On some Multidimensional Fractional Integral Operators Involving Multivariable I-Function

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Abstract. In the present paper, we study certain multidimensional fractional integral operators involving a general I-function in their kernel. We give five basic properties of these operators, and then establish two theorems and two corollaries, which are believed to be new. These basic theorems exhibit structural relationships between the multidimensional integral transforms. The one- and two-dimensional analogues of these results, which are new and of interest in themselves, can easily be deduced. Special cases of these latter theorems will give rise to certain known results obtained from time to time by several earlier authors.

Introduction

Fractional integral operators have been defined and studied by various authors notably by Riemann and Liouville [4], Weyl [4], Erdelyi [1,2], Kober [7], Sneddon [15], Kalla [10], Saxena [13], Srivastava and Buschmann [16] etc. These operators play an important role in the theory of integral equations and problems concerning Mathematical Physics. In this paper, we shall study the following multidimensional fractional integral operators having the general function N as their kernel. Also, for the sake of brevity, we shall use the symbol $f(x)$ to represent $f(x_1, x_2, \dots, x_r)$. The multivariable I-function introduced by Prasad [12] will be define and represent it in the following manner:

$$I[z_1, \dots, z_r] = I_{p_2, q_2, \dots, p_r, q_r; (p', q') \dots (p^{(r)}, q^{(r)})}^{0, n_2, \dots, 0, n_r; (m', n') \dots (m^{(r)}, n^{(r)})} \left[\begin{matrix} z_1, \dots, z_r \\ (a_{2j}, \alpha_{2j}, \alpha_{2j}^*)_{1, p_2} \dots (\alpha_{rj}, \alpha_{rj}^*, \alpha_{rj}^{(r)})_{1, p_r} : (a_j, \alpha_j^*)_{1, p} \dots (a_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}} \\ (b_{2j}, \beta_{2j}, \beta_{2j}^*)_{1, q_2} \dots (b_{rj}, \beta_{rj}^*, \beta_{rj}^{(r)})_{1, q_r} : (b_j, \beta_j^*)_{1, q} \dots (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}} \end{matrix} \right] \\ = \frac{1}{(2\pi w)^r} \int_{L_1} \dots \int_{L_r} \phi_1(s_1) \dots \phi_r(s_r) \psi(s_1, \dots, s_r) z_1^{s_1} \dots z_r^{s_r} ds_1 \dots ds_r \quad (1)$$

Where

$$w = \sqrt{-1} \\ \phi_i(s_i) = \frac{\prod_{j=1}^{m^{(i)}} \Gamma(b_j^{(i)} - \beta_j^{(i)} s_i) \prod_{j=1}^{n^{(i)}} \Gamma(1 - a_j^{(i)} + \alpha_j^{(i)} s_i)}{\prod_{j=m^{(i)}+1}^{q^{(i)}} \Gamma(1 - b_j^{(i)} + \beta_j^{(i)} s_i) \prod_{j=n^{(i)}+1}^{p^{(i)}} \Gamma(a_j^{(i)} - \alpha_j^{(i)} s_i)} \quad \forall i \in (1, 2, \dots, r) \quad (1.2)$$

$$\psi(s_1, \dots, s_r) = \frac{\prod_{j=1}^{n_2} \Gamma\left(1 - a_{2j} + \sum_{i=1}^2 \alpha_{2j}^{(i)} s_i\right) \prod_{j=1}^{n_3} \Gamma\left(1 - a_{3j} + \sum_{i=1}^3 \alpha_{3j}^{(i)} s_i\right)}{\prod_{j=n_2+1}^{p_2} \Gamma\left(a_{2j} - \sum_{i=1}^2 \alpha_{2j}^{(i)} s_i\right) \prod_{j=n_3+1}^{p_3} \Gamma\left(a_{3j} - \sum_{i=1}^3 \alpha_{3j}^{(i)} s_i\right)} \dots \prod_{j=1}^{n_r} \Gamma\left(1 - a_{rj} + \sum_{i=1}^r \alpha_{rj}^{(i)} s_i\right) \dots \prod_{j=n_r+1}^{p_r} \Gamma\left(a_{rj} - \sum_{i=1}^r \alpha_{rj}^{(i)} s_i\right) \prod_{j=1}^{q_2} \Gamma\left(1 - b_{2j} - \sum_{i=1}^2 \beta_{2j}^{(i)} s_i\right) \dots \prod_{j=1}^{q_r} \Gamma\left(1 - b_{rj} - \sum_{i=1}^r \beta_{rj}^{(i)} s_i\right) \quad (1.3)$$

