ON $\hat{a}g$ COMPACT SPACES

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ABSTRACT. We introduce and study some subsets of a topological space $X$ called $\hat{a}g$ compact sets; $\hat{a}g$ compact spaces are defined and their properties are studied.

1. INTRODUCTION

Throughout the paper $(X,\tau)$ denotes a topological space in which no separation axiom is assumed. $(X,\tau)$ will be simply denoted by $X$. Levine [3] initiated semi open sets in topological spaces. Several spaces are defined in terms of semi open sets such as $S-C$ompact [5], $S-$closed space [10], $s$-closed space [2] etc. In section 3 of the present work, we introduce the notion of $\hat{a}g$ compact spaces and study their properties.

2. PRELIMINARIES

Definition 2.1: A subset $A$ of a topological space $X$ is said to be
1. pre open [4] if $A \subseteq \text{int cl } A$ and pre closed, if $\text{cl int } A \subseteq A$.
2. regular open [4] if $A = \text{int cl } A$ and regular closed if $A = \text{cl int } A$.
3. semi open [4] if $A \subseteq \text{cl int } A$ and semi closed if $\text{int cl } A \subseteq A$.

Definition 2.2: A subset $A$ of a topological space $X$ is said to be $\hat{a}g$ closed if and only if $\text{int cl int } A \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
The complement of $\hat{a}g$ closed set in $X$ is called $\hat{a}g$ open set in $X$.
The union of all $\hat{a}g$ open sets contained in $A$ is called $\hat{a}g$ interior of $A$ and is denoted by $\hat{a}g \text{ int } A$.
In general $\hat{a}g \text{ int } A$ is not $\hat{a}g$ open.
The intersection of all $\hat{a}g$ closed sets containing $A$ is called the $\hat{a}g$ closure of $A$ and is denoted by $\hat{a}g \text{ cl } A$.
In general $\hat{a}g \text{ cl } A$ is not $\hat{a}g$ closed.
In what follows we assume the intersection of $\hat{a}g$ closed sets is $\hat{a}g$ closed. Then $\hat{a}g \text{ int } A$ is $\hat{a}g$ open and $\hat{a}g \text{ cl } A$ is $\hat{a}g$ closed. Also we assume the union of $\hat{a}g$ closed sets is $\hat{a}g$ closed.

3. $\hat{a}g$ COMPACT SPACES

Definition 3.1: By a $\hat{a}g$ open cover (or covering) of a subset $A$ of a topological space $X$, we mean a collection $C = \{ G\lambda : \lambda \in \mathbb{V} \}$ of $\hat{a}g$ open subsets of $X$ such that $A \subseteq \bigcup \{ G\lambda : \lambda \in \mathbb{V} \}$

Definition 3.2: A subset $A$ of a topological space $X$ is said to be $\hat{a}g$ compact if and only if every $\hat{a}g$ open cover of $A$ has a finite subcover.
In particular, $X$ is said to be $\hat{a}g$ compact if and only if for every collection $\{ G\lambda : \lambda \in \mathbb{V} \}$ of $\hat{a}g$ open sets for which $X = \bigcup \{ G\lambda : \lambda \in \mathbb{V} \}$ there exists finitely many sets $G\lambda_1, G\lambda_2, \ldots \ldots G\lambda_n$ among the $G\lambda$ such that $X = \bigcup \{ G\lambda_i : i = 1, 2, \ldots n \}$
Remark 3.3: Finite union of a g compact sets is compact.

Definition 3.4: A topological space is said to be a g Hausdorff if for distinct points x, y of X, there exist disjoint a g open sets M and N containing x and y respectively.

Theorem 3.5: Every a g compact subset of a a g Hausdorff space is closed.

Proof: Let A be a a g compact subset of a a g Hausdorff space X.

Let us prove A' is a g open. Let p ∈ A'.

As X is a g Hausdorff, for every q ∈ A, there exist disjoint a g open sets M(q) and N(q) of p and q respectively.

The collection \{N(q): q ∈ A\} is a a g open cover of A. Since A is a g compact, there exists a finite number of qi, s, i = 1, 2, …, n such that

A ⊂ ∪{N(qi): i = 1, 2, …, n}

Let M = ∩ {M(qi): i = 1, 2, …, n}

N = ∪{N(qi): i = 1, 2, …, n}

M is a a g open set containing p. We have x ∈ N ⇒ x ∈ N(qi), for some i. Therefore x ∈ M and hence x ∉ M.

