INDEX SUMMABILITY OF AN INFINITE SERIES USING $\delta$-QUASI MONOTONE SEQUENCE

Mahendra Misra\(^1\) B.P.Padhy\(^2\) Santosh kumar Nayak\(^3\) U.K.Misra\(^4\)

\(^1\)P.G.Department of Mathematics
N.C. College(Autonomous)
Jajpur, Odisha

\(^2\)Roland Institute of Technology
Golanthara-761008, Odisha, India.

\(^3\)Department of Mathematics
Jeevan Jyoti Mahavidyalaya
Raikia, Khandhamal, Odisha, India

\(^4\)Department of Mathematics
National Institute of Science and Technology
Pallur Hills, Golanthara-761008, Odisha, India

**Keywords**: Quasi-increasing, Quasi - f - power increasing, $\delta$ - Quasi monotone, index absolute Summability, summability factor.

**ABSTRACT**: A result concerning absolute indexed Summability factor of an infinite series using $\delta$ - Quasi monotone sequence has been established.

1. **INTRODUCTION:**

A positive sequence $\{a_n\}$ is said to be almost increasing if there exists a positive sequence $\{b_n\}$ and two positive constants $A$ and $B$ such that

$$Ab_n \leq a_n \leq Bb_n, \text{ for all } n.$$  \hfill (1.1)

The sequence $\{a_n\}$ is said to be quasi- $\beta$ -power increasing, if there exists a constant $K$ depending upon $\beta$ with $K \geq 1$ such that

$$Kn^\beta a_n \geq m^\beta a_n,$$  \hfill (1.2)

for all $n \geq m$. In particular, if $\beta = 0$, then $\{a_n\}$ is said to be quasi-increasing sequence. It is clear that every almost increasing sequence is a quasi- $\beta$ -power increasing sequence for any non-negative $\beta$. But the converse is not true as $(n^{-\beta})$ is quasi- $\beta$ -power increasing but not almost increasing.

Let $f = (f_n)$ be a positive sequence of numbers. Then the positive sequence $\{a_n\}$ is said to be quasi- $f$ -power increasing, if there exists a constant $K$ depending upon $f$ with $K \geq 1$ such that

$$Kf_n a_n \geq f_m a_m,$$  \hfill (1.3)

for $n \geq m \geq 1([6])$. Clearly, if $\{a_n\}$ is a quasi- $f$ -power increasing sequence, then the $(a_n f_n)$ is a quasi-increasing sequence.
Let \( \delta = (\delta_n) \) be a positive sequence of numbers. Then a positive sequence \((a_n)\) is said to be \(\delta\) - quasi monotone [1], if \(a_n \to 0\), \(a_n > 0\) ultimately and \(\Delta d_n \geq -\delta_n\), where \(\Delta d_n = d_n - d_{n-1}\).

Let \(\sum a_n\) be an infinite series with sequence of partial sums \(\{s_n\}\). Let \(\{p_n\}\) be a sequence of positive numbers such that

\[
P_n = \sum_{v=0}^{n} p_v \to \infty, \text{ as } n \to \infty.
\]

Then the sequence-to-sequence transformation

\[
T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v, \quad P_n \neq 0.
\]

defines the \(\overline{\{N.p_s\}}\) mean of the sequence \(\{s_n\}\) generated by the sequence of coefficients \(\{p_n\}\). The series \(\sum a_n\) is said to be summable \(\overline{\{N.p_s\} \delta}, k \geq 1([2])\), if

\[
\sum_{n=1}^{\infty} \left(\frac{p_n}{P_n}\right)^{n-k} |T_n - T_{n-1}|^k < \infty.
\]

(1.5)

The series \(\sum a_n\) is said to be summable \(\overline{\{N.p_s\} \delta}, k \geq 1, \delta \geq 0,\) if

\[
\sum_{n=1}^{\infty} \left(\frac{p_n}{P_n}\right)^{n-k-1} |T_n - T_{n-1}|^k < \infty.
\]

(1.6)

2. KNOWN THEOREMS:

Dealing with quasi- \(\beta\) -power increasing sequence Bor and Debnath [3] have established the following theorem:

2.1. THEOREM:

Let \((X_n)\) be a quasi-\(\beta\) -power increasing sequence for \(0 < \beta < 1\) and \((\lambda_n)\) be a real sequence. If the conditions

\[
\sum_{n=1}^{n} \frac{P_n}{n} = O(P_n),
\]

(2.1.1)

\[
\lambda_n X_n = O(1),
\]

(2.1.2)
are satisfied, where \( t_n \) is the \((C,1)\) mean of the sequence \((na_n)\). Then the series \( \sum a_n \lambda_n \) is summable \( \|N_p a_n\|_k, k \geq 1 \).

