CENTRALIZING AND COMMUTING LEFT GENERALIZED DERIVATIONS ON PRIME RINGS

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Abstract: Let $R$ be a prime ring and $d$ a derivation on $R$. If $f$ is a left generalized derivation on $R$ such that $f$ is centralizing on a left ideal $U$ of $R$, then $R$ is commutative.

Introduction:

H.E Bell and W.S. Martindale III, [1] proved that a semiprime ring $R$ must have a nontrivial central ideal if it admits an appropriate endomorphism or derivation which is centralizing on $U$. Then $R$ is commutative. Bresar, M [2] proved that some concrete additive mappings cannot be centralizing on certain subsets of non-commutative prime rings. He also described the structure of an arbitrary additive mapping which is centralizing on a prime ring. J.H. Mayne [4] had discussed about the existence of a non-trivial automorphism or derivation which is centralizing on a nonzero ideal in a prime ring and then shown that the ring is commutative. Asif Ali and Tariqshah [5] has considered $f$ as a generalized derivation on $R$ such that $f$ is centralizing on a left ideal $U$ of $R$, then proved that $R$ is commutative.

Preliminaries:

A ring $R$ is said to be a prime if $aRb = 0$ implies that either $a = 0$ or $b = 0$. An additive mapping $d: R \rightarrow R$ is said to be derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $d: R \rightarrow R$ is said to be left derivation if $d(xy) = d(x)y + xd(y)$ for $x, y \in R$ all $A$ mapping $f$ is said to commuting on a left ideal $U$ of $R$ if $f(xy) = 0$ for all $x \in U$ and $f$ is said to be centralizing if $f(x, y) \in Z(R)$ for all $x \in U$. An additive map $f: R \rightarrow R$ is called a generalized derivation if there exists a derivation $d: R \rightarrow R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. A additive map $f: R \rightarrow R$ is called a left generalized derivation if $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$ where $d$ is derivation on $R$. Throughout this paper $R$ will represent a prime ring with $Z(R)$ being its centre.

Remark 1: For a nonzero element $a \in Z(R), f \in Z(R), \text{then } b \in Z(R).

To prove the main results, we find it necessary to establish the following lemma.

Lemma 1: If $f$ is an additive mapping from $R$ to $R$ such that $f$ is centralizing on a left ideal $U$ of $R$, then for all $x \in U \cap Z(R)$, $f(x) \in Z(R)$.

Proof: Since $f$ is centralizing on $U$, we have

$x, y \in U \Rightarrow x + y \in U$

$\forall x + y \in U \Rightarrow [f(x + y), x + y] \in Z(R)$

If $f$ is additive then

$[f(x) + f(y), x + y] \in Z(R)$

$\Rightarrow [f(x), x] + [f(x), y] + [f(y), x] + [f(y), y] \in Z(R)$

If $f$ is centralizing on a left ideal $U$ then

$[f(x), x] = 0, [f(y), y] = 0$

$[f(x), y] + [f(y), x] \in Z(R)$

(1)
Now if \( x \in Z(R) \), then equation (1) becomes
\[
[f(x), y] + [f(x), x] \in Z(R)
\]
\[
\implies [f(x), y] \in Z(R)
\] (2)
Replacing \( y \) by \( f(x)y \) in equation (2) we obtain
\[
[f(x), f(x)y] \in Z(R)
\]
\[
\implies [f(x), f(x)]y + f(x)[f(x), y] \in Z(R)
\]
\[
f(x)[f(x), y] \in Z(R)
\]
Case 1: \( [f(x), y] = 0 \)
\[
\forall y \in U \text{ we have } f(x)y - yf(x) = 0
\]
\[
\implies f(x) \in C^*_g
\]
The centralizer of \( U \) in \( R \) and hence \( \{1, \text{ identity} IV \} \)
\( f(x) \in Z(R) \).
Case 2: \( [f(x), y] \neq 0 \),
It again follows from remark 1
\( x \in Z(R), f(x)[f(x), y] \in Z(R) \implies f(x) \in Z(R) \).
Hence the lemma.

