

INDEX SUMMABILITY OF AN INFINITE SERIES USING δ -QUASI MONOTONE SEQUENCE

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ABSTRACT: A result concerning absolute indexed Summability factor of an infinite series using δ - Quasi monotone sequence has been established.

1. INTRODUCTION:

A positive sequence (a_n) is said to be almost increasing if there exists a positive sequence (b_n) and two positive constants A and B such that

$$Ab_n \leq a_n \leq Bb_n, \text{ for all } n. \quad (1.1)$$

The sequence (a_n) is said to be quasi- β -power increasing, if there exists a constant K depending upon β with $K \geq 1$ such that

$$Kn^\beta a_n \geq m^\beta a_m, \quad (1.2)$$

for all $n \geq m$. In particular, if $\beta = 0$, then (a_n) is said to be quasi-increasing sequence. It is clear that every almost increasing sequence is a quasi- β -power increasing sequence for any non-negative β . But the converse is not true as $(n^{-\beta})$ is quasi- β -power increasing but not almost increasing.

Let $f = (f_n)$ be a positive sequence of numbers. Then the positive sequence (a_n) is said to be quasi- f -power increasing, if there exists a constant K depending upon f with $K \geq 1$ such that

$$Kf_n a_n \geq f_m a_m, \quad (1.3)$$

for $n \geq m \geq 1$ ([6]). Clearly, if (a_n) is a quasi- f -power increasing sequence, then the $(a_n f_n)$ is a quasi-increasing sequence.

Let $\delta = (\delta_n)$ be a positive sequence of numbers. Then a positive sequence (a_n) is said to be δ -quasi monotone [1], if $a_n \rightarrow 0$, $a_n > 0$ ultimately and $\Delta d_n \geq -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$.

Let $\sum a_n$ be an infinite series with sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty, \text{ as } n \rightarrow \infty$$

Then the sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v, P_n \neq 0, \quad (1.4)$$

defines the (\overline{N}, p_n) mean of the sequence (s_n) generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $\left| \overline{N}, p_n \right|_k, k \geq 1$ [2], if

$$\sum_{n=1}^{\infty} \left(\frac{p_n}{P_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty. \quad (1.5)$$

The series $\sum a_n$ is said to be summable $\left| \overline{N}, p_n; \delta \right|_k, k \geq 1, \delta \geq 0$, if

$$\sum_{n=1}^{\infty} \left(\frac{p_n}{P_n} \right)^{\delta k + k - 1} |T_n - T_{n-1}|^k < \infty. \quad (1.6)$$

2. KNOWN THEOREMS:

Dealing with quasi- β -power increasing sequence Bor and Debnath [3] have established the following theorem:

2.1. THEOREM:

Let (X_n) be a quasi- β -power increasing sequence for $0 < \beta < 1$ and (λ_n) be a real sequence. If the conditions

$$\sum_{n=1}^m \frac{P_n}{n} = O(P_m), \quad (2.1.1)$$

$$\lambda_n X_n = O(1), \quad (2.1.2)$$

$$\sum_{n=1}^m \frac{|t_n|^k}{n} = O(X_m), \tag{2.1.3}$$

$$\sum_{n=1}^m \frac{p_n |t_n|^k}{P_n} = O(X_m) \tag{2.1.4}$$

and

$$\sum_{n=1}^m nX_n |\Delta^2 \lambda_n| < \infty \tag{2.1.5}$$

are satisfied, where t_n is the $(C,1)$ mean of the sequence (na_n) . Then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k, k \geq 1$

Subsequently Leindler [4] established a similar result reducing certain condition of Bor. He established:

2.2. THEOREM:

Let the sequence (X_n) be a quasi- β -power increasing sequence for $0 < \beta < 1$, and the real sequence (λ_n) satisfies the conditions

$$\sum_{n=1}^m \lambda_n = O(m) \tag{2.2.2}$$

and

$$\sum_{n=1}^m |\Delta \lambda_n| = O(m). \tag{2.2.3}$$

hold, where $X_n(\beta) = \max(n^\beta X_n, \log n)$. Then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k, k \geq 1$.

