

## A criterion for (non-)planarity of the block-transformation graph $G^{\alpha\beta\gamma}$ when $\alpha\beta\gamma = 101$

B. Basavanagoud<sup>1</sup> and Jaishri B. Veeragoudar<sup>2</sup>

<sup>1</sup>Department of Mathematics, Karnatak University, Dharwad-580 003, India.

<sup>2</sup>Department of Mathematics, Jain College of Engineering, Belgaum-590 014, India.

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**Abstract:** The general concept of the block-transformation graph  $G^{\alpha\beta\gamma}$  was introduced in [1]. The vertices and blocks of a graph are its members. The block-transformation graph  $G^{101}$  of a graph  $G$  is the graph, whose vertex set is the union of vertices and blocks of  $G$ , in which two vertices are adjacent whenever the corresponding vertices of  $G$  are adjacent or the corresponding blocks of  $G$  are nonadjacent or the corresponding members of  $G$  are incident. In this paper, we present characterizations of graphs whose block-transformation graphs  $G^{101}$  are planar, outerplanar or minimally nonouterplanar. Further we establish a necessary and sufficient condition for the block-transformation graph  $G^{101}$  to have crossing number one.

### 1 Introduction

The graphs considered are finite, simple and connected. We follow the terminology of [4]. Let  $G$  be a graph. Let  $V(G)$ ,  $E(G)$ ,  $\bar{G}$  and  $B(G)$  denote the vertex set, edge set, complement and block graph of  $G$ , respectively. A block is connected, nontrivial graph having no cutvertices. A block  $B$  of a graph  $G$  is called an endblock if it contains exactly one cutvertex of  $G$ . A block of a graph  $G$  is called a nonendblock if it contains at least two cutvertices of  $G$ . A graph  $G$  is called a chain graph if every nonendblock has only two cutvertices. A graph  $G$  is unicyclic if it is connected and has only one cycle. The block-star graph, denoted by  $B_{1,n}$ , is a graph obtained by replacing at least one edge of  $K_{1,n}$  by a complete block. Note that the cutvertex of  $B_{1,n}$  is same as that of  $K_{1,n}$  and star graph is a special case of block-star graph introduced in [1]. A graph  $G$  is called a planar graph if  $G$  can be drawn in a plane so that no two of its edges cross each other. A graph that is not planar is called nonplanar. A set of vertices of a planar graph  $G$  is called an inner vertex set  $S$  of  $G$  if  $G$  can be drawn on the plane in such a way that each vertex of  $S$  lies only on the interior region and  $S$  contains the minimum possible number of vertices of  $G$ . The number of vertices of  $S$  is said to be the inner vertex number of  $G$  and is denoted by  $i(G)$ . A graph  $G$  is said to be  $k$ -minimally nonouterplanar if  $i(G) = k$ ,  $k \geq 1$ . A 1-minimally nonouterplanar graph is called minimally nonouterplanar. These concepts are introduced in [5]. A graph  $G$  is outerplanar if and only if  $i(G) = 0$ . This concept is introduced in [2].

The crossing number  $Cr(G)$  of a graph  $G$  is the minimum number of pairwise intersections of its edges when  $G$  is drawn in the plane. Obviously,  $Cr(G) = 0$  if and only if  $G$  is planar. A graph  $G$  has crossing number 1 if and only if  $Cr(G) = 1$ , (see in [3]). A graph  $G$  is called a theta wheel if  $G$  is a cycle  $C$  together with a new vertex adjacent with exactly two nonadjacent vertices of  $C$  or adjacent with at least 3 vertices of  $C$ . A minimally-nonouterplanar graph  $G$  is called theta-minimally nonouterplanar if  $G$  has a theta wheel such that at least one edge of a theta wheel is a boundary edge. If  $B$  is a block of  $G$  with its vertex set  $V(B) = \{u_1, u_2, \dots, u_r; r \geq 2\}$ , then we say that  $u_i$ ,  $1 \leq i \leq r$ , and  $B$  are incident with each other. If two blocks  $B_i$  and  $B_j$  of  $G$  have a common cutvertex, then we say that  $B_i$  and  $B_j$  are adjacent blocks of  $G$ . Let  $U(G)$  denote the block set of  $G$  and is the set  $\{B_i : B_i \text{ is a block of } G\}$ . In [6] the concept of total-block graph  $TB(G)$  was introduced. The total-block graph  $TB(G)$  of a graph  $G$  is the graph whose set of vertices is the union of set of vertices and set of blocks of  $G$  in which two vertices are adjacent if and only if the corresponding members of  $G$  are adjacent or incident. Basavanagoud et al. [1] introduced some new

graphical transformations which generalize the concept of total-block graph. In this paper, we present the characterizations of graphs whose block-transformation graphs  $G^{101}$  are planar, outerplanar or minimally nonouterplanar. Further, we establish a necessary and sufficient condition for the block-transformation graph  $G^{101}$  to have crossing number 1.