$\alpha_j^{(i)}, \beta_j^{(i)}, \alpha_{kj}^{(i)}, \beta_{kj}^{(i)} (i=1, \dots, r) (k=1, \dots, r)$ are positive numbers, $a_j^{(i)}, b_j^{(i)} (i=1, \dots, r), a_{kj}, b_{kj} (k=2, \dots, r)$ are complex numbers and here $m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)} (i=1, \dots, r), n_k, p_k, q_k (k=2, \dots, r)$ are non-negative integers where $0 \leq m^{(i)} \leq q^{(i)}, 0 \leq n^{(i)} \leq p^{(i)}, q^k \geq 0, 0 \leq n_k \leq p_k$. Here (i) denotes the numbers of dashes. The contours L_i in the complex s_i -plane is of the Mellin-Barnes type which runs from $-w \infty$ to $+w \infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma(b_j^{(i)} - \beta_j^{(i)} s_i) (j=1, \dots, m^{(i)})$ are separated from those of $\Gamma\left(1 - a_{rj} + \sum_{i=1}^r \alpha_{rj}^{(i)} s_i\right) (j=1, \dots, n_r)$.

For further details and asymptotic expansion of the I -function one can refer by Prasad [12].

In what follows, the multivariable I -function defined by [12] will be represented in the contracted notation:

$$I_{p_2, q_2, \dots, p_r, q_r; (m', n') : \dots : (m^{(r)}, n^{(r)})}^{0, n_2 : \dots : 0, n_r; (m', n') : \dots : (m^{(r)}, n^{(r)})} [Z_1, \dots, Z_r]$$

Or simply by $I[z_1, \dots, z_r]$

We introduce the fractional integration operators by means of the following integral equations:

$$Y\{f(x)\} = Y\{f(x); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r\} = \prod_{i=1}^r (t_i^{\gamma_i-1}) \int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r \left\{ (x_i)^{\gamma_i} \phi\left(\frac{x_1}{t_1}, \dots, \frac{x_r}{t_r}\right) \right\} f(x) dx_1 \dots dx_r \quad (1.4)$$

And

$$N\{f(x)\} = N\{f(x); t_1, \dots, t_r; \delta_1, \dots, \delta_r\} = \prod_{i=1}^r (t_i^{\delta_i}) \int_{t_1}^{\infty} \dots \int_{t_r}^{\infty} \prod_{i=1}^r \left\{ (x_i)^{-\delta_i-1} \phi\left(\frac{t_1}{x_1}, \dots, \frac{t_r}{x_r}\right) \right\} f(x) dx_1 \dots dx_r \quad (1.5)$$

Where the kernel ϕ is such that the above integrals make sense. The above operators exist under the following conditions:

- (i) $p_i \leq 1, q_i < \infty, \frac{1}{p_i} + \frac{1}{q_i} = 1, i=1, 2, \dots, r$
- (ii) $\text{Re}(\gamma_i) > -\frac{1}{q_i}; \text{Re}(\delta_i) > -\frac{1}{p_i}; i=1, 2, \dots, r$
- (iii) $f(x) \in L_{p_i}((0, \infty), \dots, (0, \infty)); i=1, 2, \dots, r$
- (iv) $0 \leq m^{(i)} \leq q^{(i)}, 0 \leq n^{(i)} \leq p^{(i)}, q^k \geq 0, 0 \leq n_k \leq p_k$

The following special case of the multidimensional fractional integral operators involving product of Gauss's hypergeometric functions ([14], p.153, eq. (i) and (ii)) will be used in Section 3.

$$I\{f(x)\} = \prod_{i=1}^r \left\{ \frac{t_i^{\gamma_i-1}}{\Gamma(1-a_i)} \right\} \int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r \left\{ (x_i)^{\gamma_i} {}_2F_1\left(\alpha_i, \beta_i + m; \beta_i; \frac{x_i}{t_i}\right) \right\} f(x) dx_1 \dots dx_r \quad (1.6)$$

And

$$K\{f(x)\} = \prod_{i=1}^r \left\{ \frac{t_i^{\delta_i}}{\Gamma(1-\alpha_i)} \right\} \int_{t_1}^{\infty} \dots \int_{t_r}^{\infty} \prod_{i=1}^r \left\{ (x_i)^{-\delta_i-1} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta_i; \frac{t_i}{x_i} \right) \right\} f(x) dx_1 \dots dx_r \quad (1.7)$$

The conditions of existence of these operators follow easily from the conditions given in the paper referred to above.