Recently, extending the above results to quasi- f -power increasing sequence, Sulaiman [7] have established the following theorem:

2.3. THEOREM:

Let $f = (f_n) = (n^\beta \log^\gamma n), 0 \leq \beta < 1, \gamma \geq 0$ be a sequence. Let (X_n) be a quasi- f -power sequence and (λ_n) a sequence of constants satisfying the conditions

$$\lambda_n \rightarrow 0 \text{ as } n \rightarrow \infty, \tag{2.3.1}$$

$$\sum_{n=1}^{\infty} nX_n |\Delta \lambda_n| < \infty, \tag{2.3.2}$$

$$|\lambda_n|X_n = O(1), \quad (2.3.3)$$

$$\sum_{n=1}^{\infty} \frac{1}{nX_n^{k-1}} |t_n|^k = O(X_m) \quad (2.3.4)$$

and

$$\sum_{n=1}^{\infty} \frac{P_n}{P_n} \frac{1}{X_n^{k-1}} |t_n|^k = O(X_m), \quad (2.3.5)$$

where t_n is the $(C,1)$ mean of the sequence (na_n) . Then the series $\sum a_n \lambda_n$ is summable $\left[\overline{N}, P_n \right]_k, k \geq 1$.

Very recently, Misra et al [5] established the following theorem:

2.4. THEOREM:

Let $f = (f_n) = (n^\beta (\log n)^\gamma)$ be a sequence and (X_n) be a quasi- f -power sequence. Let (λ_n) a sequence of constants such that

$$\lambda_n \rightarrow 0, \text{ as } n \rightarrow \infty, \quad (2.4.1)$$

$$\sum_{n=1}^{\infty} nX_n |\Delta \lambda_n| < \infty, \quad (2.4.2)$$

$$|\lambda_n|X_n = O(1), \quad (2.4.3)$$

$$\sum_{n=v+1}^m \left(\frac{P_n}{P_n} \right)^{\delta k-1} \frac{1}{P_{n-1}} = O \left(\frac{P_m}{P_m} \right)^{\delta k-1}, \quad (2.4.4)$$

$$\sum_{n=1}^m \left(\frac{P_n}{P_n} \right)^{\delta k-1} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m), \quad (2.4.5)$$

$$\sum_{n=1}^m \left(\frac{P_n}{P_n} \right)^{\delta k} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m). \quad (2.4.6)$$

Then the series $\sum a_n \lambda_n$ is summable $\left[\overline{N}, P_n; \delta \right]_k, k \geq 1, \delta \geq 0$.

In what follows in this paper, using δ -quasi monotone sequence, we prove the following theorem.

3. MAIN THEOREM

Let $f = (f_n) = (n^\beta (\log n)^\gamma)$ be a sequence and (X_n) be a quasi- f -power sequence. Suppose also that there exists a sequence of numbers (A_n) such that it is δ -quasi-monotone with

$$\sum n \delta_n X_n < \infty \tag{3.1}$$

$$\Delta A_n \leq \delta_n \text{ for all } n. \tag{3.2}$$

Let (λ_n) a sequence of constants such that

$$\lambda_n \rightarrow 0, \text{ as } n \rightarrow \infty, \tag{3.3}$$

$$|\lambda_n| X_n = O(1), \tag{3.4}$$

and

$$|\Delta \lambda_n| \leq |A_n| \text{ for all } n. \tag{3.5}$$

Then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n; \sigma|_k, k \geq 1, \sigma \geq 0.$, if

$$\sum_{n=v+1}^m \left(\frac{P_n}{P_v}\right)^{\sigma k-1} \frac{1}{P_{n-1}} = O\left(\frac{P_v}{P_v}\right)^{\sigma k-1}, \tag{3.6}$$

$$\sum_{n=1}^m \left(\frac{P_n}{P_n}\right)^{\sigma k-1} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m), \tag{3.7}$$

$$\sum_{n=1}^m \left(\frac{P_n}{P_n}\right)^{\sigma k} \frac{|t_n|^k}{n X_n^{k-1}} = O(X_m), \tag{3.8}$$

where (t_n) is the n th $(C,1)$ mean of the sequence $(n a_n)$.