Hence M ∩ N = ∅. A ⊂ N. So, M ∩ A = ∅.

So, M ∩ A' contains a g neighbourhood of each of its points. So, A' is a g open and hence A is a g compact.

Theorem 3.6: Every a g closed subset of a compact space is a g compact.

Proof: Let F be a a g closed subset of a a g compact space X. Let C = {Gλ: λ ∈ V} be a a g open cover of F. Then the collection D = {Gλ: λ ∈ V} ∪ \{X - F\} forms a a g open cover of X. Since X is compact, there is a finite subcollection D' of D that covers X.

If X - F is a member of D', remove it from D'. The remaining finite collection will cover F. So F is a g compact.

Theorem 3.7: A topological space X is a g compact if and only if every collection of a g closed subsets of X with finite intersection property (FIP) has a nonempty intersection.

Proof: Let X be a g compact. Let F = \{Fλ: λ ∈ V\} be a collection of a g closed subsets of X with the FIP. Suppose, ∩\{Fλ: λ ∈ V\} = ∅. Then \∪\{Fλ: λ ∈ V\} = X, by De Morgan’s law.

This means the collection \{Fλ: λ ∈ V\} is a a g open cover of X. Since X is compact, we have X = ∪\{Fλi: i = 1, 2, …, n\}, where n is finite.

Again by De Morgan’s law (∩\{Fλi: i = 1, 2, …, n\})' = X.

This implies ∩\{Fλi: i = 1, 2, …, n\} = ∅ contradicting FIP. Hence ∩\{Fλ: λ ∈ V\} ≠ ∅.

Conversely, let every collection of a g closed subsets of X with FIP have nonempty intersection.

Let C = \{Gλ: λ ∈ V\} be a a g open cover of X. X = ∪\{Gλ: λ ∈ V\}. ∅ = ∩\{Gλ: λ ∈ V\}.

Thus \{Gλ: λ ∈ V\} is a collection of a g closed sets with empty intersection. So, this collection does not have the FIP. Hence there exist a finite number of sets Gλi, i = 1, 2, …, n such that

∅ = ∩\{Gλi: i = 1, 2, …, n\} = (∪\{Gλi: i = 1, 2, …, n\})'

This implies X = ∪\{Gλi: i = 1, 2, …, n\}.

Hence X is a g compact.

Theorem 3.8: A topological space X is a g compact if and only if every class of a g closed sets with empty intersection has a finite sub class with empty intersection.

Proof: Let X be a g compact space. Let C = \{Fa : a ∈ V\} be a collection of a g closed subsets of X such that ∩\{Fa : a ∈ V\} = ∅.

Taking complements \∪\{Fa : a ∈ V\} = X.

The collection \{Fa : a ∈ V\} is a a g open covering of X. As X is a g compact, there exist finite number of indices a1, a2, …, an such that

\∪\{Fa:i=1,2,…,n\} = X.

Taking complements ∩\{Fai: i = 1, 2, …, n\} = ∅.

Conversely, suppose that every class of a g closed subsets of X with empty intersection has a finite subclass with empty intersection. Let C* = \{Ga : a ∈ V\} be an arbitrary a g open covering of X.
\[ \bigcup \{ G_\alpha : \alpha \in V \} = X. \]
Taking complements, \[ \bigcap \{ G'\alpha : \alpha \in V \} = \phi. \]
The collection \( \{ G'\alpha : \alpha \in V \} \) has an empty intersection. Hence, there exist finite number of indices \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that, \[ \bigcap \{ G'\alpha_i : i = 1,2,\ldots,n \} = \phi. \]
Taking complements. \[ \bigcup \{ G\alpha_i : i = 1,2,\ldots,n \} = X. \]
Hence \( X \) is \( \tilde{a} g \) compact.

**Definition 3.9:** A topological space \( X \) is said to have \( \tilde{a} g \) Bolzano – Weierstrass property (\( \tilde{a} g \) BWP) if and only if every infinite set in \( X \) has a \( \tilde{a} g \) limit point.

**Theorem 3.10:** If \( A \) is an infinite subset of a \( \tilde{a} g \) compact space \( X \), then \( A \) has a \( \tilde{a} g \) limit point.

