

ON $\hat{\alpha}g$ COMPACT SPACES

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Keywords: $\hat{\alpha}g$ compact sets, $\hat{\alpha}g$ compact spaces, $\hat{\alpha}g$ open sets.

ABSTRACT. We introduce and study some subsets of a topological space X called $\hat{\alpha}g$ compact sets; $\hat{\alpha}g$ compact spaces are defined and their properties are studied.

1. INTRODUCTION

Throughout the paper (X, τ) denotes a topological space in which no separation axiom is assumed. (X, τ) will be simply denoted by X . Levine [3] initiated semi open sets in topological spaces. Several spaces are defined in terms of semi open sets such as S – Compact [5], S – closed space [10], s -closed space [2] etc. In section 3 of the present work, we introduce the notion of αg compact spaces and study their properties.

2. PRELIMINARIES

Definition 2.1: A subset A of a topological space X is said to be

1. pre open [4] if $A \subset \text{int cl } A$ and pre closed, if $\text{cl int } A \subset A$.
2. regular open [4] if $A = \text{int cl } A$ and regular closed if $A = \text{cl int } A$.
3. semi open [4] if $A \subset \text{cl int } A$ and semi closed if $\text{int cl } A \subset A$.

Definition 2.2: A subset A of a topological space X is said to be $\hat{\alpha}g$ closed if $\text{int cl int } A \subset U$ whenever $A \subset U$ and U is open in X .

The complement of $\hat{\alpha}g$ closed set in X is called $\hat{\alpha}g$ open set in X .

The union of all $\hat{\alpha}g$ open sets contained in A is called $\hat{\alpha}g$ interior of A and is denoted by $\hat{\alpha}g \text{ int } A$. In general $\hat{\alpha}g \text{ int } A$ is not $\hat{\alpha}g$ open.

The intersection of all $\hat{\alpha}g$ closed sets containing A is called the $\hat{\alpha}g$ closure of A and is denoted by $\hat{\alpha}g \text{ cl } A$. In general $\hat{\alpha}g \text{ cl } A$ is not $\hat{\alpha}g$ closed.

In what follows we assume the intersection of $\hat{\alpha}g$ closed sets is $\hat{\alpha}g$ closed. Then $\hat{\alpha}g \text{ int } A$ is $\hat{\alpha}g$ open and $\hat{\alpha}g \text{ cl } A$ is $\hat{\alpha}g$ closed. Also we assume the union of $\hat{\alpha}g$ closed sets is $\hat{\alpha}g$ closed.

3. $\hat{\alpha}g$ COMPACT SPACES

Definition 3.1: By a $\hat{\alpha}g$ open cover (or covering) of a subset A of a topological space X , we mean a collection $C = \{ G_\lambda : \lambda \in \nabla \}$ of $\hat{\alpha}g$ open subsets of X such that $A \subset \cup \{ G_\lambda : \lambda \in \nabla \}$

Definition 3.2: A subset A of a topological space X is said to be $\hat{\alpha}g$ compact if and only if every $\hat{\alpha}g$ open cover of A has a finite subcover.

In particular, X is said to be $\hat{\alpha}g$ compact if and only if for every collection $\{ G_\lambda : \lambda \in \nabla \}$ of $\hat{\alpha}g$ open sets for which $X = \cup \{ G_\lambda : \lambda \in \nabla \}$ there exists finitely many sets $G_{\lambda_1}, G_{\lambda_2}, \dots, G_{\lambda_n}$ among the G_λ such that $X = \cup \{ G_{\lambda_i} : i = 1, 2, \dots, n \}$

Remark 3.3: Finite union of $\hat{a}g$ compact sets is compact.

Definition 3.4: A topological space is said to be $\hat{a}g$ Hausdorff if for distinct points x, y of X , there exist disjoint $\hat{a}g$ open sets M and N containing x and y respectively.

Theorem 3.5: Every $\hat{a}g$ compact subset of a $\hat{a}g$ Hausdorff space is closed.

Proof: Let A be a $\hat{a}g$ compact subset of a $\hat{a}g$ Hausdorff space X .

Let us prove A' is $\hat{a}g$ open. Let $p \in A'$.

As X is $\hat{a}g$ Hausdorff, for every $q \in A$, there exist disjoint $\hat{a}g$ open sets $M(q)$ and $N(q)$ of p and q respectively.

The collection $\{N(q) : q \in A\}$ is a $\hat{a}g$ open cover of A . Since A is $\hat{a}g$ compact, there exist finite number of $q_i, i=1, 2, \dots, n$ such that

$$A \subset \cup \{N(q_i) : i=1, 2, \dots, n\}$$

$$\text{Let } M = \cap \{M(q_i) : i=1, 2, \dots, n\}$$

$$N = \cup \{N(q_i) : i=1, 2, \dots, n\}$$

M is a $\hat{a}g$ open set containing p . We have $x \in N \Rightarrow x \in N(q_i)$, for some i

Therefore $x \notin M(q_i)$ and hence $x \notin M$

Hence $M \cap N = \phi$. $A \subset N$. So, $M \cap A = \phi$

So, $M \subset A'$. A' contains $\hat{a}g$ neighbourhood of each of its points. So, A' is $\hat{a}g$ open and hence A is $\hat{a}g$ closed.