The generalized multidimensional integral transform T , defined below, will also be required during the course of our study:

$$T\{f(x); s_1, \dots, s_r\} = \int_0^{\infty} \dots \int_0^{\infty} k(s_1 x_1, \dots, s_r x_r) f(x) dx_1 \dots dx_r \quad (1.8)$$

Where $k(s_1 x_1, \dots, s_r x_r)$ is the kernel of the transform T and the multiple integral occurring in the equation (1.8) is assumed to be convergent.

The following multivariable I -function transform will also be used in the sequel:

$$I\{f(x); s_1, \dots, s_r\} = \int_0^{\infty} \dots \int_0^{\infty} I_{p_2, q_2, \dots, p_r, q_r; (p', q') \dots (p^{(r)}, q^{(r)})}^{0, n_2, \dots, 0, n_r; (m', n') \dots (m^{(r)}, n^{(r)})} \left[\begin{matrix} s_1 x_1 \\ \vdots \\ s_r x_r \end{matrix} \middle| \begin{matrix} (a_{2j}, \alpha'_{2j}, \alpha''_{2j})_{1, p_2} \dots (\alpha'_{rj}, \alpha''_{rj}, \alpha'''_{rj})_{1, p_r} : (a'_j, \alpha'_j)_{1, p} \dots (a_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}} \\ (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1, q_2} \dots (\beta'_{rj}, \beta''_{rj}, \beta'''_{rj})_{1, q_r} : (b'_j, \beta'_j)_{1, q} \dots (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}} \end{matrix} \right] f(x) dx_1 \dots dx_r \quad (1.9)$$

The transform defined above will be denoted symbolically as follows:

$$I \left\{ f(x); \begin{matrix} 0, n_2, \dots, 0, n_r; (m', n') \dots (m^{(r)}, n^{(r)}), \\ p_2, q_2, \dots, p_r, q_r; (p', q') \dots (p^{(r)}, q^{(r)}), \\ (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1, q_2} \dots (\beta'_{rj}, \beta''_{rj}, \beta'''_{rj})_{1, q_r} : (b'_j, \beta'_j)_{1, q} \dots (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}} \end{matrix} ; \begin{matrix} s_1 x_1 \\ \vdots \\ s_r x_r \end{matrix} \right\} \quad (1.10)$$

2. Some Basic Properties

Property 1. If $f(x) \in L_{p_i}((0, \infty), \dots, (0, \infty)); 1 \leq p_i \leq 2$ (or $f(x) \in M_{p_i}((0, \infty), \dots, (0, \infty)); p_i > 2$, where $M_{p_i}((0, \infty), \dots, (0, \infty))$ denotes the class of all functions $f(x) \in L_{p_i}((0, \infty), \dots, (0, \infty)); p_i > 2$, which are the inverse Mellin transforms of functions belonging to $L_{q_i}((-\infty, \infty), \dots, (-\infty, \infty)); \operatorname{Re}(\gamma_i) > \frac{1}{q_i}, \operatorname{Re}(\delta_i) > \frac{1}{p_i}, \frac{1}{p} + \frac{1}{q} = 1 (i = 1, 2, \dots, r)$ and the

multidimensional Mellin transform of the function $f(x)$ exists, then

$$(a) M[Y\{f(x); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r\}; s_1, \dots, s_r] = M[f(x); s_1, \dots, s_r] N\{1; t_1, \dots, t_r; \gamma_1 - s_1 + 1, \dots, \gamma_r - s_r + 1\} \quad (2.1)$$

$$(b) M[N\{f(x); t_1, \dots, t_r; \delta_1, \dots, \delta_r\}; s_1, \dots, s_r] = M[f(x); s_1, \dots, s_r] Y\{1; t_1, \dots, t_r; \delta_1 + s_1 + 1, \dots, \delta_r + s_r + 1\} \quad (2.2)$$

Where the symbol M occurring in (2.1) and (2.2) stands for the multidimensional Mellin transform defined in the following way:

$$M\{f(x); s_1, \dots, s_r\} = \int_0^{\infty} \dots \int_0^{\infty} f(x) \prod_{i=1}^r (x_i^{s_i-1}) dx_1 \dots dx_r \quad (2.3)$$

Provided that the multiple integral involved in (2.3) exists.