In order to prove the theorem we require the following lemma.

4. LEMMA:

Let $f = (f_n) = (n^\beta (\log n)^\gamma), 0 \leq \beta < 1, \gamma \geq 0$ be a sequence and (X_n) be a quasi- f -power increasing sequence. Let (A_n) be a sequence of numbers such that it is δ -quasi-monotone satisfying (3.1) and (3.2). then

$$n X_n |A_n| = O(1) \tag{4.1}$$

and

$$\sum_{n=1}^m X_n |A_n| < \infty, \text{ as } m \rightarrow \infty. \tag{4.2}$$

4.1. PROOF OF THE LEMMA:

As $A_n \rightarrow 0$ and $n^\beta (\log n)^\gamma X_n$ is non-decreasing, we have

$$\begin{aligned}
 n X_n |A_n| &= n^{1-\beta} (\log n)^{-\gamma} \left(n^\beta (\log n)^\gamma X_n \right) \sum_{\nu=n}^{\infty} \Delta |A_\nu| \\
 &= O(1) n^{1-\beta} (\log n)^\gamma n \sum_{\nu=n}^{\infty} \nu^\beta (\log \nu)^\gamma X_\nu |\Delta A_\nu| \\
 &= O(1) \sum_{\nu=n}^{\infty} \nu^{1-\beta} (\log \nu)^{-\gamma} \nu^\beta (\log \nu)^\gamma X_\nu |\Delta A_\nu| \\
 &= O(1) \sum_{\nu=n}^{\infty} \nu X_\nu |\Delta A_\nu|. \\
 &= O(1) \sum_{\nu=n}^{\infty} \nu X_\nu |\delta_\nu| \\
 &= O(1)
 \end{aligned}$$

This establishes (4.1).

Next

$$\begin{aligned}
 \sum_{n=1}^m X_n |A_n| &= \sum_{n=1}^{m-1} \left(\sum_{r=1}^n X_r \right) \Delta |A_n| + |A_m| \left(\sum_{r=1}^m X_r \right) \\
 &= O(1) \sum_{n=1}^{m-1} \left(\sum_{r=1}^n r^{-\beta} (\log r)^{-\gamma} r^\beta (\log r)^\gamma X_r \right) \Delta |A_n| \\
 &\quad + O(1) \left(\sum_{r=1}^m r^{-\beta} (\log r)^{-\gamma} r^\beta (\log r)^\gamma X_r \right) |A_m| \\
 &= O(1) \sum_{n=1}^{m-1} \left(n^\beta (\log n)^\gamma X_n \right) \Delta |A_n| \sum_{r=1}^n r^{-\beta-\varepsilon} (\log r)^{-\gamma} r^\varepsilon \\
 &\quad + O(1) m^\beta X_m |A_m| (\log m)^\gamma \sum_{r=1}^m r^{-\beta-\varepsilon} (\log r)^{-\gamma} r^\varepsilon, \varepsilon < 1 - \beta. \\
 &= O(1) \sum_{n=1}^{m-1} \left(n^\beta (\log n)^\gamma X_n \right) \Delta |A_n| n^\varepsilon (\log n)^{-\gamma} \sum_{r=1}^n r^{-\beta-\varepsilon} \\
 &\quad + O(1) m^\beta X_m |A_m| (\log m)^\gamma m^\varepsilon (\log m)^{-\gamma} \sum_{r=1}^m r^{-\beta-\varepsilon} \\
 &= O(1) \sum_{n=1}^m n^{\beta+\varepsilon} X_n \Delta |A_n| \left(\int_1^n u^{-\beta-\varepsilon} du \right) + O(1) m^{\beta+\varepsilon} X_m |A_m| \left(\int_1^m u^{-\beta-\varepsilon} du \right) \\
 &= O(1) \sum_{n=1}^m n X_n \Delta |A_n| + O(1) m X_m |A_m| \\
 &= O(1).
 \end{aligned}$$

This establishes (4.2).