Theorem 3.6: Every $\hat{a}g$ closed subset of a compact space is $\hat{a}g$ compact.

Proof: Let F be a $\hat{a}g$ closed subset of a $\hat{a}g$ compact space X . Let $\mathbf{C} = \{G_\lambda : \lambda \in \nabla\}$ be a $\hat{a}g$ open cover of F . Then the collection $\mathbf{D} = \{G_\lambda : \lambda \in \nabla\} \cup \{X - F\}$ forms $\hat{a}g$ open cover of X . Since X is compact, there is a finite subcollection \mathbf{D}' of \mathbf{D} that covers X .

If $X - F$ is a member of \mathbf{D}' , remove it from \mathbf{D}' . The remaining finite collection will cover F . So F is $\hat{a}g$ compact.

Theorem 3.7: A topological space X is $\hat{a}g$ compact if and only if every collection of $\hat{a}g$ closed subsets of X with finite intersection property (FIP) has a nonempty intersection.

Proof: Let X be $\hat{a}g$ compact. Let $F = \{F_\lambda : \lambda \in \nabla\}$ be a collection of $\hat{a}g$ closed subsets of X with the FIP. Suppose, $\cap \{F_\lambda : \lambda \in \nabla\} = \phi$. Then $\cup \{F'_\lambda : \lambda \in \nabla\} = X$, by De Morgan's law.

This means the collection $\{F'_\lambda : \lambda \in \nabla\}$ is a $\hat{a}g$ open cover of X . Since X is compact, we have $X = \cup \{F'_i : i=1, 2, \dots, n\}$, where n is finite.

Again by De Morgan's law $(\cap \{F_\lambda : \lambda \in \nabla\})' = X$

This implies $\cap \{F_\lambda : \lambda \in \nabla\} = \phi$ contradicting FIP. Hence $\cap \{F_\lambda : \lambda \in \nabla\} \neq \phi$

Conversely, let every collection of $\hat{a}g$ closed subsets of X with FIP have nonempty intersection.

Let $\mathbf{C} = \{G_\lambda : \lambda \in \nabla\}$ be a $\hat{a}g$ open cover of X . $X = \cup \{G_\lambda : \lambda \in \nabla\}$. $\phi = \cap \{G'_\lambda : \lambda \in \nabla\}$.

Thus $\{G'_\lambda : \lambda \in \nabla\}$ is a collection of $\hat{a}g$ closed sets with empty intersection. So, this collection does not have the FIP. Hence there exist a finite number of sets $G'_i, i=1, 2, \dots, n$ such that $\phi = \cap \{G'_i : i=1, 2, \dots, n\} = (\cup \{G_i : i=1, 2, \dots, n\})'$

This implies $X = \cup \{G_i : i=1, 2, \dots, n\}$.

Hence X is $\hat{a}g$ compact.

Theorem 3.8: A topological space X is $\hat{a}g$ compact if and only if every class of $\hat{a}g$ closed sets with empty intersection has a finite sub class with empty intersection.

Proof: Let X be a $\hat{a}g$ compact space. Let $\mathbf{C} = \{F_\alpha : \alpha \in \nabla\}$ be a collection of $\hat{a}g$ closed subsets of X such that $\cap \{F_\alpha : \alpha \in \nabla\} = \phi$

Taking complements $\cup \{F'_\alpha : \alpha \in \nabla\} = X$

The collection $\{F'_\alpha : \alpha \in \nabla\}$ is a $\hat{a}g$ open covering of X . As X is $\hat{a}g$ compact, there exist finite number of indices $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$\cup \{F'_{\alpha_i} : i=1, 2, \dots, n\} = X$$

Taking complements $\cap \{F_{\alpha_i} : i=1, 2, \dots, n\} = \phi$

Conversely, suppose that every class of $\hat{a}g$ closed subsets of X with empty intersection has a finite subclass with empty intersection. Let $\mathbf{C}^* = \{G_\alpha : \alpha \in \nabla\}$ be an arbitrary $\hat{a}g$ open covering of X .

$$\cup \{G_\alpha : \alpha \in \nabla\} = X.$$

Taking complements, $\cap \{G'_\alpha : \alpha \in \nabla\} = \emptyset$

The collection $\{G'_\alpha : \alpha \in \nabla\}$ has a empty intersection. Hence, there exist finite number of indices $\alpha_1, \alpha_2, \dots, \alpha_n$ such that,

$$\cap \{G'_{\alpha_i} : i = 1, 2, \dots, n\} = \emptyset. \text{ Taking complements. } \cup \{G_{\alpha_i} : i = 1, 2, \dots, n\} = X.$$

Hence X is $\hat{a}g$ compact.

Definition 3.9: A topological space X is said to have αg Bolzano – Weierstrass property ($\hat{a}g$ BWP) if and only if every infinite set in X has a $\hat{a}g$ limit point.

Theorem 3.10: If A is an infinite subset of a $\hat{a}g$ compact space X , then A has a $\hat{a}g$ limit point.

