ON \(\alpha\)G COMPACT SPACES

V.Senthilkumaran, R.Krishnakumar, Y.Palaniappan*

1Associate Professor of Mathematics, Arignar Anna Government Arts College, Musiri, Tamilnadu, India
2Assistant Professor of Mathematics, UrumuDhanalakshmi College Kattur, Trichy, Tamilnadu, India
3Associate Professor of Mathematics (Retd), Arignar Anna Government Arts College, Musiri, Tamilnadu, India

*Corresponding author

Keywords: \(\alpha\)G compact sets, \(\alpha\)G compact spaces, \(\alpha\)G open sets.

ABSTRACT. We introduce and study some subsets of a topological space \(X\) called \(\alpha\)G compact sets; \(\alpha\)G compact spaces are defined and their properties are studied.

1. INTRODUCTION

Throughout the paper \((X,\tau)\) denotes a topological space in which no separation axiom is assumed. \((X,\tau)\) will be simply denoted by \(X\). Levine [3] initiated semi open sets in topological spaces. Several spaces are defined in terms of semi open sets such as \(S\)–Compact [5], \(S\)–closed space [10], \(s\)–closed space [2] etc. In section 3 of the present work, we introduce the notion of \(\alpha\)G compact spaces and study their properties.

2. PRELIMINARIES

Definition 2.1: A subset \(A\) of a topological space \(X\) is said to be
1. pre open [4] if \(A \subset \text{int} \text{ cl} A\) and pre closed, if \(\text{cl} \text{ int} A \subset A\).
2. regular open [4] if \(A = \text{int} \text{ cl} A\) and regular closed if \(A = \text{cl} \text{ int} A\).
3. semi open [4] if \(A \subset \text{cl} \text{ int} A\) and semi closed if \(\text{int} \text{ cl} A \subset A\).

Definition 2.2: A subset \(A\) of a topological space \(X\) is said to be \(\alpha\)G closed if \(\text{int} \text{ cl} \text{ int} A \subset U\) whenever \(A \subset U\) and \(U\) is open in \(X\).
The complement of \(\alpha\)G closed set in \(X\) is called \(\alpha\)G open set in \(X\).
The union of all \(\alpha\)G open sets contained in \(A\) is called \(\alpha\)G interior of \(A\) and is denoted by \(\alpha\)G int \(A\).
In general \(\alpha\)G int \(A\) is not \(\alpha\)G open.
The intersection of all \(\alpha\)G closed sets containing \(A\) is called the \(\alpha\)G closure of \(A\) and is denoted by \(\alpha\)G cl \(A\).
In general \(\alpha\)G cl \(A\) is not \(\alpha\)G closed.
In what follows we assume the intersection of \(\alpha\)G closed sets is \(\alpha\)G closed. Then \(\alpha\)G int \(A\) is \(\alpha\)G open and \(\alpha\)G cl \(A\) is \(\alpha\)G closed. Also we assume the union of \(\alpha\)G closed sets is \(\alpha\)G closed.

3. \(\alpha\)G COMPACT SPACES

Definition 3.1: By a \(\alpha\)G open cover ( or covering ) of a subset \(A\) of a topological space \(X\), we mean a collection \(C = \{ G_\lambda: \lambda \in \mathbb{V} \}\) of \(\alpha\)G open subsets of \(X\) such that \(A \subset \mathbb{\cup} \{ G_\lambda: \lambda \in \mathbb{V} \}\)

Definition 3.2: A subset \(A\) of a topological space \(X\) is said to be \(\alpha\)G compact if and only if every \(\alpha\)G open cover of \(A\) has a finite subcover.
In particular, \(X\) is said to be \(\alpha\)G compact if and only if for every collection \(\{ G_\lambda: \lambda \in \mathbb{V} \}\) of \(\alpha\)G open sets for which \(X = \mathbb{\cup} \{ G_\lambda: \lambda \in \mathbb{V} \}\) there exists finitely many sets \(G_\lambda 1,G_\lambda 2,........G_\lambda n\) among the \(G_\lambda\) such that \(X = \mathbb{\cup} \{ G_\lambda i: i = 1,2,....n\}\)
Remark 3.3: Finite union of āg compact sets is compact.

Definition 3.4: A topological space is said to be āg Hausdorff if for distinct points x, y of X, there exist disjoint āg open sets M and N containing x and y respectively.

Theorem 3.5: Every āg compact subset of a āg Hausdorff space is closed.

Proof: Let A be a āg compact subset of a āg Hausdorff space X. Let us prove A’ is āg open. Let p ∈ A’. As X is āg Hausdorff, for every q ∈ A, there exist disjoint āg open sets M (q) and N(q) of p and q respectively.

The collection \{ N (q) : q ∈ A \} is a āg open cover of A. Since A is āg compact, there exists finite number of qi, s, i = 1, 2, … , n such that

A ⊆ ∪{N(qi) : i = 1, 2, … , n}

Let M = ∩{M(qi) : i = 1, 2, … , n}

N = ∪{N(qi) : i = 1, 2, … , n}

M is a āg open set containing p. We have x ∈ N ⇒ x ∈ N(qi), for some i

Therefore x ∈ M(qi) and hence x ∈ M

Hence M ∩ N = φ. A ⊆ N. So, M ∩ A = φ

So, McaA’. A’ contains āg neighbourhood of each of its points. So, A’ is āg open and hence A is āg closed.

Theorem 3.6: Every āg closed subset of a compact space is āg compact.

Proof: Let F be a āg closed subset of a āg compact space X. Let C = { Gλ : λ ∈ V } be a āg open cover of F. Then the collection D = { Gλ : λ ∈ V } ∪ { X − F } forms āg open cover of X. Since X is compact, there is a finite subcollection D’ of D that covers X.