Subsequently Leindler [4] established a similar result reducing certain condition of Bor. He established:

2.2. THEOREM:

Let the sequence \((X_n)\) be a quasi-\( \beta \)-power increasing sequence for \(0 < \beta < 1\), and the real sequence \((\lambda_n)\) satisfies the conditions

\[
\sum_{n=1}^{\infty} \lambda_n = O(m) \tag{2.2.2}
\]

and

\[
\sum_{n=1}^{\infty} |\Delta^2 \lambda_n| = O(m) \tag{2.2.3}
\]

hold, where \( X_n(\beta) = \max(n^\beta X_n, \log n) \). Then the series \( \sum q_n \lambda_n \) is summable \( \|N_p q_n\|_k, k \geq 1 \).

Recently, extending the above results to quasi-\( f \)-power increasing sequence, Sulaiman [7] have established the following theorem:

2.3. THEOREM:

Let \( f = (f_n) = (n^\beta \log r n) \) for \(0 \leq \beta < 1, r \geq 0\) be a sequence. Let \((X_n)\) be a quasi-\( f \)-power sequence and \((\lambda_n)\) a sequence of constants satisfying the conditions

\[
\lambda_n \to 0 \text{ as } n \to \infty, \tag{2.3.1}
\]

\[
\sum_{n=1}^{\infty} nX_n |\Delta^2 \lambda_n| < \infty, \tag{2.3.2}
\]
\[ |\lambda_n| X_n = O(1), \quad (2.3.3) \]
\[ \sum_{n=1}^{\infty} \frac{1}{nX_n^{k-1}} |t_n|^k = O(X_m) \quad (2.3.4) \]

and
\[ \sum_{n=1}^{\infty} \frac{P_n}{P_n} \frac{1}{X_n^{k-1}} |t_n|^k = O(X_m), \quad (2.3.5) \]

where \( t_n \) is the \((C,1)\)mean of the sequence \( (na_n) \). Then the series \( \sum a_n \lambda_n \) is summable \( \lim_{N\to\infty} |N, P_n|^k \), \( k \geq 1 \).

Very recently, Misra et al [5] established the following theorem:

2.4. THEOREM:

Let \( f = (f_n) = (n^{\alpha}(\log n)^\beta) \) be a sequence and \((X_n)\) be a quasi- \( f \)-power sequence. Let \((\lambda_n)\) a sequence of constants such that

\[ \lambda_n \to 0, \text{ as } n \to \infty, \quad (2.4.1) \]
\[ \sum_{n=1}^{\infty} nX_n |\Delta^2 \lambda_n| < \infty, \quad (2.4.2) \]
\[ |\lambda_n| X_n = O(1), \quad (2.4.3) \]
\[ \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k-1} \frac{1}{P_{n-1}} = O \left( \frac{P_m}{P_m} \right)^{\delta k-1}, \quad (2.4.4) \]
\[ \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k-1} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m), \quad (2.4.5) \]
\[ \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m). \quad (2.4.6) \]

Then the series \( \sum a_n \lambda_n \) is summable \( \lim_{N\to\infty} |N, P_n|^k \), \( k \geq 1, \delta \geq 0 \).