Proof: To prove (2.1), we use (2.3) and (1.4) to obtain

$$M[Y\{f(x)\}] = \int_0^{\infty} \dots \int_0^{\infty} (t_i^{s_i-1}) \left\{ \int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r (x_i^{\gamma_i}) f(x) \phi\left(\frac{x_1}{t_1}, \dots, \frac{x_r}{t_r}\right) \right\} dt_1 \dots dt_r \quad (2.4)$$

Now interchanging the orders of t_i and $x_i (i = 1, 2, \dots, r)$ integrals in (2.4), which is easily seen to be permissible under the conditions stated with (2.1) and interpreting the results thus obtained with the help of (1.5), we arrive at the required result (2.1).

The result (2.2) can be established in a similar manner.

Property 2. If $f(x) \in L_{p_i}((0, \infty), \dots, (0, \infty)); 1 \leq p_i \leq 2$, $g(x) \in L_{q_i}((0, \infty), \dots, (0, \infty));$,
 $\operatorname{Re}(\gamma_i) > \max\left(-\frac{1}{p_i}, -\frac{1}{q_i}\right), (i = 1, 2, \dots, r)$, then

$$\int_0^\infty \dots \int_0^\infty f(x) Y\{g(x)\} dx_1 \dots dx_r = \int_0^\infty \dots \int_0^\infty g(x) N\{f(x)\} dx_1 \dots dx_r \quad (2.5)$$

Provided that the multiple integrals involved in (2.5) are absolutely convergent.

Property 3.

$$(a) Y\{f(wx); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r\} = Y\{f(x); t_1 w_1, \dots, t_r w_r; \gamma_1, \dots, \gamma_r\} \quad (2.6)$$

$$(b) N\{f(wx); t_1, \dots, t_r; \delta_1, \dots, \delta_r\} = W\{f(x); t_1 w_1, \dots, t_r w_r; \delta_1, \dots, \delta_r\} \quad (2.7)$$

Property 4.

$$(a) Y\{f(1/x); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r\} = Y\{f(x); 1/t_1, \dots, 1/t_r; \gamma_1 + 1, \dots, \gamma_r + 1\} \quad (2.8)$$

$$(b) N\{f(1/x); t_1, \dots, t_r; \delta_1, \dots, \delta_r\} = W\{f(x); 1/t_1, \dots, 1/t_r; \delta_1 - 1, \dots, \delta_r - 1\} \quad (2.9)$$

Property 5.

$$(a) Y\left\{\prod_{i=1}^r (x_i^{\beta_i}) f(x); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r\right\} = \prod_{i=1}^r (t_i^{\beta_i}) Y\{f(x); t_1, \dots, t_r; \gamma_1 + \beta_1, \dots, \gamma_r + \beta_r\} \quad (2.10)$$

$$(b) N\left\{\prod_{i=1}^r (x_i^{\beta_i}) f(x); t_1, \dots, t_r; \delta_1, \dots, \delta_r\right\} = \prod_{i=1}^r (t_i^{\beta_i}) N\{f(x); t_1, \dots, t_r; \delta_1 - \beta_1, \dots, \delta_r - \beta_r\} \quad (2.11)$$

Provided that the multiple integrals involved in (2.10) and (2.11) are absolutely convergent.

The proofs of the properties 2 to 5 are straight forward and follow from the definitions (1.4) and (1.5) of the multidimensional fractional integral operators involved therein

3. Relationship Between Multidimensional Fractional Integral operators and Multidimensional Integral Transforms

In this section, we shall establish two most general theorems exhibiting interconnections between the fractional integral operators Y and N defined by (1.4) and (1.5) respectively and the integral transform T defined by (1.8). Next, we give two interesting corollaries interconnecting the multidimensional fractional integral operators defined by (1.6) and (1.7) and the multidimensional I -function transform defined by (1.9).