**Definition 1.1.** The block-transformation graph  $G^{101}$  of a graph  $G$  is the graph with vertex set  $V(G) \cup U(G)$  in which the vertices  $x$  and  $y$  are joined by an edge if one of the following conditions holds

- (i)  $x, y \in V(G)$ , and  $x$  and  $y$  are adjacent in  $G$ .
- (ii)  $x, y \in U(G)$ , and  $x$  and  $y$  are not adjacent in  $G$ .
- (iii) One of  $x$  and  $y$  is in  $V(G)$  and the other is in  $U(G)$ , and they are incident in  $G$ , (see, Figure 1.1).

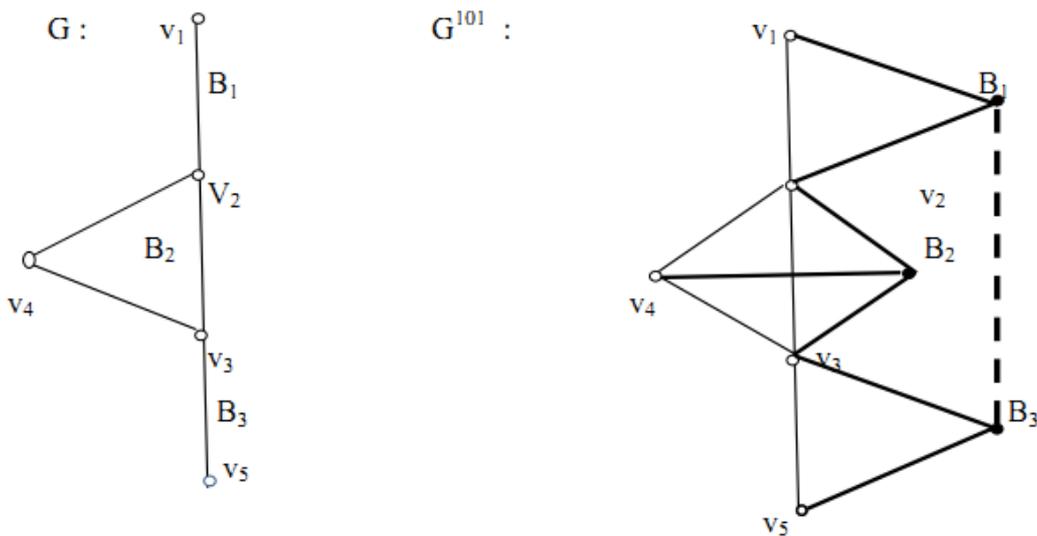


Figure 1.1

In  $G^{101}$ , the light vertices correspond to the vertices  $G$  and dark vertices correspond to blocks of  $G$  and are called block-vertices

## 2 Planarity of Graphs $G^{101}$

Now we characterize graphs whose block-transformation graph  $G^{101}$  is planar, outerplanar or minimally nonouterplanar. We make use of the following well-known results in our later study.

The vertices and blocks of a graph  $G$  are its members. The semitotal-block graph  $T_b(G)$  introduced in [6], is the graph whose set of vertices is the union of the set of vertices and blocks of  $G$  and in which two vertices are adjacent if and only if the corresponding vertices of  $G$  are adjacent or corresponding members are incident.

**Remark 2.1**  $T_b(G) \subseteq G^{101}$ .

**Remark 2.2**  $G^{101} = T_b(G)$  if and only if  $G$  is a block or has a single cutvertex.

**Theorem 2.A** [4] A graph is planar if and only if it has no subgraph homeomorphic to  $K_5$  or  $K_{3,3}$ .

**Theorem 2.B** [4] A graph is outerplanar if and only if it has no subgraph homeomorphic to  $K_4$  or  $K_{2,3}$  except  $K_4 - x$  (where  $x$  is any edge in  $K_4$ ).

**Theorem 2.C** [7] The semitotal-block graph  $T_b(G)$  of a connected graph  $G$  is minimally nonouterplanar if and only if  $G$  is an unicyclic graph.

**Theorem 2.D** [5] If  $G$  is a finite nonplanar graph and  $G - x$  is planar for some edge  $x$  of  $G$ , then  $i(G - x) \geq 2$ .

**Theorem 2.E** [8] The semitotal-block graph  $T_b(G)$  of a graph  $G$  has crossing number 1 if and only if  $G$  is theta-minimally nonouterplanar.

**Theorem 2.1.** The block-transformation graph  $G^{101}$  of a connected graph  $G$  is planar if and only if  $G$  is outerplanar with at most 3 cutvertices and

(i) if  $G$  has two cutvertices, say  $c_1$  and  $c_2$ , then only one of the cutvertices can be incident with more than two blocks and the nonendblock is an edge or is a block such that  $c_1$  and  $c_2$  are in the same region of the block.

(ii) if  $G$  has 3 cutvertices then  $G$  is a cycle whose any three vertices each of which is adjoined to an endblock or  $G$  is a chain graph with cutvertices, say  $c_1, c_2, c_3$  such that  $c_1c_2$  and  $c_2c_3$  form the boundary edges of the nonendblocks.

