

## CENTRALIZING AND COMMUTING LEFT GENERALIZED DERIVATIONS ON PRIME RINGS

Dr. C. Jaya Subba Reddy<sup>1</sup>, S.MallikarjunaRao<sup>2</sup>, V.Vijaya Kumar<sup>3</sup>

<sup>1,2,3</sup>Department of Mathematics, S.V.University, Tirupati-517502, AndhraPradesh, India.

**Keywords:** Derivations, left derivation, generalized derivation, left generalized derivation, prime ring, centralizing and commuting.

**Abstract:** Let  $R$  be a prime ring and  $d$  a derivation on  $R$ . If  $f$  is a left generalized derivation on  $R$  such that  $f$  is centralizing on a left ideal  $U$  of  $R$ , then  $R$  is commutative.

### Introduction:

H.E Bell and W.S.Martindale III, [1] proved that a semiprime ring  $R$  must have a nontrivial central ideal if it admits an appropriate endomorphism or derivation which is centralizing on  $U$ . Then  $R$  is commutative. Bresar.M [2] proved that some concrete additive mappings cannot be centralizing on certain subsets of non-commutative prime rings. He also described the structure of an arbitrary additive mapping which is centralizing on a prime ring. J.H. Mayne [4] had discussed about the existence of a non-trivial automorphism or derivation which is centralizing on a nonzero ideal in a prime ring and then shown that the ring is commutative. Asif Ali and Tariqshah [5] has considered  $f$  as a generalized derivation on  $R$  such that  $f$  is centralizing on a left ideal  $U$  of  $R$ , then proved that  $R$  is commutative.

### Preliminaries:

A ring  $R$  is said to be a prime if  $aRb = 0$  implies that either  $a=0$  or  $b=0$ . An additive mapping  $d:R \rightarrow R$  is said to be derivation if  $d(xy)=d(x)y+xd(y)$  for all  $x,y \in R$ . An additive mapping  $d:R \rightarrow R$  is said to be left derivation if  $d(xy)=d(x)y+xd(y)$  for  $x,y \in R$  all. A mapping  $f$  is said to commuting on a left ideal  $U$  of  $R$  if  $[f(x),x]=0$  for all  $x \in U$  and  $f$  is said to be centralizing if  $[f(x),x] \in Z(R)$  for all  $x \in U$ . An additive map  $f:R \rightarrow R$  is called a generalized derivation if there exists a derivation  $R \rightarrow R$  such that  $f(xy)=f(x)y+xd(y)$  for all  $x,y \in R$ . A additive map  $f:R \rightarrow R$  is called a left generalized derivation if  $f(xy)=f(x)y+xd(y)$  for all  $x,y \in R$  where  $d$  is derivation on  $R$ . Throughout this paper  $R$  will represent a prime ring with  $Z(R)$  being its centre.

**Remark1:** For a nonzero element  $a \in Z(R)$ ,  $f ab \in Z(R)$ , then  $b \in Z(R)$ .

To prove the main results, we find it necessary to establish the following lemma.

**Lemma 1:** If  $f$  is an additive mapping from  $R$  to  $R$  such that  $f$  is centralizing on a left ideal  $U$  of  $R$ , then for all  $x \in U \cap Z(R)$ ,  $f(x) \in Z(R)$ .

**Proof:** Since  $f$  is centralizing on  $U$ , we have

$$x, y \in U \Rightarrow x + y \in U$$

$$\forall x + y \in U \Rightarrow [f(x + y), x + y] \in Z(R)$$

$f$  is additive then

$$[f(x) + f(y), x + y] \in Z(R)$$

$$\Rightarrow [f(x), x] + [f(x), y] + [f(y), x] + [f(y), y] \in Z(R)$$

$f$  is centralizing on left ideal  $U$  then

$$[f(x), x] = 0, [f(y), y] = 0$$

$$[f(x), y] + [f(y), x] \in Z(R)$$

(1)

Now if  $x \in Z(R)$ , then equation (1) becomes

$$\begin{aligned} [f(x), y] + [f(x), x] &\in Z(R) \\ \Rightarrow [f(x), y] &\in Z(R) \end{aligned} \quad (2)$$

Replacing  $y$  by  $f(x)y$  in equation (2) we obtain

$$\begin{aligned} [f(x), f(x)y] &\in Z(R) \\ \Rightarrow [f(x), f(x)]y + f(x)[f(x), y] &\in Z(R) \\ f(x)[f(x), y] &\in Z(R) \end{aligned}$$

Case1:  $[f(x), y] = 0$

$$\begin{aligned} \forall y \in U \text{ we have } f(x)y - yf(x) &= 0 \\ \Rightarrow f(x) &\in C_R^{(U)} \end{aligned}$$

The centralizer of  $U$  in  $R$  and hence  $\{[1, \text{identity } IV]\} f(x) \in Z(R)$ .

Case2:  $[f(x), y] \neq 0$ ,

It again follows from remark 1

$$x \in Z(R), f(x)[f(x), y] \in Z(R) \Rightarrow f(x) \in Z(R).$$

Hence the lemma.