Theorem 1. If

$$\tau(s_1, \dots, s_r) = T\{\psi(u^\rho)g(u); s_1, \dots, s_r\} = \int_0^\infty \dots \int_0^\infty k(su)\psi(u^\rho)g(u)du_1 \dots du_r \quad (3.1)$$

And

$$\psi(t_1, \dots, t_r) = Y\{f(x^\sigma); t_1, \dots, t_r\} = \prod_{i=1}^r (x_i^{-\gamma_i-1}) \int_0^{t_1} \dots \int_0^{t_r} f(x^\sigma) \prod_{i=1}^r (x_i^{\gamma_i}) \phi\left(\frac{x_1}{t_1}, \dots, \frac{x_r}{t_r}\right) dx_1 \dots dx_r \quad (3.2)$$

Then

$$\tau(s_1, \dots, s_r) = \frac{1}{\rho_1 \dots \rho_r} \int_0^\infty \dots \int_0^\infty f(x^\sigma) \theta(s_1, \dots, s_r, x_1, \dots, x_r, \rho_1, \dots, \rho_r) dx_1 \dots dx_r \quad (3.3)$$

Where

$$\begin{aligned} &\theta(s_1, \dots, s_r, x_1, \dots, x_r, \rho_1, \dots, \rho_r) = \\ &Y \left\{ \prod_{i=1}^r \left(x_i^{\frac{1}{\rho_i}-1} \right) g \left(x^{\frac{1}{\rho}} \right) k \left(s x^{\frac{1}{\rho}} \right); t_1, \dots, t_r \right\} = \\ &\prod_{i=1}^r \left(t_i^{\gamma_i} \right) \int_{t_1}^{\infty} \dots \int_{t_r}^{\infty} \prod_{i=1}^r \left(x_i^{\frac{1}{\rho_i}-\gamma_i-2} \right) g \left(x^{\frac{1}{\rho}} \right) k \left(s x^{\frac{1}{\rho}} \right) \phi \left(\frac{t_1}{x_1}, \dots, \frac{t_r}{x_r} \right) dx_1 \dots dx_r \end{aligned} \quad (3.4)$$

Where ρ_i and $\sigma_i (i = 1, 2, \dots, r)$ are non zero real numbers of the same sign and all the integrals involved in equations (3.1) to (3.4) are assumed to be absolutely convergent. Also in (3.4) $k \left(s x^{\frac{1}{\rho}} \right)$

stands for $k \left(s_1 x_1^{\frac{1}{\rho_1}}, \dots, s_r x_r^{\frac{1}{\rho_r}} \right)$ and so on.

Proof: Applying the formula (2.5) to the pair of equations (3.2) and (3.4), we get

$$\begin{aligned} &\int_0^{\infty} \dots \int_0^{\infty} f(x^{\sigma}) \theta(s_1, \dots, s_r, x_1, \dots, x_r, \rho_1, \dots, \rho_r) dx_1 \dots dx_r = \\ &\int_0^{\infty} \dots \int_0^{\infty} \prod_{i=1}^r \left(x_i^{\frac{1}{\rho_i}-1} \right) g \left(x^{\frac{1}{\rho}} \right) k \left(s x^{\frac{1}{\rho}} \right) \psi(x) dx_1 \dots dx_r \end{aligned} \quad (3.5)$$

Now changing the variables of integrations on the right hand side of (3.5) slightly and interpreting the result thus obtained with the help of (3.1), we easily arrive at (3.3) after a little simplification.

If in the above theorem, we replace ρ_i and $h_i (i = 1, 2, \dots, r)$ and take

$$g(x) = \prod_{i=1}^r \left\{ x_i^{h_i(c_i+1)-1} \right\},$$

$$\phi(x) = \prod_{i=1}^r \left\{ \frac{1}{\Gamma(1-\alpha_i)} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta_i; x_i \right) \right\}$$

And T to be multidimensional I -function transform defined by (1.8), the right hand side of equation (3.4) assumes the following form:

$$\begin{aligned} &\prod_{i=1}^r \left\{ \frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)} \right\} \int_{t_1}^{\infty} \dots \int_{t_r}^{\infty} \prod_{i=1}^r \left\{ x_i^{c_i-\gamma_i-1} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta_i; \frac{t_i}{x_i} \right) \right\} \\ &I \left(s_1 x_1^{\frac{1}{h_1}}, \dots, s_r x_r^{\frac{1}{h_r}} \right) dx_1 \dots dx_r \end{aligned} \quad (3.6)$$

On evaluation the above integral with the help of known results ([4], p.398, eq.(2)) and ([3], p.105, eq.(1)) and the definition of multivariable I -function (1.1) and substituting the values of $\theta(s_1, \dots, s_r, x_1, \dots, x_r, \rho_1, \dots, \rho_r)$ thus obtained, in the right hand side of equation (3.3), we obtain the following interesting corollary with the help of equations (3.1) and (3.3).