**Proof.** Suppose  $G^{101}$  is planar. If possible, assume that  $G$  is nonouterplanar. Then  $G$  contains at least 1 block  $B$  which is nonouterplanar. Consequently,  $B$  contains a subgraph homeomorphic to  $K_4$  or  $K_{2,3}$ . By definition of  $G^{101}$ , the block-vertex  $B$  is adjacent to each incident vertex of  $B$ , thereby produces a subgraph homeomorphic to  $K_5$  or  $K_{3,3}$  respectively. So by Theorem 2.A,  $G^{101}$  is nonplanar, a contradiction. Let  $G$  has at least 4 cutvertices. Then  $G$  has subgraph homeomorphic to  $H_1, H_2$  or  $H_3$ , (see, Figure 1.2). We see that  $H_1^{101}, H_2^{101}$  and  $H_3^{101}$  have subgraphs homeomorphic to  $K_5, K_5$  and  $K_{3,3}$  respectively such that  $Cr(H_1^{101}) = Cr(H_2^{101}) = Cr(H_3^{101}) = 1$ , a contradiction. Hence  $G$  is outerplanar with at most 3 cutvertices. We consider the following cases:

**Case 1.**  $G$  has 2 cutvertices. We observe the following subcases:

**Subcase 1.1.** Both the cutvertices are incident with at least three blocks. Then a graph homeomorphic to  $H_4$ , (see, Figure 1.2) is a subgraph of  $G$  and  $H_4^{101}$  has a subgraph homeomorphic to  $K_{3,3}$ , such that  $Cr(H_4^{101}) = 1$ . Hence  $Cr(G^{101}) \geq 1$ , a contradiction.

**Subcase 1.2.** Only one cutvertex is incident with more than 2 blocks and the nonendblock has a diagonal edge such that  $c_1$  and  $c_2$  are in different regions of the block. Then in any planar drawing of  $G$  if the endblocks are taken inside the interior region of the nonendblock then the block-vertices corresponding to these endblocks are adjacent forming crossing with the diagonal edge. If the endblocks are kept in the exterior region of the nonendblock then the block-vertex corresponding to the nonendblock is adjacent with the vertices of the nonendblock giving at least 1 crossing, a contradiction.

**Case 2.**  $G$  has 3 cutvertices. We observe the following subcases:

**Subcase 2.1.** All the cutvertices lie on the same block. Assume at least one of the cutvertices is incident with at least 3 blocks. Then a graph homeomorphic to  $H_5$  is a subgraph in  $G$ , (see, Figure 1.2). We see that  $H_5^{101}$  has a subgraph homeomorphic to  $K_{3,3}$  such that crossing number of  $H_5^{101}$  is 1. Hence  $G^{101}$  is nonplanar, a contradiction. Assume the block  $B$  with 3 cutvertices is not a cycle and hence has a diagonal edge. Then, in  $G^{101}$ , since the other 3 blocks are outerplanar, along with their corresponding block-vertices they form planar blocks and the three block-vertices are adjacent to each other forming  $K_3$ . The block-vertex  $B$  is adjacent to all the vertices of block  $B$  with at least one crossing because of the diagonal edge. Therefore,  $G^{101}$  is nonplanar, a contradiction.

**Subcase 2.2.**  $G$  is a graph with 3 cutvertices such that each block has at most 2 cutvertices incident with it. Assume one of the cutvertices is incident with at least three blocks. Then a graph homeomorphic to  $H_6$  or  $H_7$  is a subgraph of  $G$ , (see, Figure 1.2). We see that  $H_6^{101}$  and  $H_7^{101}$  have subgraphs homeomorphic to  $K_{3,3}$  such that crossing number of  $H_6^{101}$  and  $H_7^{101}$  is 1. Therefore  $G^{101}$  is nonplanar, a contradiction. Assume each cutvertex is incident with at most 2 blocks and at least one of the blocks with cutvertices  $c_1$  and  $c_2$  is such that  $c_1c_2$  is not its outer edge. Then a graph homeomorphic to  $H_8$  is a subgraph of  $G$  and we see that  $H_8^{101}$  has a subgraph homeomorphic to  $K_{3,3}$  such that crossing number of  $H_8^{101}$  is 1. Hence,  $G^{101}$  is nonplanar, a contradiction.