**Proof:** Let \( A \) have no \( \tilde{a} g \) limit point in \( X \). Then, for every \( x \in X \), there exists a \( \tilde{a} g \) neighbourhood \( N_x \) of \( x \) which contains no point of \( A \) other than (possibly) \( x \). Now the collection \( \{ N_x : x \in X \} \) forms a \( \tilde{a} g \) open cover of \( X \). Since, \( X \) is \( \tilde{a} g \) compact, there exist finitely many points \( x_1,x_2,\ldots,x_n \) in \( X \) such that \( X = \bigcup \{ N_{x_i} : i= 1,2,\ldots,n \} \). Consequently \( A \subset \bigcup \{ N_{x_i} : i= 1,2,\ldots,n \} \). As each \( N_x \) contains atmost one point of \( A \), the above relation shows \( A \) has atmost \( n \) points and hence \( A \) is finite, which is a contradiction.

This completes the proof.

**Remark 3.11:** No infinite discrete space is \( \tilde{a} g \) compact.

**Theorem 3.12:** Let \( A \) and \( B \) be disjoint \( \tilde{a} g \) compact subsets of a \( \tilde{a} g \) Hausdorff space \( X \). Then there exist disjoint \( \tilde{a} g \) open subsets \( G \) and \( H \) such that \( A \subset G, B \subset H \).

**Proof:** Let \( a \in A \) be arbitrary but fixed. Let \( x \in B \) be arbitrary. As \( A \cap B = \phi \), \( a \neq x \). There exist disjoint \( \tilde{a} g \) open subsets \( G_x \) and \( H_x \) such that \( a \in G_x, x \in H_x \). The collection \( \{ H_x : x \in B \} \) is a \( \tilde{a} g \) open cover of \( B \). As \( B \) is \( \tilde{a} g \) compact, there exist finitely many points \( x_1,x_2,\ldots,x_n \) in \( B \) such that \( B \subset \bigcup \{ H_{x_i} : i= 1,2,\ldots,n \} = H_a \) (say).

Let \( G_a = \bigcap \{ G_{x_i} : i= 1,2,\ldots,n \} \). Then \( G_a \) and \( H_a \) are disjoint \( \tilde{a} g \) open sets such that \( a \in G_a, B \subset H_a \).

\( \{ G_a : a \in A \} \) is a \( \tilde{a} g \) open cover of \( A \). As \( A \) is \( \tilde{a} g \) compact, there exist finitely many points \( a_1,a_2,\ldots,am \in A \) such that \( A \subset \bigcup \{ G_{a_i} : i= 1,2,\ldots,m \} = G \) (say).

Let \( H = \bigcap \{ H_{a_i} : i= 1,2,\ldots,m \} \). Then \( G \) and \( H \) are disjoint \( \tilde{a} g \) open sets such that \( A \subset G \) and \( B \subset H \).

**Theorem 3.13:** Let \( X \) be a \( \tilde{a} g \) connected \( \tilde{a} g \) Hausdorff space[8]. Then no nonempty proper \( \tilde{a} g \) open subset of \( X \) is \( \tilde{a} g \) compact.

**Proof:** \( X \) is \( \tilde{a} g \) connected. No nonempty \( \tilde{a} g \) open proper subset of \( X \) can be \( \tilde{a} g \) closed. Thus, if \( G \) is a nonempty proper \( \tilde{a} g \) open subset of \( X \), it is not \( \tilde{a} g \) closed. A \( \tilde{a} g \) compact subset of a \( \tilde{a} g \) Hausdroff space is closed. Since \( G \) is not \( \tilde{a} g \) closed, it is not \( \tilde{a} g \) compact.

**Theorem 3.14:** If the product of two nonempty spaces is \( \tilde{a} g \) compact, then each factor space is \( \tilde{a} g \) compact.

**Proof:** Let \( X \times Y \) be the product space of the nonempty spaces \( X \) and \( Y \) and let \( X \times Y \) be \( \tilde{a} g \) compact.

Then the projection \( \pi_1 : X \times Y \to X \) is a \( \tilde{a} g \) irresolute map. Hence \( \pi_1 \) (\( X \times Y \)) = \( X \) is \( \alpha g \) compact. For the space \( Y \), the proof is similar.

**REFERENCES**