If X − F is a member of D’, remove it from D’. The remaining finite collection will cover F. So F is āg compact.

Theorem 3.7: A topological space X is āg compact if and only if every collection of āg closed subsets of X with finite intersection property (FIP) has a nonempty intersection.

Proof: Let X be āg compact. Let F = { Fλ : λ ∈ V } be a collection of āg closed subsets of X with the FIP. Suppose, ∩{Fλ : λ ∈ V } = φ. Then ∪{F’λ : λ ∈ V } = X, by De Morgan’s law.

This means the collection {F’λ : λ ∈ V } is a āg open cover of X. Since X is compact, we have X = ∪{F’λi : i = 1, 2, … , n}, where n is finite.

Again by De Morgan’s law (∩{Fλi : i = 1, 2, … , n})’ = X

This implies ∩{Fλi : i = 1, 2, … , n} = φ contradicting FIP. Hence ∩{Fλ : λ ∈ V } ≠ φ

Conversely, let every collection of āg closed subsets of X with FIP have nonempty intersection.

Let C = { Gλ : λ ∈ V } be a āg open cover of X. X = ∪{ Gλ : λ ∈ V }. φ = ∩{ G’λ : λ ∈ V }.

Thus { G’λ : λ ∈ V } is a collection of āg closed sets with empty intersection. This collection does not have the FIP. Hence there exists a finite number of sets Gλi , i = 1, 2, … , n such that

φ = ∩{ G’λi : i = 1, 2, … , n} = (∩{ Gλi : i = 1, 2, … , n})’

This implies X = ∪{ Gλi : i = 1, 2, … , n}.

Hence X is āg compact.

Theorem 3.8: A topological space X is āg compact if and only if every class of āg closed sets with empty intersection has a finite sub class with empty intersection.

Proof: Let X be a āg compact space. Let C = { Fa : α ∈ V } be a collection of āg closed subsets of X such that ∩{ Fa : α ∈ V } = φ

Taking complements ∪{ F’α : α ∈ V } = X

The collection { F’α : α ∈ V } is a āg open covering of X. As X is āg compact, there exist finite number of indices a1, a2, … , an such that

∪{ F’αi : i = 1, 2, … , n } = X

Taking complements ∩{ Fai : i = 1, 2, … , n } = φ

Conversely, suppose that every class of āg closed subsets of X with empty intersection has a finite subclass with empty intersection. Let C* = { Ga : λ ∈ V } be an arbitrary āg open covering of X.
\[ \bigcup \{ G\alpha : \alpha \in \nabla \} = X. \]
Taking complements, \( \bigcap \{ G'\alpha : \alpha \in \nabla \} = \phi \)
The collection \( \{ G'\alpha : \alpha \in \nabla \} \) has an empty intersection. Hence, there exist finite number of indices \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that,
\[ \bigcap \{ G'\alpha_i : i = 1,2,\ldots,n \} = \phi. \]
Taking complements. \( \bigcup \{ G\alpha_i : i = 1,2,\ldots,n \} = X. \)
Hence X is a compact.

**Definition 3.9:** A topological space X is said to have a Bolzano–Weierstrass property (a BWP) if and only if every infinite subset in X has a limit point.

**Theorem 3.10:** If A is an infinite subset of a compact space X, then A has a limit point.

**Proof:** Let A have no limit point in X. Then, for every \( x \in X \), there exists a neighbourhood \( N_x \) of x which contains no point of A other than (possibly) x. Now the collection \( \{ N_x : x \in X \} \) forms an open cover of X. Since, X is compact, there exist finitely many points \( x_1,x_2,\ldots,x_n \) in X such that \( X = \bigcup \{ N_{x_i} : i = 1,2,\ldots,n \} \).
Consequently A \( \subseteq \bigcup \{ N_{x_i} : i = 1,2,\ldots,n \} \). As each \( N_{x_i} \) contains at most one point of A, the above relation shows A has at most n points and hence A is finite, which is a contradiction.
This completes the proof.

**Remark 3.11:** No infinite discrete space is compact.

**Theorem 3.12:** Let A and B be disjoint compact subsets of a Hausdorff space X. Then there exist disjoint open subsets \( G \) and \( H \) such that \( A \subseteq G, B \subseteq H. \)

**Proof:** Let \( a \in A \) be arbitrary but fixed. Let \( x \in B \) be arbitrary. As \( A \cap B = \phi, a \neq x. \)
There exist disjoint open subsets \( G_x \) and \( H_x \) such that \( a \in G_x, x \in H_x. \)

The collection \( \{ H_x : x \in B \} \) is a open cover of B. As B is compact, there exist finitely many points \( x_1,x_2,\ldots,x_n \in B \) such that \( B \subseteq \bigcup \{ H_{x_i} : i = 1,2,\ldots,n \} = H (say) \).
Let \( G = \bigcap \{ G_x : i = 1,2,\ldots,n \}. \) Then G and H are disjoint open sets such that \( a \in G, B \subseteq H. \)

**Theorem 3.13:** Let X be a connected Hausdorff space. Then no nonempty proper open subset of X is compact.

**Proof:** X is a connected. No nonempty open proper subset of X can be closed. Thus, if \( G \) is a nonempty proper open subset of X, it is not closed. A compact subset of a Hausdorff space is closed. Since G is not closed, it is not compact.

**Theorem 3.14:** If the product of two nonempty spaces is compact, then each factor space is compact.

**Proof:** Let \( X \times Y \) be the product space of the nonempty spaces X and Y and let \( X \times Y \) be compact.
Then the projection \( \pi_1:X \times Y \to X \) is a irresolute map. Hence \( \pi_1 (X \times Y) = X \) is compact.
For the space Y, the proof is similar.

**REFERENCES**