Corollary 1. If $h_i > 0$, $\text{Re}(1 - \alpha_i) > m$, $m \in \mathbb{W}$, (the set of whole number) $\sigma_i (i = 1, 2, \dots, r)$ are non-zero real numbers of the same sign.

$\beta_i \neq 0, -1, -2, \dots, \text{Re}(C_i) \geq 0$

$$f(x) = \frac{o(x_i^{A_i}), \text{ for small values of } x_i}{o(x_i^{B_i} e^{-C_i x_i}) \text{ for large values of } x_i}; i = 1, 2, \dots, r$$

The multivariable I -function satisfy the conditions corresponding appropriately to those given by Srivastava and Panda ([18], p.130, eqs. (1.6) and (1.10)), then

$$I \left\{ \prod_{i=1}^r x_i^{h_i(c_i+1)-1} \psi(x^h); \begin{matrix} 0, n_2, \dots, 0, n_r; (m', n') \dots (m^{(r)}, n^{(r)}), \\ p_2, q_2, \dots, p_r, q_r; (p', q') \dots (p^{(r)}, q^{(r)}), \\ (a_{2j}, \alpha'_{2j}, \alpha''_{2j})_{1,p_2} \dots (\alpha_{rj}, \alpha'_{rj}, \alpha''_{rj})_{1,p_r}; (a'_j, \alpha'_j)_{1,p'-1} \dots (a_j^{(r)}, \alpha_j^{(r)})_{1,p^{(r)}-1} \\ (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1,q_2} \dots (\beta_{rj}, \beta'_{rj}, \beta''_{rj})_{1,q_r}; (b'_j, \beta'_j)_{1,q'-1} \dots (b_j^{(r)}, \beta_j^{(r)})_{1,q^{(r)}-1} \\ \left(1 - \alpha_1 + \gamma_1 - c_1, \frac{1}{h_1}\right), \dots, \left(1 - \alpha_r + \gamma_r - c_r, \frac{1}{h_r}\right) \\ \left(1 - \beta_1 + \gamma_1 - m - c_1, \frac{1}{h_1}\right), \dots, \left(1 - \beta_r + \gamma_r - m - c_r, \frac{1}{h_r}\right); S_1, \dots, S_r \end{matrix} \right\} =$$

$$\frac{1}{\prod_{i=1}^r (\beta_i)_m} I \left\{ \prod_{i=1}^r x_i^{h_i(c_i+1)-1} f(x^{h\sigma}); \begin{matrix} 0, n_2, \dots, 0, n_r; (m', n') \dots (m^{(r)}, n^{(r)}), \\ p_2, q_2, \dots, p_r, q_r; (p', q') \dots (p^{(r)}, q^{(r)}), \\ (a_{2j}, \alpha'_{2j}, \alpha''_{2j})_{1,p_2} \dots (\alpha_{rj}, \alpha'_{rj}, \alpha''_{rj})_{1,p_r}; (a'_j, \alpha'_j)_{1,p'-1} \dots (a_j^{(r)}, \alpha_j^{(r)})_{1,p^{(r)}-1} \\ (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1,q_2} \dots (\beta_{rj}, \beta'_{rj}, \beta''_{rj})_{1,q_r}; (b'_j, \beta'_j)_{1,q'-1} \dots (b_j^{(r)}, \beta_j^{(r)})_{1,q^{(r)}-1} \\ \left(\gamma_1 - c_1, \frac{1}{h_1}\right), \dots, \left(\gamma_r - c_r, \frac{1}{h_r}\right) \\ \left(1 - \beta_1 + \gamma_1 - c_1, \frac{1}{h_1}\right), \dots, \left(1 - \beta_r + \gamma_r - c_r, \frac{1}{h_r}\right); S_1, \dots, S_r \end{matrix} \right\} \quad (3.7)$$

Where

$$\psi(t_1, \dots, t_r) = I \{ f(x); t_1, \dots, t_r \} = \prod_{i=1}^r \left\{ \frac{t_i^{\gamma_i}}{\Gamma(1 - \alpha_i)} \right\}$$

$$\int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r \left\{ x_i^{\gamma_i} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta_i; \frac{x_i}{t_i} \right) \right\} f(x^\sigma) dx_1 \dots dx_r \quad (3.8)$$

