1)}\right) \geq \frac{q_1}{2}$,

$\forall\, i = 1,2,\dots,n_1$, $\forall\, j = 1,2,\dots,m_1$;

$g\left(u_{i,j}^{(2)}\right) = f\left(u_{i,j}^{(2)}\right)$, when $f\left(u_{i,j}^{(2)}\right) < \frac{q_2}{2}$

$= f\left(u_{i,j}^{(2)}\right) + q_3$, when $f\left(u_{i,j}^{(2)}\right) \geq \frac{q_2}{2}$,

$\forall\, i = 1,2,\dots,n_2$, $\forall\, j = 1,2,\dots,m_2$;

$g\left(u_{i,j}^{(3)}\right) = f_3\left(u_{i,j}^{(3)}\right) + \frac{q_1+q_2+4}{2}$,

$\forall\, i = 1,2,\dots,n_3$, $\forall\, j = 1,2,\dots,m_3$.

Above labeling pattern give rise a graceful labeling to given graph $U_{l=1}^{3}(P_{n_l} \times P_{m_l})$ and so it is a graceful graph.

Illustration–2.4 : $P_3 \times P_3 \cup P_3 \times P_5 \cup P_2 \times P_4$ and its graceful labeling shown in *figure*–4. While *figure*–2 shown graceful labeling of graphs $P_3 \times P_3$, $P_3 \times P_5$, $P_2 \times P_4$ and *figure*–3 shows graceful labeling for $P_3 \times P_3 \cup P_3 \times P_5$. Here $q_1 = 12$, $q_2 = 22$, $q_3 = 10$ and q= q_1+ q_2+ q_3 =44.

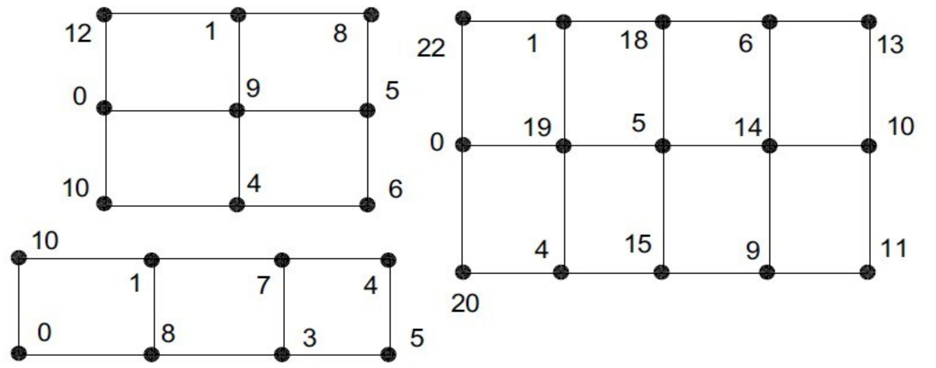

Figure 2 Graceful labeling for $P3\times P3$, $P3\times P5$ and $P2\times P4$ under functions $f1, f2, f3$ respectively.

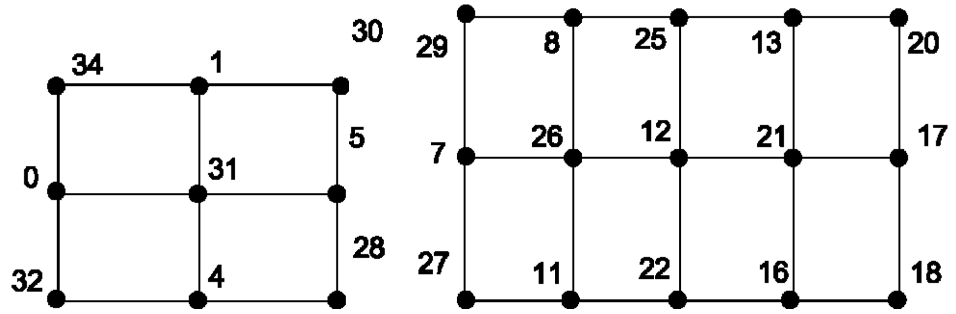

Figure 3 Disconnected graph $P3\times P3 \cup P3\times P5$ and its graceful labeling under f.

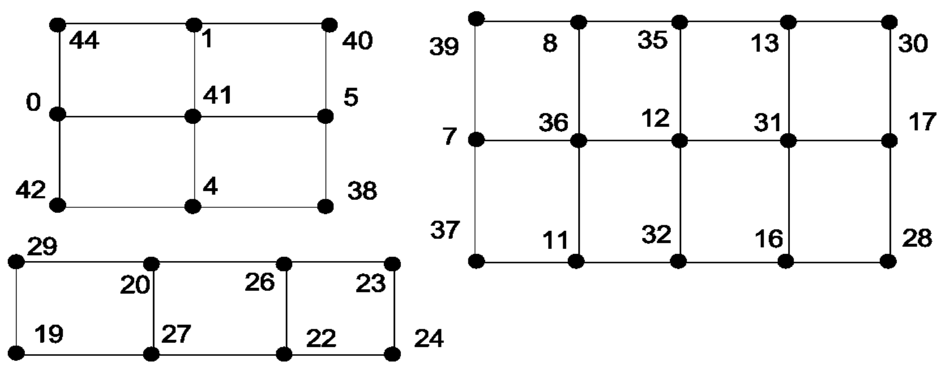

Figure 4 Disconnected graph $P_3\times P_3 \cup P_3\times P_5 \cup P_2\times P_4$ and its graceful labeling.

3 Concluding Remarks:

We have discussed two distinct graceful labeling patterns by suitable function for a grid graph. We also proved union of finite copies of a grid graph and union of three grid graphs are graceful. The labeling pattern is demonstrated by means of illustrations, which provide better understanding of derived results. We raise open question to get graceful labeling for the disconnected graph $U_{i=1}^{t}(P_{n_i}\times P_{m_i})$ where $t \geq 4$.

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