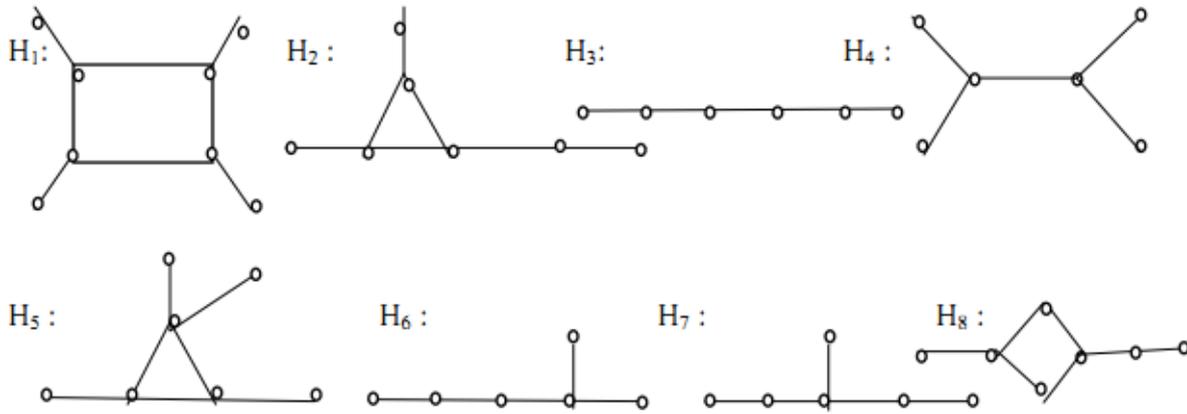


Figure 1.2

Conversely, let  $G$  is outerplanar. Since every block of  $G$  is outerplanar, each can be drawn on the plane in such a way that all its vertices lie on the exterior region. Let  $B_1$  be a block of  $G$  and  $u_i \in V(B_1)$ ,  $1 \leq i \leq p$ , where  $p$  is the number of vertices of block  $B_1$ . The block  $B_1$  together with the edges  $B_1u_i$  form a subgraph  $G_1$  of  $G^{101}$ . Clearly it is planar. If  $G = B_1$ , then of course  $G^{101}$  is planar. If  $G$  has a single cutvertex with  $n$  number of blocks, then as in case of  $B_1$ , we have  $n$  planar blocks adjacent at the common cutvertex which itself is  $G^{101}$  and hence planar. Assume  $G$  has 2 cutvertices, say  $c_1$  and  $c_2$ , such that  $c_1$  is incident with blocks  $B_1$  and  $B_2$ ,  $c_2$  is incident with  $B_2, B_3, \dots, B_p$  and  $c_1c_2$  is the boundary edge. As above all the block-vertices with their corresponding blocks form planar blocks. The block-vertex  $B_1$  is adjacent to the block-vertices  $B_3, B_4, \dots, B_p$  and hence planar. Assume  $G$  has two cutvertices, say  $c_1$  and  $c_2$ , such that the nonendblock has  $c_1$  and  $c_2$  in the same region. The endblocks can be drawn on the plane in such a way that all lie entirely in the region of the nonendblock  $B_2$  containing  $c_1$  and  $c_2$ . We take the block-vertices of these endblocks also in the same region but not on any interior region of the corresponding blocks. Each block-vertex form a planar block with the corresponding incident vertices of the blocks. The block-vertex  $B_1$  is made to be adjacent with the block-vertices  $B_3, B_4, \dots, B_p$ . The block-vertex  $B_2$  is taken in the exterior region of block  $B_2$  and is joined with all the incident vertices of block  $B_2$ . Thus we have a planar graph  $G^{101}$ .

Let  $G$  satisfies condition 2. Since  $c_1c_2$  and  $c_2c_3$  are the boundary edges of the nonendblocks we see that each block with its block-vertices form planar subgraph of  $G^{101}$  and the block-vertices of the nonadjacent blocks can be easily made adjacent without any crossings.

**Theorem 2.2.** The block-transformation graph  $G^{101}$  of a connected graph  $G$  is outerplanar if and only if  $G = K_{1,n}$ ,  $n \geq 1$ , or  $P_4$ .

**Proof.** Suppose  $G^{101}$  of a connected graph  $G$  is outerplanar. Assume at least one of the blocks, say  $B$ , of  $G$  contains a cycle. Then in  $G^{101}$ , the vertex corresponding to the block is adjacent with the incident vertices of block  $B$  forming a subgraph homeomorphic to a wheel, hence  $G^{101}$  is not outerplanar, a contradiction. Therefore, each block of  $G$  is an edge. By Theorem 2.1,  $G$  has at most three cutvertices. Let  $G$  has 3 cutvertices. By condition (2) of Theorem 2.1, the only possible graph

for which  $G^{101}$  is planar is  $G = P_5$ . But one can easily see that  $P_5^{101}$  has a subgraph homeomorphic to  $K_{2,3}$  such that  $i(P_5^{101}) = 4$ , a contradiction. Hence we conclude  $G$  has at most 2 cutvertices and each block of  $G$  is an edge. Let  $G$  has two cutvertices. Then by Theorem 2.1, only one of the cutvertices can be incident with more than 2 blocks. Suppose  $G \neq P_4$  and has exactly two cutvertices. Then  $G'$  is the subgraph of  $G$ , where  $G'$  is the graph  $P_4$  together with a vertex adjoined to one of the nonendvertices of  $P_4$ . Then we see that  $(G')^{101}$  has a subgraph  $K_{2,3}$  such that  $i(K_{2,3}^{101}) = 2$ . Hence  $G^{101}$  is not outerplanar, a contradiction.

Conversely, we see that  $K_2^{101} = K_3$  and  $K_{1,n}^{101}$  for  $n \geq 2$  is the block-stargraph  $B_{1,n}$  where each block is a triangle. Also  $P_4^{101}$  is a planar graph with innervertex number 0.