Provided that

$$\operatorname{Re} \left(c_i - \gamma_i + \frac{1}{h_i} \frac{b_j^{(i)} - 1}{B_j^{(i)}} \right) < 0; 1 \leq j \leq n^{(i)}$$

$$\operatorname{Re} \left(\sigma_i B_i + c_i + 1 + \frac{1}{h_i} \frac{b_j^{(i)} - 1}{B_j^{(i)}} \right) < 0; 1 \leq j \leq n^{(i)}$$

$$\operatorname{Re} \left(\sigma_i A_i + c_i + 1 + \frac{1}{h_i} \frac{d_k^{(i)} - 1}{D_k^{(i)}} \right) < 0; 1 \leq k \leq m^{(i)}$$

$$\operatorname{Re}(\sigma_i A_i + \gamma_i + 1) > 0; i = 1, 2, \dots, r$$

Theorem 2. If

$$\tau(s_1, \dots, s_r) = T \{ \psi(u^\rho) g(u); s_1, \dots, s_r \} =$$

$$\int_0^\infty \dots \int_0^\infty k(su) \psi(u^\rho) g(u) du_1 \dots du_r \quad (3.9)$$

And

$$\psi(t_1, \dots, t_r) = N \{ f(x^\sigma); t_1, \dots, t_r \} =$$

$$\prod_{i=1}^r (t_i^{\gamma_i}) \int_{t_1}^\infty \dots \int_{t_r}^\infty f(x^\sigma) \prod_{i=1}^r (x_i^{-\gamma_i-1}) \phi \left(\frac{t_1}{x_1}, \dots, \frac{t_r}{x_r} \right) dx_1 \dots dx_r \quad (3.10)$$

then

$$\tau(s_1, \dots, s_r) = \frac{1}{\rho_1 \dots \rho_r} \int_0^\infty \dots \int_0^\infty f(x^\sigma) \theta(s_1, \dots, s_r, t_1, \dots, t_r, \rho_1, \dots, \rho_r) dx_1 \dots dx_r \quad (3.11)$$

Where

$$\theta(s_1, \dots, s_r, t_1, \dots, t_r, \rho_1, \dots, \rho_r) = Y \left\{ \prod_{i=1}^r \left(x_i^{\frac{1}{\rho_i}-1} \right) g \left(x^{\frac{1}{\rho}} \right) k \left(s x^{\frac{1}{\rho}} \right); t_1, \dots, t_r \right\} = \prod_{i=1}^r \left(t_i^{-\gamma_i-1} \right) \int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r \left(x_i^{\frac{1}{\rho_i}-1} \right) g \left(x^{\frac{1}{\rho}} \right) k \left(s x^{\frac{1}{\rho}} \right) \phi \left(\frac{x_1}{t_1}, \dots, \frac{x_r}{t_r} \right) dx_1 \dots dx_r \quad (3.12)$$

Where ρ_i and $\sigma_i (i = 1, 2, \dots, r)$ are non zero real numbers of the same sign and all the integrals involved in equations (3.1) to (3.4) are assumed to be absolutely convergent.

Proof: The proof of the above theorem can be easily developed on the lines similar to those of theorem 1.

Again, if in theorem 2, we replace ρ_i by $h_i (i = 1, 2, \dots, r)$ and take

$$g(x) = \prod_{i=1}^r \left\{ x_i^{h_i(c_i+1)-1} \right\} \phi(x) = \prod_{i=1}^r \left\{ \frac{1}{\Gamma(1-\alpha_i)} {}_2F_1(\alpha_i, \beta_i + m; \beta_i; x_i) \right\}$$

And T to be multidimensional I -function transform defined by (1.8), then proceeding on the lines similar to those of corollary 1, we obtain the following interesting corollary:

Corollary 2. If $h_i > 0$, $\text{Re}(1 - \alpha_i) > m$, $m \in \mathbb{W}$, (the set of whole number) $\sigma_i (i = 1, 2, \dots, r)$ are non-zero real numbers of the same sign.

$$\beta_i \neq 0, -1, -2, \dots; \text{Re}(c_i) \geq 0$$

$$f(x) = \begin{cases} o(x_i^{A_i}), \text{ for small values of } x_i \\ o(x_i^{B_i} e^{-C_i x_i}), \text{ for large values of } x_i \end{cases}; i = 1, 2, \dots, r$$

