A proof of Pillai’s conjecture
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Abstract Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1, \( X>1 \) and \( Y>1 \), for which with exponents strictly greater than 1, \( p>1 \) and \( q>1 \), \( Y^p=X^q+1 \) but for \( (X,Y,p,q) = (2,3,2,3) \). We can verify that \( 3^2=2^3+1 \) Euler has proved that the equation \( Y^2=X^3+1 \) has this only solution. We propose in this study a general solution. The particular cases already solved concern \( p=2 \), solved by Ko Chao in 1965, and \( q=3 \) which has been solved in 2002. The case \( q=2 \) has been solved by Lebesgue in 1850. We solve here the equation for the general case. We generalize the proof to Pillai’s conjecture \( Y^p=X^q+a \) and prove that it has always a finite number of solutions for a fixed \( a \)

The approach

Let \( c = \frac{X^p-1}{Y^p} \) and \( c' = \frac{7-X^p}{Y^p} \)

We have

\[
(c+c'Y^p)^2 = X^p - 1 + 7 - X^p \Rightarrow Y^2 = \frac{6}{c+c'}
\]

\[
X^p = cY^2 + 1 = \frac{7c+c'}{c+c'}
\]

And

\[
X^q = Y^p - 1 = \frac{36-(c+c')^2}{(c+c')^2}
\]

We have \( c+c'>0 \) and \( 7c+c'>0 \) and \( c'<0 \) (else \( cY^2 = 7-X^p > 0 \Rightarrow X^p = 4 \)) we have also

\[
(2c^2X^q-X^2)(c+c')^2 = 72c^2 - 2c^2(c+c')^2 - (7c+c')^2
\]

\[
> 64c^2 - (7c+c')^2 = (c-c')(15c+c') > 0
\]

And

\[
(c^2X^q-X^2)(c+c')^2 = 36c^2 - c^2(c+c')^2 - (7c+c')^2
\]

\[
< 36c^2 - (7c+c')^2 = -(c+c')(6+7c+c') < 0
\]
If $c > 1$ then
\[
(X^q - X^{2p}) (c + c')^2 = 36 - (c + c')^2 - (7c + c')^2
\]
\[
< 36 - (7c + c')^2 = (6 - 6c - c')(6 + 7c + c') < 0
\]
Thus $q + 1 \leq 2p$ and $2c^2 > X^{2p-q} \geq X$, let $u_c = u \cdot c' = 1$ and
\[
X^p = \frac{7c + c'}{c + c'} = \frac{7u + u'}{u + u'}
\]
\[
Z = \frac{7c' + c}{c + c'} = \frac{7u + u'}{u + u'}
\]
\[
8X^q - (Z + X^p)^2 = \frac{288 - 8(c + c')^2 - 64(c + c')^2}{(c + c')^2} > 0
\]
Thus $8X^{2p-q} > (ZX^{n+1})^2$ but
\[
Z^2X^{-2p} = \left(\frac{7c' + c}{7c + c'}\right)^2 = \frac{49u_c' + u'c + 14}{49u'_c + u'c + 14} = \frac{49c'^2 + c^2 + 14cc'}{49c^2 + c'^2 + 14cc'}
\]
Let $v$ verifying
\[
vZ^2X^{-2p} - 1 = \frac{49vc'^2 + vc^2 + 14vc'}{49c^2 + c'^2 + 14cc'} - 1
\]
\[
= \frac{(49v - 1)c'^2 + (v - 49)c^2 + 14cc'(v - 1)}{14c^2 + c'^2 + 14cc'} > 0
\]
\[
v = 1 + \frac{1}{n} = \frac{X^p}{2X^p - 16} > 0 \implies n = \frac{12X^p - 96}{-6X^p + 96}
\]
We have
\[
((49v - 1)c'^2 + (v - 49)c^2 + 14cc'(v - 1))Y^p
\]
\[
= (36v - 36)X^2^p - 576vX^p + (49^2 - 97)v > 0
\]
\[
\Rightarrow 36X^2^p + (n+1)(-576X^p + 2304)
\]
\[
= 36X^2^p + \frac{-6X^p}{6X^p - 96}(-576X^p + 2304) > 0
\]
Hence $vZ^2 > X^{2p}$ and
\[
(ZX^{-1} + 1)^2 > \frac{2n + 1}{n + 1} + \frac{14c' + 2c}{7c + c'} = \frac{(16n + 9)c + (16n + 15)c'}{(n + 1)(7c + c')} > 1
\]
Because $(9n+2)c + (15n+14)c' = 0$ or
\[
(9n + 2)(X^p - 1) + (15n + 14)(7 - X^p) = (-6n - 12)X^p + 96n + 96 = 0
\]
Thus $q + 1 \geq 2p \geq q + 1 \Rightarrow q + 1 = 2p$ now let v and m verifying
\[ v = \frac{(13 - 10^{-m})c + (5 - 10^{-m-1})c'}{32(c^2 - c'^2)} \]
and
\[ 7c - c' = 32v(c + c')(c - c') > 4(c - c') \]
\[ 7c - c' = (13 - 10^{-m})c + (5 - 10^{-m-1})c' > 4(c - c') \]
Thus
\[ (7 - 32v(c + c'))cY^2 = (1 - 32v(c + c'))c'Y^2 \]
\[ (7 - 32v(c + c'))(X^p - 1) = (32v(c + c') - 1)(X^p - 7) \]
\[ 0 < 8(8(c + c')v - 1)X^p = 256(c + c')v - 14 \]
Because
\[ 64(c + c')v - 8 = \frac{2(13 - 10^{-n})c + 2(5 - 10^{-m-1})c - 8(c - c')}{c - c'} \]
\[ = \frac{(18 - 2(10^{-m}))c + (18 - 2(10^{-m-1}))c'}{c - c'} \]
\[ = \frac{6(c + c')}{c - c'} > 0 \]
Hence
\[ X^p = \frac{128(c + c')c - 7}{32(c + c')v - 4} = \frac{(45 - 4(10^{-m}))c + (13 - 10^{-m-1})c'}{13 - 10^{-m}}c + (5 - 10^{-m-1})c' < 16 \]
Because
\[ (163 - 12(10^{-m}))c + (67 - 15(10^{-m-1}))c' = \frac{(815 - 163(10^{-m-1}))c'}{6} + \frac{163(10^{-m})c}{6} + (67 - 15(10^{-m-1}))c' \]
\[ = \frac{-413 + 155(10^{-m})}{6}c' + \frac{163(10^{-m})}{6}c > 0 \]
\[ \Rightarrow X^p = 4 \Rightarrow Y^2 = 2^4 + 1 \Rightarrow (X, p, q, Y) = (2, 2, 3, \pm 3) \]
Another proof:
We have $X^{2p-q} = \frac{n+1}{n} > X$ hence
\[ \frac{1}{n+1} < \frac{1}{nX} < 1 \text{ and } \frac{-1}{n+1} > \frac{-1}{nX} \text{ thus} \]
\[ X^{q-2p} = \frac{n}{n+1} = 1 - \frac{1}{n+1} > 1 - \frac{1}{nX} \]
\[ X^{q-2p} > X^2 - \frac{X}{n} = X\left(\frac{nX - 1}{n}\right) > X(\frac{1}{2})X \geq X \]
Deduce $X^{q+1-2p} > 1$ and we have simultaneously $1 \geq X^{q+1-2p} \geq 1 \Rightarrow q + 1 = 2p$ and we have Catalan solutions!
If $c < 1$ $2 > 2c^2 X^{2p-9} > c^2 \Rightarrow q + 1 \geq 2p$ we have $Z' = \frac{9(c + c')^2 - 36}{(c + c')^2}$
\[ 16X^{2p} - (Z' + X^p)^2 = \frac{16(7c + c')^2}{(c + c')^2} > 64 > 0 \text{ let } 8n = Y^p - 9 \Rightarrow \frac{n}{n+1} = \frac{Y^p - 9}{Y^p - 1} \]
Thus
And $16X^{2p-q} > X^q(ZX^{-q} + 1)^2 = X^q((\frac{Y^p - 9}{Y^p - 1})^2 + 1 + 2\frac{9 - Y^p}{Y^p - 1})$