**Theorem 2.3.** The block-transformation graph  $G^{101}$  of a connected graph  $G$  is minimally nonouterplanar if and only if it satisfies one of the following conditions:

1.  $G$  is a cycle.
2.  $G$  is a cycle whose any one vertex is adjoined with  $n$  endedges,  $n \geq 1$ .
3.  $G$  is a cycle whose any two adjacent vertices each of which is adjoined with an endedge.

**Proof.** Suppose  $G^{101}$  is minimally nonouterplanar. Then by Theorem 2.1,  $G$  has at most three cutvertices. Assume one of the blocks  $B$  has a diagonal edge. Then in any planar drawing of  $G^{101}$ , the block-vertex  $B_2$  is adjacent to all the incident vertices of block  $B$  giving at least 2 inner vertices. Therefore each block of  $G$  is an edge or a cycle. Assume all the blocks of  $G$  are edges. Then by Theorem 2.2,  $G$  is outerplanar. Therefore,  $G$  has at least 1 cycle. Let  $G$  has at least 2 cycles. Then in any planar drawing of  $G^{101}$  we have two subgraphs homeomorphic to wheels, hence giving at least two inner vertices. Therefore  $G$  has exactly one cycle. Hence we conclude  $G$  has at most three cutvertices such that  $G$  has exactly 1 cycle and the other blocks if any are edges. Next we observe the following cases :

**Case 1.**  $G$  has 2 cutvertices. Then by Theorem 2.1, only one of the cutvertices can be incident with more than 2 blocks. Assume  $G$  has one of the cutvertices incident with at least 3 blocks. As in the proof of Theorem 2.2,  $i(G^{101}) \geq 2$ , a contradiction. Assume  $G$  is a cycle together with 2 vertices adjoined to 2 nonadjacent vertices of the cycle. Then we see that  $i(G^{101}) \geq 2$ , a contradiction.

**Case 2.**  $G$  has 3 cutvertices. By Theorem 2.1, a cutvertex is incident with at most 2 cutvertices. Then  $G$  is a cycle whose any three vertices each of which is adjoined with an endedge or  $G$  is a chain graph with 4 blocks such that only one block is a cycle and the others are edges. It can be easily verified that in any optimal outerplanar drawing of  $G^{101}$  we have  $i(G^{101}) > 1$ , a contradiction.

Conversely,  $C_n^{101} = W_{n+1}$ . If  $G$  satisfies condition (2), then  $G^{101}$  is a wheel whose one vertex is adjoined with  $n$  triangles,  $n \geq 1$ . If  $G$  satisfies condition (3), then it can be easily verified that in any outerplanar drawing of  $G^{101}$  has only one inner vertex.

### 3 Graphs $G^{101}$ and crossing number 1

The following theorem characterize the graphs  $G$  for which  $G^{101}$  has crossing number 1.

**Theorem 3.1.** The block-transformation graph  $G^{101}$  of a connected outerplanar graph  $G$  has crossing number 1 if and only if  $G$  satisfies one of the following conditions :

1.  $G$  has 2 cutvertices, say  $c_1$  and  $c_2$ , such that

(i) only one of the cutvertices is incident with more than 2 blocks and the nonendblock is a cycle with a diagonal edge forming a triangle with one of the cutvertices or,

(ii) both the cutvertices are incident with 3 blocks and the nonendblock is an edge or is a block with  $c_1$  and  $c_2$  in the same region.

2.  $G$  has 3 cutvertices then,

- (i)  $G$  is a cycle with a diagonal edge forming a triangle and any three vertices of the triangle each of which are adjoined with an endblock or,
- (ii)  $G$  is a cycle whose any one vertex is adjoined to 2 endblocks and any two vertices of the cycle each of which is adjoined with an endedge or,
- (iii)  $G$  is a graph with cutvertices, say  $c_1, c_2, c_3$  such that  $c_1c_2$  and  $c_2c_3$  form the outer edges of the nonendblocks and only one of the cutvertices is incident with 3 blocks or,
- (iv)  $G$  is a chain graph with cutvertices, say  $c_1, c_2, c_3$ , such that  $c_1c_2$  or  $c_2c_3$  is not the boundary edge of the corresponding nonendblock and one of the nonendblock is a cycle.

3.  $G$  has 4 cutvertices and 5 blocks such that the cutvertices lie on the boundary edge of the nonendblocks.

**Proof.** Suppose  $G^{101}$  has crossing number 1 where  $G$  is an outerplanar graph. Assume  $G$  is a block. Then  $G^{101} = G + K_1$ , which is planar, a contradiction. Let  $G$  has a single cutvertex. Then in  $G^{101}$ , the block-vertex corresponding to each block is adjacent to all its incident vertices forming a planar block and these planar blocks are incident at the cutvertex. There are no more edges. Thus  $G^{101}$  is planar, a contradiction. Assume  $G$  has at least 5 cutvertices. Then  $G$  has at least 6 blocks. It is easy to verify that for any connected graph with 6 blocks, the block-vertices corresponding to these blocks form a subgraph homeomorphic to  $K_6$ , a contradiction. Therefore,  $G$  is outerplanar graph with only 2,3 or 4 cutvertices. Now we observe the following cases :

**Case 1.**  $G$  has 2 cutvertices. We observe the following subcases:

**Subcase 1.1.** Only one of the cutvertices is incident with more than 2 blocks and the nonendblock is an edge or a block with the cutvertices in the same region. Then by Theorem 2.1,  $G^{101}$  is planar, a contradiction.