5. PROOF OF THE THEOREM:

Let (T_n) be the sequence of (\bar{N}, p_n) mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n$, then

$$\begin{aligned}
 T_n &= \frac{1}{P_n} \sum_{\nu=0}^n p_\nu \sum_{r=0}^{\nu} a_r \lambda_r \\
 &= \frac{1}{P_n} \sum_{\nu=0}^n (P_n - P_{\nu-1}) a_\nu \lambda_\nu
 \end{aligned}$$

Hence for $n \geq 1$

$$\begin{aligned} T_n - T_{n-1} &= \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^n P_{\nu-1} a_\nu \lambda_\nu \\ &= \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^n \nu a_\nu \left(\frac{1}{\nu} P_{\nu-1} \lambda_\nu \right) \\ &= \frac{(n+1)}{n} \frac{P_n}{P_n} t_n \lambda_n + \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu-1} t_\nu \lambda_\nu \frac{\nu+1}{\nu} + \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu t_\nu \frac{\nu+1}{\nu} \Delta \lambda_\nu \\ &\quad + \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu t_\nu \frac{\lambda_{\nu+1}}{\nu} \\ &= T_{n1} + T_{n2} + T_{n3} + T_{n4} \text{ (say).} \end{aligned}$$

In order to prove the theorem, using Minkowski's inequality it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{ck+k-1} |T_{nj}| < \infty, \quad j = 1, 2, 3, 4.$$

Applying Hölder's inequality, we have

$$\begin{aligned} \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck+k-1} |T_{n1}|^k &= \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck+k-1} \left| \frac{n+1}{n} \frac{P_n}{P_n} t_n \lambda_n \right|^k \\ &= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck-1} \frac{|t_n|^k}{X_n^{k-1}} (X_n |\lambda_n|)^{k-1} |\lambda_n| \\ &= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck-1} \frac{|t_n|^k}{X_n^{k-1}} |\lambda_n| \\ &= O(1) \sum_{n=1}^{m-1} \left(\sum_{\nu=1}^n \left(\frac{P_\nu}{p_\nu} \right)^{ck-1} \frac{|t_\nu|^k}{X_\nu^{k-1}} \right) \Delta |\lambda_n| + O(1) \sum_{\nu=1}^m \left(\frac{P_\nu}{p_\nu} \right)^{ck-1} \frac{|t_\nu|^k}{X_\nu^{k-1}} |\lambda_m| \\ &\leq O(1) \sum_{n=1}^{m-1} X_n |A_n| + O(1) X_m |\lambda_m| \\ &= O(1). \end{aligned}$$

Next

$$\begin{aligned} \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck+k-1} |T_{n2}|^k &= \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck+k-1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu-1} t_\nu \lambda_\nu \frac{\nu+1}{\nu} \right|^k \\ &= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck-1} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu-1} |t_\nu|^k |\lambda_\nu|^k \left(\sum_{\nu=1}^{n-1} \frac{P_\nu}{P_{n-1}} \right)^{k-1} \\ &= O(1) \sum_{\nu=1}^m P_{\nu-1} |t_\nu|^k |\lambda_\nu|^k \sum_{n=\nu+1}^m \left(\frac{P_n}{p_n} \right)^{ck-1} \frac{1}{P_{n-1}} \\ &= O(1) \sum_{\nu=1}^m \left(\frac{P_\nu}{p_\nu} \right)^{ck-1} |t_\nu|^k |\lambda_\nu|^k \\ &= O(1), \text{ as in the case of } T_{n1}. \end{aligned}$$