Proof: Let A have no $\hat{a}g$ limit point in X . Then, for every $x \in X$, there exists a $\hat{a}g$ neighbourhood N_x of x which contains no point of A other than (possibly) x . Now the collection $\{N_x : x \in X\}$ forms a $\hat{a}g$ open cover of X . Since, X is $\hat{a}g$ compact, there exist finitely many points x_1, x_2, \dots, x_n in X such that $X = \cup \{N_{x_i} : i = 1, 2, \dots, n\}$

Consequently $A \subset \cup \{N_{x_i} : i = 1, 2, \dots, n\}$. As each N_x contains at most one point of A , the above relation shows A has at most n points and hence A is finite, which is a contradiction.

This completes the proof.

Remark 3.11 : No infinite discrete space is $\hat{a}g$ compact.

Theorem 3.12 : Let A and B be disjoint $\hat{a}g$ compact subsets of a $\hat{a}g$ Hausdorff space X . Then there exist disjoint $\hat{a}g$ open subsets G and H such that $A \subset G, B \subset H$.

Proof: Let $a \in A$ be arbitrary but fixed. Let $x \in B$ be arbitrary. As $A \cap B = \emptyset, a \neq x$.

There exist disjoint $\hat{a}g$ open subsets G_x and H_x such that $a \in G_x, x \in H_x$. The collection $\{H_x : x \in B\}$ is a $\hat{a}g$ open cover of B . As B is $\hat{a}g$ compact, there exist finitely many points $x_1, x_2, \dots, x_n \in B$ such that $B \subset \cup \{H_{x_i} : i = 1, 2, \dots, n\} = H_a$ (say)

Let $G_a = \cap \{G_{x_i} : i = 1, 2, \dots, n\}$. Then G_a and H_a are disjoint $\hat{a}g$ open sets such that $a \in G_a, B \subset H_a$.

$\{G_a : a \in A\}$ is a $\hat{a}g$ open cover of A . As A is $\hat{a}g$ compact, there exist finitely many points $a_1, a_2, \dots, a_m \in A$ such that

$$A \subset \cup \{G_{a_i} : i = 1, 2, \dots, m\} = G \text{ (say).}$$

Let $H = \cap \{H_{a_i} : i = 1, 2, \dots, m\}$. Then G and H are disjoint $\hat{a}g$ open sets such that $A \subset G$ and $B \subset H$.

Theorem 3.13 ; Let X be a $\hat{a}g$ connected $\hat{a}g$ Hausdorff space [8]. Then no nonempty proper $\hat{a}g$ open subset of X is $\hat{a}g$ compact.

Proof : X is $\hat{a}g$ connected. No nonempty $\hat{a}g$ open proper subset of X can be $\hat{a}g$ closed. Thus, if G is a nonempty proper $\hat{a}g$ open subset of X , it is not $\hat{a}g$ closed. A $\hat{a}g$ compact subset of a $\hat{a}g$ Hausdorff space is closed. Since G is not $\hat{a}g$ closed, it is not $\hat{a}g$ compact.

Theorem 3.14 : If the product of two nonempty spaces is $\hat{a}g$ compact, then each factor space is $\hat{a}g$ compact.

Proof : Let $X \times Y$ be the product space of the nonempty spaces X and Y and let $X \times Y$ be $\hat{a}g$ compact.

Then the projection $\pi_1 : X \times Y \rightarrow X$ is a $\hat{a}g$ irresolute map. Hence $\pi_1 (X \times Y) = X$ is αg compact.

For the space Y , the proof is similar.

REFERENCES

- [1]. Y. Gnanambal, On gpr continuous function in topological spaces, Indian J. pure Appl. Math 30 (6) (1999) 581 – 593.
- [2] J.E. Joseph and M.H. Kwach, On s - closed spaces, Proc. Amer. Math. Soc. 80(2):341- 348, 1980.
- [3] N. Levine, Semi open sets and semi continuity in topological spaces, Amer. Math. Monthly 70 (1963) 36-41.
- [4] A.S. Mashhour, M.E. Abd El-Monsef and S.N. ElDeeb, On pre – continuous and weak pre – continuous mappings, Proc. Math and Phy. Soc. Egypt 53 (1982) 47- 53.

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- [5] S.S.Mohammad, On Semi compact sets and associated properties, Int.J.Math.Math. Sci. Article ID 465387: 1-8,2009.
- [6] O.Njastad, On some classes of nearly open sets, Pacific.J.Math 15 (3) (1965) 961-970.
- [7] V.Senthilkumaran, R.Krishnakumar and Y.Palaniappan, On α generalized closed sets, Int.J.Math Archive. Feb 2014, 187-191.
- [8] V.Senthilkumaran, R.Krishnakumar and Y.Palaniappan, α g connected spaces ,Bulletin of Society for Mathematical Services&Standards,vol(3) Issue2(2014) 19-28.
- [9] L.A.Steen and J.A.Seebach, Counter examples in topology, Holt, Rinehart and Winston Inc. U.S.A.1970.
- [10] T.Thompson, S- closed spaces, Proc.Amer.Math.Soc.60 (61): 335-338, 1976.