**Theorem1:** Let  $R$  be a prime ring. Let  $d: R \rightarrow R$  be a non zero derivation and  $f$  be a left generalized derivation on a left ideal  $U$  of  $R$ . If  $f$  is commuting on  $U$  then  $R$  is commutative.

**Proof:** since  $f$  is commuting on  $U$ , we have  $[f(x), x] = 0 \forall x \in U$ .

Replacing  $x$  by  $x+y$  we get

$$\begin{aligned} [f(x+y), x+y] &= 0 \\ [f(x), x] + [f(x), y] + [f(y), x] + [f(y), y] &= 0 \\ \Rightarrow [f(x), y] + [f(y), x] &= 0 \end{aligned} \quad (3)$$

Now by replacing  $y$  by  $xy$  in equation (3) we get

$$\begin{aligned} [f(x), xy] + [f(xy), x] &= 0 \\ [f(x), x]y + x[f(x), y] + [d(x)y + xf(y), x] &= 0 \\ [f(x), x]y + x[f(x), y] + [d(x)y, x] + [xf(y), x] &= 0 \\ [f(x), x]y + x[f(x), y] + [d(x)y, x] + [x, x]f(y) + x[f(y), x] &= 0 \end{aligned}$$

$f$  is centralizer then,  $[f(x), x]y = 0, [x, x]f(y) = 0$ .

$$x([f(x), y] + [f(y), x]) + [d(x)y, x] = 0 \quad (4)$$

From (3) we get,  $[d(x)y, x] = 0$ .

Replacing  $y$  by  $yr$  in equation (4) we get

$$\begin{aligned} [d(x)yr, x] &= 0 \\ [d(x)y, x]r + d(x)y[r, x] &= 0 \end{aligned}$$

From (3) we get

$$d(x)y[r, x] = 0$$

$y \in U$  we generalized

$$d(x)U[r, x] = 0$$

$r \in R$  is prime then  $d(x)=0$  or  $[r, x]=0$

$$\Rightarrow \forall x \in U \text{ either } x \in Z(R) \text{ or } d(x)=0$$

Since  $d$  is non-zero on  $R$  then by [4, Lemma 2]  $\Rightarrow d$  is non zero on  $U$

Suppose  $d(x) \neq 0$  for some  $x \in U$

Then  $x \in Z(R)$

Suppose  $x \in U$  is such  $z \notin Z(R)$  iat then  $d(z)=0$   
and  $x+z \notin Z(R) \Rightarrow d(x+z) = 0 \Rightarrow d(x) = 0$

It is contradiction

$$\Rightarrow z \in Z(R) \text{ for } z \in U \text{ or } z \in U$$

Thus  $U$  is commutative

$$\Rightarrow R \text{ is commutative}$$

Hence the theorem.

**Theorem 2:** Let  $U$  be a left ideal of a prime ring  $R$  such that  $U \cap Z(R) \neq 0$ . Let  $d$  be a non zero derivation and  $f$  be a left generalized derivation on  $R$  such that  $f$  is centralizing on  $U$ . Then  $R$  is commutative.

**Proof:** We assume that

Because other wise  $f$  is  $Z(R) \neq 0$  ng on  $U$  and there is nothing to prove

Now for anon-zero  $z \in Z(R)$

We replaced  $x$  by  $xy$  in equation (1) we get

$$[f(yz), y] + [f(y), yz] \in Z(R)$$

$$[d(y)z + yf(z), y] + [f(y), yz] \in Z(R)$$

$$[d(y)z, y] + [yf(z), y] + [f(y), yz] \in Z(R)$$

$$[d(y), y]z + d(y)[z, y] + [y, y]f(z) + y[f(z), y] + [f(y), y]z + y[f(y), z] \in Z(R)$$

$$z \in Z(R) \Rightarrow [z, y] = 0 \forall y \in R, [y, y] = 0$$

Since  $z \in Z(R) \Rightarrow f$  is centralizer

$$f(z) \in Z(R) \Rightarrow [f(z), y] = 0 \forall y \in R, \text{nd } [f(y), y] = 0, [f(y), z] = 0$$

$$\Rightarrow [d(y), y]z \in Z(R)$$

Since  $z$  is non zero it follows that

$$[d(y), y] \in Z(R)$$

This implies  $d$  is centralizing on  $U$  and hence by [1, Theorem4] we conclude that  $R$  is commutative.

#### References:

- [1] H.E. Bell and W.S. Martindale, III Centralizing mapping of semiprime rings, Cand.Math. Bull.30(1987) 92-101
- [2] M.Bresar, Centralizing mappings and derivations in prime rings, J.Algebra 156(193)35-394
- [3] B.Hvala, Generalized derivation in rings, Comm. Algebra, 26(4)(1998), 1147-1166
- [4] J.H.Mayne, Centralizing mappings of prime rings, Cand. Math. Bull., 27(1)(1984), 122-126.
- [5] Asif Ali and Tariq shah, centralizing and commuting genoralized derivation on prime rings, MATHYKM BECHNK 60(2008), 1-2