In what follows in this paper, using \( \delta \)-quasi monotone sequence, we prove the following theorem.
3. MAIN THEOREM

Let \( f = (f_n) = (n^\beta \log n)^\gamma \) be a sequence and \((X_n)\) be a quasi-\( f \)-power sequence. Suppose also that there exists a sequence of numbers \((A_n)\) such that it is \( \delta \) – quasi – monotone with

\[
\sum n \delta_n X_n < \infty \quad \text{for all } n.
\]

(3.1)

\[
\Delta A_n \leq \delta_n \quad \text{for all } n.
\]

(3.2)

Let \((\lambda_n)\) a sequence of constants such that

\[
\lambda_n \to 0, \text{ as } n \to \infty,
\]

(3.3)

\[
|\lambda_n X_n| = O(1),
\]

(3.4)

and

\[
|\Delta \lambda_n| \leq |A_n| \quad \text{for all } n.
\]

(3.5)

Then the series \( \sum a_n \lambda_n \) is summable \(N, p_n; \sigma, k \geq 1, \sigma \geq 0\). If

\[
\sum_{n=1}^m \frac{P_n}{P_{n-1}} = O\left(\frac{P_m}{P_1}\right)
\]

(3.6)

\[
\sum_{n=1}^m \frac{P_n}{P_{n-1}} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m),
\]

(3.7)

\[
\sum_{n=1}^m \frac{P_n}{P_{n-1}} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m),
\]

(3.8)

where \( (t_n) \) is the \( n \)th \((C,1)\) mean of the sequence \((a_n)\).

In order to prove the theorem we require the following lemma.

4. LEMMA:

Let \( f = (f_n) = (n^\beta \log n)^\gamma \), \( \delta \leq \beta < 1, \gamma \geq 0 \) be a sequence and \((X_n)\) be a quasi -\( f \) – power increasing sequence. Let \((A_n)\) be a sequence of numbers such that it is \( \delta \) – quasi – monotone satisfying (3.1) and (3.2), then

\[
h X_n |A_n| = O(1)
\]

(4.1)

and

\[
\sum_{n=1}^m X_n |A_n| < \infty, \text{ as } m \to \infty.
\]

(4.2)
4.1. PROOF OF THE LEMMA:

As $A_n \to 0$ and $n^\beta (\log n)^\gamma X_n$ is non-decreasing, we have

$$nX_n |A_n| = n^{1-\beta} (\log n)^\gamma (n^\beta (\log n)^\gamma X_n) \sum_{v=1}^\infty \Delta |A_v|$$

$$= O(1)n^{1-\beta} (\log n)^\gamma n \sum_{v=1}^\infty v^\beta (\log v)^\gamma X_v |\Delta A_v|$$

$$= O(1) \sum_{v=1}^\infty v^{1-\beta} (\log v)^\gamma v^\beta (\log v)^\gamma X_v |\Delta A_v|$$

$$= O(1) \sum_{v=1}^\infty v X_v |\Delta A_v|.$$ 

$$= O(1) \sum_{v=1}^\infty v X_v |\delta_v|$$

This establishes (4.1).

Next

$$\sum_{n=1} ^\infty X_n |A_n| = \sum_{n=1} ^\infty \left( \sum_{r=1} ^n X_r \right) |\Delta |A_n| + \Delta |A_n| \left( \sum_{r=1} ^n X_r \right)$$

$$= O(1) \sum_{n=1} ^\infty \left( \sum_{r=1} ^n r^{-\beta} (\log r)^{-\gamma} r^\beta (\log r)^\gamma X_r \right) |\Delta |A_n|$$

$$+ O(1) \left( \sum_{n=1} ^\infty r^{-\beta} (\log r)^{-\gamma} r^\beta (\log r)^\gamma X_n \right) |\Delta |A_m|$$

$$= O(1) \sum_{n=1} ^\infty n^{-\beta} (\log n)^\gamma X_n |\Delta |A_n| \sum_{r=1} ^n r^{-\beta-\varepsilon} (\log r)^{-\gamma} r^\varepsilon$$

$$+ O(1) m^\delta X_m |A_m| \sum_{r=1} ^m r^{-\beta-\varepsilon}$$

$$= O(1) \sum_{n=1} ^m h^{\delta \varepsilon} X_n |\Delta |A_n| + O(1) \sum_{r=1} ^n \left( n^{\delta \varepsilon} - \int_1^n u^{\delta \varepsilon} du \right) + O(1) m^{\delta \varepsilon - \varepsilon} X_m |A_m|$$

$$= O(1) \sum_{r=1} ^m \left( P_n - P_{r-1} \right) a_r \lambda_r$$

This establishes (4.2).