**Subcase 1.2.** Both the cutvertices are incident with 2 blocks and the nonendblock is an edge or is a block with  $c_1$  and  $c_2$  in the same region. By Theorem 2.1,  $G^{101}$  is planar, a contradiction. If both the cutvertices are incident with at least 3 blocks then in  $G^{101}$  the block-vertices corresponding to the endblocks form a subgraph  $K_{m,n}$ , where  $m, n \geq 4$ , a contradiction.

**Case 2.**  $G$  has 3 cutvertices. We observe the following subcases:

**Subcase 2.1.**  $G$  is a cycle whose any three vertices each of which is adjoined with an endblock. Then by Theorem 2.1,  $G^{101}$  is planar, a contradiction. If the cycle is adjacent with at least 5 blocks, then the block-vertices corresponding to these blocks will be adjacent to each other forming  $K_n$ ,  $n \geq 6$ , in  $G^{101}$ , a contradiction.

**Subcase 2.2.**  $G$  is a block with a diagonal edge such that there is a cycle of length at least 4 and any three vertices of the cycle each of which is adjoined with an endblock, then in  $G^{101}$ , the block-vertex corresponding to this block is adjacent with all the vertices of the block giving at least 2 crossings, a contradiction.

**Subcase 2.3.**  $G$  is a chain graph with cutvertices, say  $c_1, c_2, c_3$  such that  $c_1c_2$  and  $c_2c_3$  is the boundary edge of the corresponding nonendblock. Then by Theorem 2.1,  $G^{101}$  is planar.

**Subcase 2.4.**  $G$  is a chain graph with cutvertices, say  $c_1, c_2, c_3$  such that  $c_1c_2$  and  $c_2c_3$  are not the boundary edge of the corresponding nonendblocks. Then in  $G^{101}$  we have at least 2 crossings, a contradiction.

**Case 3.**  $G$  has 4 cutvertices. We observe the following subcases :

**Subcase 3.1.** Assume  $G$  has 4 cutvertices and at least 6 blocks. Then we have subgraph homeomorphic to  $K_m$ ,  $m \geq 6$ , in  $G^{101}$ , a contradiction.

**Subcase 3.2.** Assume  $G$  has 4 cutvertices and 5 blocks such that at least one of the nonendblock is such that its cutvertices do not form the boundary edge. Then in  $G^{101}$ , the block-vertices corresponding to these blocks form a graph homeomorphic to  $K_5$  and the block-vertex corresponding to the nonendblock is adjacent to the incident vertices of its block giving at least one crossing. Thus crossing number of  $G^{101} \geq 2$ , a contradiction.

Conversely, let  $G$  be an outerplanar graph. Since  $G$  is outerplanar, each block can be drawn in such a way that the incident vertices lie on the boundary edge of the blocks. The block-vertices corresponding to the endblocks are taken in the exterior region of the blocks. Let  $G$  satisfies condition 1(i), then in  $G^{101}$  the block-vertices corresponding to endblocks are placed in the exterior region of the blocks and are made adjacent with the incident vertices of the corresponding blocks forming planar blocks. Since only one cutvertex is incident with more than 2 blocks, these block-vertices form a subgraph  $K_{1,n}$  in the block graph  $B(G)$ . The block-vertex corresponding to nonendblock is kept in the interior region of the block and is made adjacent with all the incident vertices giving exactly one crossing with the diagonal edge. Let  $G$  satisfies condition 1(ii). Then, the endblocks are drawn in the interior region of the nonendblock containing  $c_1$  and  $c_2$ . The block-vertices corresponding to these blocks are kept inside the interior region of the nonendblock but in the exterior region of the corresponding blocks. The block-vertex corresponding to nonendblock is kept in the exterior region of the block and edges are drawn according to the definition of  $G^{101}$ . We see that the block-vertices corresponding to the endblocks form a subgraph  $K_{3,3}$  and  $G^{101}$  has only one set of edges giving a crossing. Let  $G$  satisfies condition 2(i). Then the endblocks are drawn in the exterior region of the cycle and the block-vertex corresponding to the cycle is kept in the region not containing the triangle. Then we see that  $G^{101}$  has a subgraph homeomorphic to  $K_{3,3}$  and has only one crossing. Let  $G$  satisfies condition 2(ii). Then it is easy to verify that in any optimum planar drawing of  $G^{101}$ , the block-vertices form a subgraph homeomorphic to  $K_5$  and there is only one crossing. Let  $G$  satisfies condition 2(iii). Then we see that the block-vertices along with the cutvertex incident with three blocks form a subgraph homeomorphic to  $K_{3,3}$ . Let  $G$  satisfies condition 2(iv).