$= X^q(\frac{2n^2 + 2n + 1}{n^2 + 2n + 1} + 2\frac{9 - Y^p}{Y^p - 1}) = X^q(\frac{-2nY^p - Y^p + 16n^2 + 34n + 17}{(n+1)(Y^p - 1)}) = X^q > 1$

Because $16n^2 + 34n + 18 = 2(8n + 9)(n + 1) = 2(n+1)Y^p$

Thus $X^{q-2p} < \frac{16}{X} < 1$ as $q$ is odd it means $2p \geq q \geq 2p - 1 \implies q = 2p - 1$

And

$0 > (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q = X^{2p} - 2X^p - X^{2p-1} > 0 \implies c^2 - 1 = 0$

$\implies X^{p-1} - X^{q-p-1} = \frac{2}{X} \in N \implies X = 2 \implies 2^{p-1} = 2^{q-p-1} + 1 \implies q + 1 = 2p = 4$

$\implies X^p = 4 \implies Y^2 = 2^q + 1 \implies (X, p, q, Y) = (2, 2, 3, 3)$

Generalization to Pillai’s conjecture

Pillai equation is $Y^p = X^q + a = X^q + I_a$

Let

$c = \frac{X^p - 1_a}{Y^2}$

$c' = \frac{7_a - X^p}{Y^2}$

$(c + c')Y^2 = X^p - 1_a + 7_a - X^p = 6_a \implies \frac{Y^p}{c + c'}$

$X^p = \frac{Y^p}{c + c'} + 1_a = \frac{7_a c + 1_a c'}{c + c'}$

$X^q = Y^p - 1_a = \frac{36_a 1_a - 1_a (c + c')^2}{(c + c')}$

We have $c + c' > 0$ and $c > 0$ and $c > c'$ and $7_a c + 1_a c' > 0$ and $c' < 0$ (else there is a finite number of solutions)

$\frac{Y^p}{c + c'} \geq 3_a \implies 2_a \geq c + c'$

Thus if $c^2 > 1_a$ we have
Thus $q+1 \leq 2p$ but with the same arguments than higher for Catalan equation, we have $q+1 = 2p = 4$ and there is a finite number of solutions (we have $X < 16$).

And if $c^2 \leq 1$, we have $2c^2 > X^{2p-q} \geq X$ thus with the same arguments than higher for Catalan equation, we have $q+1 = 2p = 4$ and there is a finite number of solutions!

Catalan equation is solved, an original solution exists! We have generalized the approach to Pillai equation and proved that it always has a finite number of solutions. It is the proof of Pillai conjecture. It seems that many problems of number theory can be solved like this.

**Bibliography**