**Theorem 1:** Let \( R \) be a prime ring. Let \( d: R \to R \) be a non zero derivation and \( f \) be a left generalized derivation on a left ideal \( U \) of \( R \). If \( f \) is commuting on \( U \) then \( R \) is commutative.

**Proof:** since \( f \) is commuting on \( U \), we have \( [f(x), x] = 0 \) \( \forall x \in U \).
Replacing \( x \) by \( x+y \) we get
\[
[f(x+y), x+y] = 0
\]
\[
[f(x), x] + [f(x), x] + [f(y), x] + [f(y), y] = 0
\]
\[
\implies [f(x), y] + [f(y), x] = 0
\] (3)
Now by replacing \( y \) by \( xy \) in equation (3) we get
\[
[f(x), xy] + [f(xy), x] = 0
\]
\[
[f(x), x]y + x[f(x), y] + [d(x)y + xf(y), x] = 0
\]
\[
[f(x), x]y + x[f(x), y] + [d(x)y, x] + [xf(y), x] = 0
\]
\[
[f(x), x]y + x[f(x), y] + [d(x), y, x] + [x, y]f(y) + x[f(y), x] = 0
\]
\( f \) is centralizer then, \( [f(x), x]y = 0, [x, x]f(y) = 0 \).
\( x([f(x), y] + [f(y), x]) + [d(x), y, x] = 0 \)
From (3) we get, \( [d(x), y, x] = 0 \).
Replacing \( y \) by \( yr \) in equation (4) we get
\[
[d(x), yr, x] = 0
\]
\[
[d(x)y, x]r + d(x)yx[r, x] = 0
\]
From (3) we get
\[
d(x)yx[r, x] = 0
\]
y \( \in \) \( U \) we generalized
\[
d(x)[U]r, x] = 0
\]
\( r \in \) \( \mathbb{R} \) is prime then \( d(x) = 0 \) or \( [r, x] = 0 \)
\[
\implies \forall x \in U \text{ either } x \in Z(R) \text{ or } d(x) = 0
\]
Since \( d \) is non-zero on \( R \) then by [4, Lemma 2] \( \implies d \) is non zero on \( U \)
Suppose \( d(x) \neq 0 \) for some \( x \in U \)
Then \( x \in Z(R) \)
Suppose \( x \in U \) is such \( z \in Z(R) \) that \( d(z) = 0 \)
and \( x + z \in Z(R) \implies d(x + z) = 0 \implies d(x) = 0 \)
It is contradiction
\[
\implies z \in Z(R) \text{ for } z \in U \text{ or } z \in U
\]
Thus \( U \) is commutative
\[
\implies R \text{ is commutative}
\]
Hence the theorem.
**Theorem 2:** Let $U$ be a left ideal of a prime ring $R$ such that $U \cap z(R) \neq 0$. Let $d$ be a non zero derivation and $f$ be a left generalized derivation on $R$ such that $f$ is centralizing on $U$. Then $R$ is commutative.

**Proof:** We assume that
Because otherwise $f$ is commutating on $U$ and there is nothing to prove
Now for anon-zero $z \in Z(R)$
We replaced $x$ by $xy$ in equation (1) we get

$$[f(yz), y] + [f(y), yz] \in Z(R)$$
$$[d(y)z + zf(z), y] + [f(y), yz] \in Z(R)$$
$$[d(y)z, y] + [zf(z), y] + [f(y), yz] \in Z(R)$$
$$[d(y), y]z + d(y)[z, y] + [y, y]f(z) + y[f(z), y] + [f(y), y]z + y[f(y), z] \in Z(R)$$

$z \in Z(R) \Rightarrow [z, y] = 0 \forall y \in R, [y, y] = 0$
Since $z \in Z(R) \Rightarrow f$ is centralizer

$$f(z) \in Z(R) \Rightarrow [f(z), y] = 0 \forall y \in R, nd \ f(y), y = 0, [f(y), z] = 0$$
$$\Rightarrow [d(y), y]z \in Z(R)$$
Since $z$ is non zero it follows that

$$[d(y), y] \in Z(R)$$
This implies $d$ is centralizing on $U$ and hence by [1,Theorem4] we conclude that $R$ is commutative.

**References:**