Next

$$\sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck+k-1} |T_{n3}|^k = \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck+k-1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu t_\nu \frac{\nu+1}{\nu} \Delta \lambda_\nu \right|^k$$

$$\begin{aligned}
&= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck-1} \frac{1}{P_{n-1}^k} \sum_{v=1}^{n-1} P_v^k \frac{|t_v|^k}{X_v^{k-1}} |\Delta \lambda_v| \left(\sum_{v=1}^{n-1} X_v |\Delta \lambda_v| \right)^{k-1} \\
&\leq O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck-1} \frac{1}{P_{n-1}^k} \sum_{v=1}^{n-1} P_v^k \frac{|t_v|^k}{X_v^{k-1}} |A_v| \left(\sum_{v=1}^{n-1} X_v |A_v| \right)^{k-1} \\
&= O(1) \sum_{v=1}^m P_v^k \frac{|t_v|^k}{X_v^{k-1}} |A_v| \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n} \right)^{ck-1} \frac{1}{P_{n-1}^k} \\
&= O(1) \sum_{v=1}^m \left(\frac{P_v}{p_v} \right)^{ck} \frac{1}{v} \frac{|t_v|^k}{X_v^{k-1}} (v |A_v|) \\
&= O(1) \sum_{v=1}^{m-1} \sum_{r=1}^v \left(\frac{P_r}{p_r} \right)^{ck} \frac{1}{r} \frac{|t_r|^k}{X_r^{k-1}} \Delta(v |A_v|) + O(m |A_m|) \sum_{r=1}^m \left(\frac{P_r}{p_r} \right)^{ck} \frac{1}{r} \frac{|t_r|^k}{X_r^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} (-|A_v| + (v+1) |\Delta| |A_v|) X_v + O(m X_m |A_m|). \\
&= O(1) \sum_{v=1}^n X_v |A_v| + O(1) \sum_{v=1}^n v X_v |\Delta A_v| + O(m X_m |A_m|) \\
&\leq O(1) \sum_{v=1}^n X_v |A_v| + O(1) \sum_{v=1}^n v X_v \delta_v + O(m X_m |A_m|) \\
&= O(1).
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck+k-1} |T_{n4}|^k &= \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck+k-1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \frac{\lambda_{v+1}}{v} \right|^k \\
&= O(1) \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{ck-1} \frac{1}{P_{n-1}^k} \sum_{v=1}^{n-1} \frac{P_v}{v} |t_v|^k |\lambda_v|^k \left(\sum_{v=1}^{n-1} \frac{P_v}{v} \right)^{k-1} \\
&= O(1) \sum_{v=1}^m \frac{P_v}{v} |t_v|^k |\lambda_v|^k \sum_{n=v+1}^m \left(\frac{P_n}{p_n} \right)^{ck-1} \frac{1}{P_{n-1}^k} \\
&= O(1) \sum_{v=1}^m \left(\frac{P_v}{p_v} \right)^{ck} \frac{|t_v|^k}{X_v^{k-1}} (X_v |\lambda_v|)^{k-1} |\lambda_v| \\
&= O(1) \sum_{v=1}^m \left(\frac{P_v}{p_v} \right)^{ck} \frac{|t_v|^k |\lambda_v|}{v X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^v \left(\frac{P_r}{p_r} \right)^{ck} \frac{|t_r|^k}{r X_r^{k-1}} \right) |\Delta \lambda_v| + O(1) \sum_{r=1}^m \left(\frac{P_r}{p_r} \right)^{ck} \frac{|t_r|^k}{r X_r^{k-1}} \\
&\leq O(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^v \left(\frac{P_r}{p_r} \right)^{ck} \frac{|t_r|^k}{r X_r^{k-1}} \right) |A_v| + O(1) \sum_{r=1}^m \left(\frac{P_r}{p_r} \right)^{ck} \frac{|t_r|^k}{r X_r^{k-1}} \\
&= O(1) \sum_{v=1}^m X_v |A_v| + O(1) X_m |\lambda_m| \\
&= O(1).
\end{aligned}$$

This completes the proof of the theorem.

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