Then in any planar drawing of  $G^{101}$ , the block-vertices corresponding to the blocks other than cycle form planar blocks in  $G^{101}$  and the adjacency of these block vertices do not form any crossing. The block-vertex corresponding to the cycle forms a wheel with the incident vertices and the inner vertex is adjacent to a block-vertex corresponding to one of the endblocks, thus giving a crossing. Let  $G$  satisfies condition 4. Then in  $G^{101}$ , the block-vertices corresponding to these 5 blocks are adjacent to the incident vertices forming planar blocks but the adjacency of the block-vertices form a subgraph homeomorphic to  $K_5$ , producing a crossing.

**Theorem 3.2.** The block-transformation graph  $G^{101}$  of a connected nonouterplanar graph  $G$  has crossing number 1 if and only if  $G$  is theta-minimally nonouterplanar with at most three cutvertices and

1 if  $G$  has two cutvertices, say  $c_1$  and  $c_2$ , then

- (i) only one of the cutvertices is incident with more than two blocks, the nonendblock is an edge or is a block such that  $c_1$  and  $c_2$  are in the same region of the block and one of the endblocks is theta-minimally nonouterplanar or,
- (ii) only one of the cutvertices is incident with more than two blocks, the nonendblock is theta-minimally nonouterplanar such that  $c_1c_2$  is the boundary edge of the thetaminimally nonouterplanar block and if  $c_1c_2$  is the boundary edge of theta wheel, then the theta-minimally nonouterplanar block has at least one more edge which is not the boundary edge of the theta wheel.

2. if  $G$  has 3 cutvertices then,

- (i)  $G$  is a cycle whose any three vertices each of which is adjoined with an endblock and any one of these endblocks is theta-minimally nonouterplanar.
- (ii)  $G$  is a cycle whose any three vertices each of which is adjoined with an endblock and the cycle has an inner vertex adjacent to a pair of vertices of length exactly 2.
- (iii)  $G$  is a chain graph with cutvertices, say  $c_1, c_2, c_3$  such that  $c_1, c_2$  and  $c_2, c_3$  form the boundary edges of the nonendblocks and one of the endblocks is theta-minimally nonouterplanar.
- (iv)  $G$  is a chain graph with cutvertices, say  $c_1, c_2, c_3$  such that  $c_1, c_2$  and  $c_2, c_3$  form the boundary edges of the nonendblocks and, one of the nonendblocks is theta-minimally nonouterplanar block with at least one edge ( $\neq c_1, c_2$  or  $c_2, c_3$ ) is the boundary edge of the theta wheel.

**Proof.** Suppose the block-transformation graph  $G^{101}$  of a connected nonouterplanar graph  $G$  has crossing number 1. Assume  $i(G) > 1$ . Suppose  $G$  has a unique block  $B$  which is homeomorphic to any one of the graphs  $H_i, i = 13, \dots, 20$ , (see, Figure 1.3) and remaining blocks are outerplanar. Then the block-vertex  $B$  in  $G^{101}$  is adjacent to all vertices of block  $B$ . The edges joining the inner vertices and the block-vertex  $B$  cross at least 2 edges of  $B$ . Thus crossing number of  $G^{101}$  is greater than 1, a contradiction. Suppose  $G$  has at least two blocks say  $B_1$  and  $B_2$  each of which is minimally nonouterplanar and the remaining blocks are outerplanar. Then the block-vertices corresponding to these blocks are adjacent to the incident vertices of their corresponding blocks and form at least 2 blocks containing subgraphs homeomorphic to  $K_{3,3}$  or  $K_5$ , remaining blocks are outerplanar. Clearly  $G^{101}$  has crossing number exceeding 1, a contradiction. Suppose  $G$  is not theta-minimally nonouterplanar. Then a block  $B$  of  $G$  is homeomorphic to  $H_{21}$  or  $H_{22}$ , (see, Figure 1.3). Then in  $G^{101}$  the block-vertex corresponding to block  $B$  is adjacent to all the vertices of block  $B$  giving two crossings, a contradiction. Therefore  $G$  is theta-minimally nonouterplanar. Suppose  $G$  has at least 4 cutvertices.

By Theorem 2.1, any outerplanar graph with 4 cutvertices has at least one crossing. Since we have a theta-minimally nonouterplanar block in  $G$ , the block-vertex corresponding to this block gives a crossing when made to adjacent to the vertices of the block. Hence  $G^{101}$  has at least two crossing if  $G$  is theta-minimally nonouterplanar with at least 4 cutvertices. Therefore, we conclude that if  $G^{101}$  has a crossing number 1 then  $G$  is theta-minimally nonouterplanar graph with at most 3 cutvertices. Our proof relies on the fact that if  $G^{101}$  is planar when  $G$  is outerplanar, then in same graph  $G$  we replace a block with a theta-minimally nonouterplanar graph. The block-vertex corresponding to this block gives a crossing when made to adjacent with all the vertices of the block and the other edges in  $G^{101}$  produce no other crossings and any other change in this  $G$  will give at least one more crossing