5. PROOF OF THE THEOREM:

Let $(T_n)$ be the sequence of mean of the series $\sum_{n=1} ^\infty a_n \lambda_n$, then

$$T_n = \frac{1}{P_n} \sum_{r=0} ^n p_r \sum_{r=0} ^n a_r \lambda_r$$

$$= \frac{1}{P_n} \sum_{r=0} ^n \left( P_n - P_{r-1} \right) a_r \lambda_r$$
Hence for \( n \geq 1 \)

\[
T_n - T_{n-1} = \frac{p_n}{p_n p_{n-1}} \sum_{i=1}^{n} p_{i-1} a_i \lambda_i
\]

\[
= \frac{p_n}{p_n p_{n-1}} \sum_{i=1}^{n} v_{i-1} \left( \frac{1}{v} \right) p_{i-1} \lambda_i
\]

\[
= \frac{(n+1) p_n}{n} \sum_{i=1}^{n-1} p_{i-1} a_i \lambda_i + \frac{p_n}{p_n p_{n-1}} \sum_{i=1}^{n-1} p_{i-1} t_i \lambda_i \frac{v+1}{v} + \frac{p_n}{p_n p_{n-1}} \sum_{i=1}^{n-1} p_i t_i \frac{v+1}{v} \Delta \lambda_i
\]

\[
+ \frac{p_n}{p_n p_{n-1}} \sum_{i=1}^{n-1} p_i t_i \frac{\lambda_{i+1}}{v}
\]

\[= T_{n1} + T_{n2} + T_{n3} + T_{n4} \text{(say)}.\]

In order to prove the theorem, using Minkowski's inequality it is enough to show that

\[
\sum_{k=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \left| T_{n j} \right| < \infty \quad , j = 1, 2, 3, 4.
\]

Applying Hölder's inequality, we have

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \left| T_{n j} \right|^{k} = \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \frac{n+1}{n} \frac{p_n}{p_n} t_i \lambda_i
\]

\[
= O \left( \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \frac{|x_n|}{X_n^{k-1}} \lambda_i \right)
\]

\[
= O \left( \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \frac{|x_n|}{X_n^{k-1}} \lambda_i \right)
\]

\[
= O \left( \sum_{n=1}^{\infty} \frac{|x_n|}{X_n^{k-1}} \lambda_i \right) + O \left( \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \frac{|x_n|}{X_n^{k-1}} \lambda_i \right)
\]

\[
\leq O \left( \sum_{n=1}^{\infty} |x_n| A_n + O(1) |x_n| \lambda_i \right)
\]

Next

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \left| T_{n2} \right|^{k} = \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \frac{p_n}{p_n p_{n-1}} \sum_{i=1}^{n-1} p_{i-1} t_i \lambda_i \frac{v+1}{v}
\]

\[
= O \left( \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \frac{1}{p_{n-1}} \sum_{i=1}^{n-1} p_{i-1} t_i |\lambda_i| \left( \frac{p_n}{p_{n-1}} \right)^{\alpha k} \right)
\]

\[
= O \left( \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \frac{1}{p_{n-1}} \sum_{i=1}^{n-1} p_{i-1} t_i \frac{|\lambda_i|}{\left( \frac{p_n}{p_{n-1}} \right)^{\alpha k}} \right)
\]

\[
= O \left( \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \frac{1}{p_{n-1}} \sum_{i=1}^{n-1} p_{i-1} t_i \frac{|\lambda_i|}{\left( \frac{p_n}{p_{n-1}} \right)^{\alpha k}} \right)
\]

Next

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \left| T_{n3} \right|^{k} = \sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{\alpha k} \frac{p_n}{p_n p_{n-1}} \sum_{i=1}^{n-1} p_i t_i \frac{v+1}{v} \Delta \lambda_i
\]
Finally

This completes the proof of the theorem.
REFERENCES


[3]. H.Bor and L. Debnath, “Quasi-β-power increasing sequences”, International Journal of


[7]. W.T. Sulaiman, “A recent note on absolute absolute Riesz summability factors of an infinite