The multivariable I -function satisfy the conditions corresponding appropriately to those given by Srivastava and Panda ([18], p.130, eqs. (1.6) and (1.10)), then

$$I \left\{ \prod_{i=1}^r x_i^{h_i(c_i+1)-1} \psi(x^h); \begin{matrix} 0, n_2, \dots, 0, n_r; (m', n') \dots (m^{(r)}, n^{(r)}), \\ p_2, q_2, \dots, p_r, q_r; (p', q') \dots (p^{(r)}, q^{(r)}), \\ (a_{2j}, \alpha'_{2j}, \alpha''_{2j})_{1, p_2} \dots (\alpha'_{rj}, \alpha''_{rj}, \alpha^{(r)}_{rj})_{1, p_r}; (a'_j, \alpha'_j)_{1, p'-1} \dots (a_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}-1} \\ (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1, q_2} \dots (\beta'_{rj}, \beta''_{rj}, \beta^{(r)}_{rj})_{1, q_r}; (b'_j, \beta'_j)_{1, q'-1} \dots (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}-1} \end{matrix} \right. \\ \left. \left(-\gamma_1 - c_1, \frac{1}{h_1} \right) \dots \left(-\gamma_r - c_r, \frac{1}{h_r} \right) \right\}; S_1, \dots, S_r \left\{ = \frac{1}{\prod_{i=1}^r (\beta_i)_m} I \left\{ \prod_{i=1}^r x_i^{h_i(c_i+1)-1} f(x^{h\sigma}); \begin{matrix} 0, n_2, \dots, 0, n_r; (m', n') \dots (m^{(r)}, n^{(r)}), \\ p_2, q_2, \dots, p_r, q_r; (p', q') \dots (p^{(r)}, q^{(r)}), \\ (a_{2j}, \alpha'_{2j}, \alpha''_{2j})_{1, p_2} \dots (\alpha'_{rj}, \alpha''_{rj}, \alpha^{(r)}_{rj})_{1, p_r}; (a'_j, \alpha'_j)_{1, p'-1} \dots (a_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)}-1} \\ (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1, q_2} \dots (\beta'_{rj}, \beta''_{rj}, \beta^{(r)}_{rj})_{1, q_r}; (b'_j, \beta'_j)_{1, q'-1} \dots (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)}-1} \end{matrix} \right. \right. \\ \left. \left(\alpha_1 - \gamma_1 - c_1 - 1, \frac{1}{h_1} \right) \dots \left(\alpha_r - \gamma_r - c_r - 1, \frac{1}{h_r} \right) \right\}; S_1, \dots, S_r \left\{ \right. \quad (3.13)$$

Where

$$\psi(t_1, \dots, t_r) = N \{f(x); t_1, \dots, t_r\} = \prod_{i=1}^r \left\{ \frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)} \right\} \int_{t_1}^{\infty} \dots \int_{t_r}^{\infty} \prod_{i=1}^r \left\{ x_i^{-\gamma_i-1} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta_i; \frac{t_i}{x_i} \right) \right\} f(x^\sigma) dx_1 \dots dx_r \quad (3.14)$$

Provided that

$$\begin{aligned} \operatorname{Re} \left(c_i + \gamma_i + \frac{1}{h_i} \frac{d_j^{(i)}}{D_j^{(i)}} \right) &> 0; 1 \leq j \leq m^{(i)} \\ \operatorname{Re} \left(\sigma_i B_i + c_i + 1 + \frac{1}{h_i} \frac{b_j^{(i)} - 1}{B_j^{(i)}} \right) &< 0; 1 \leq j \leq n^{(i)}; i = 1, 2, \dots, r \\ \operatorname{Re} \left(\sigma_i A_i + c_i + 1 + \frac{1}{h_i} \frac{d_k^{(i)}}{D_k^{(i)}} \right) &> 0; 1 \leq k \leq m^{(i)} \\ \operatorname{Re}(\sigma_i A_i + \beta_i - \gamma_i + m) &> 0; i = 1, 2, \dots, r \end{aligned}$$

The one and two-dimensional analogous of Theorems 1 and 2 can easily be deduced but since the theorems contain a reasonably detailed analysis of the multidimensional case we prefer to omit their details. The corollaries 1 and 2 given earlier are also new. They give rise to interesting theorems involving multidimensional analogues of fractional integral operators defined by Kober [7], Riemann-Liouville [4] and Weyl [4] and simpler multidimensional integral transforms, on suitably specializing the fractional integral operators and multidimensional I -function transform involved therein. Again, the one and two-dimensional analogues of corollaries 1 and 2, yield theorems essentially similar to those given earlier by Gupta, Goyal and Handa ([5], p.165-170), Handa ([6], p. 200-204), Kalla ([8], p. 1008-1010, [9] p. 54-56) and Mathur ([11] p.108-121).

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