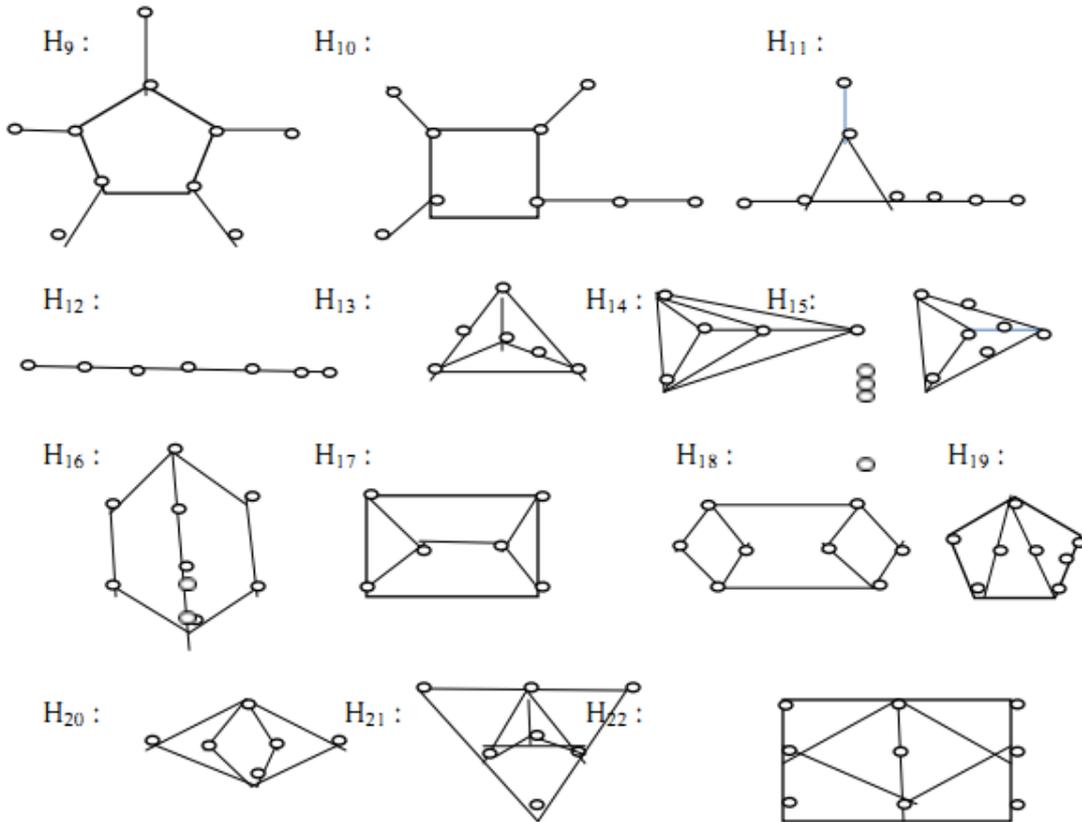


Figure 1.3

Now we consider the following cases:

**Case 1.**  $G$  has 2 cutvertices, say  $c_1$  and  $c_2$ . Suppose only one of the cutvertices is incident with more than two blocks, the nonendblock is theta-minimally nonouterplanar such that  $c_1c_2$  is the boundary edge of the theta-minimally nonouterplanar block and is the only edge of theta wheel which is boundary edge. We see that in any planar drawing of  $G^{101}$ , the block-vertex corresponding to this nonendblocks is adjacent to all the vertices of the block forming 2 crossings, a contradiction.

**Case 2.**  $G$  has 3 cutvertices. Suppose  $G$  is a cycle together with three blocks adjoined to three vertices of the cycle and the cycle has an inner vertex adjacent to a pair of vertices of length  $\geq 2$ . Then the block-vertex corresponding to these blocks is adjacent to the incident vertices of its corresponding block giving at least 2 crossings, a contradiction. Suppose  $G$  is a chain graph with cutvertices, say  $c_1, c_2, c_3$  such that  $c_1c_2$  and  $c_2c_3$  form the boundary edges of the nonendblocks and, one of the nonendblocks is theta-minimally nonouterplanar block and  $c_1c_2$  (or  $c_2c_3$ ) is the only boundary edge of the theta wheel. Then as argued in case (1), we arrive at a contradiction.

Conversely, let  $G$  is theta-minimally nonouterplanar. Since  $G$  is theta-minimally nonouterplanar, there exists exactly one block in  $G$  which is theta-minimally nonouterplanar. Then vertex corresponding to this block is adjacent with all the vertices of the block producing a subgraph as  $K_{3,3}$  or  $K_5$  in  $G^{101}$ . The remaining blocks if any of  $G$  are outerplanar. The vertices corresponding to these blocks lie on exterior region and are adjacent to the vertices of the corresponding blocks forming planar blocks in  $G^{101}$ . The vertices corresponding to nonadjacent blocks can be easily made adjacent without giving any crossing. Thus  $G^{101}$  has exactly one subgraph as  $K_{3,3}$  or  $K_5$ . Thus  $Cr(G^{101}) = 1$

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