A proof of Pillai's conjecture

Jamel Ghanouchi

6 Rue Khansa 2070 Marsa Tunisia

Keywords : Diophantine equations; Catalan equation; Pillai conjecture; Algebraic resolution

Abstract Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1, $X > 1$ and $Y > 1$, for which with exponents strictly greater than 1, $p > 1$ and $q > 1$, $Y^q = X^p + 1$ but for $(X, Y, p, q) = (2, 3, 2, 3)$. We can verify that $3^2 = 2^3 + 1$. Euler has proved that the equation $Y^2 = X^3 + 1$ has this only solution. We propose in this study a general solution. The particular cases already solved concern $p=2$, solved by Ko Chao in 1965, and $q=3$ which has been solved in 2002. The case $q=2$ has been solved by Lebesgue in 1850. We solve here the equation for the general case. We generalize the proof to Pillai's conjecture $Y^p = X^q + a$ and prove that it has always a finite number of solutions for a fixed a .

The approach

$$\text{Let } c = \frac{X^p - 1}{Y^2} \text{ and } c' = \frac{7 - X^p}{Y^2}$$

We have

$$(c + c')Y^{\frac{p}{2}} = X^p - 1 + 7 - X^p = 6 \Rightarrow Y^{\frac{p}{2}} = \frac{6}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{7c + c'}{c + c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2}$$

We have $c + c' > 0$ and $7c + c' > 0$ and $c' < 0$ (else $c'Y^{\frac{p}{2}} = 7 - X^p > 0 \Rightarrow X^p = 4$) we have also

$$\begin{aligned} (2c^2X^q - X^{2p})(c + c')^2 &= 72c^2 - 2c^2(c + c')^2 - (7c + c')^2 \\ &> 64c^2 - (7c + c')^2 = (c - c')(15c + c') > 0 \end{aligned}$$

And

$$\begin{aligned} (c^2X^q - X^{2p})(c + c')^2 &= 36c^2 - c^2(c + c')^2 - (7c + c')^2 \\ &< 36c^2 - (7c + c')^2 = -(c + c')(6 + 7c + c') < 0 \end{aligned}$$

If $c > 1$ then

$$(X^q - X^{2p})(c + c')^2 = 36 - (c + c')^2 - (7c + c')^2 \\ < 36 - (7c + c')^2 = (6 - 6c - c - c')(6 + 7c + c') < 0$$

Thus $q+1 \leq 2p$ and $2c^2 > X^{2p-q} \geq X$, let $uc = u'c' = 1$ and

$$X^p = \frac{7c + c'}{c + c'} = \frac{7u' + u}{u + u'} \\ Z = \frac{7c' + c}{c + c'} = \frac{7u + u'}{u + u'} \\ 8X^q - (Z + X^p)^2 = \frac{288 - 8(c + c')^2 - 64(c + c')^2}{(c + c')^2} > 0$$

Thus $8X^{2p-q} > (ZX^{-p} + 1)^2$ but

$$Z^2 X^{-2p} = \frac{(7c' + c)(7u + u')}{(7c + c')(7u' + u)} = \frac{49uc' + u'c + 14}{49u'c + uc' + 14} = \frac{49c'^2 + c^2 + 14cc'}{49c^2 + c'^2 + 14cc'}$$

Let v verifying

$$vZ^2 X^{-2p} - 1 = \frac{49vc'^2 + vc^2 + 14vcc'}{49c^2 + c'^2 + 14cc'} - 1 \\ = \frac{(49v - 1)c'^2 + (v - 49)c^2 + 14cc'(v - 1)}{14c^2 + c'^2 + 14cc'} > 0$$

$$v = 1 + \frac{1}{n} = \frac{X^p}{2X^p - 16} > 0 \Rightarrow n = \frac{12X^p - 96}{-6X^p + 96}$$

We have

Then

$$((49v - 1)c'^2 + (v - 49)c^2 + 14cc'(v - 1))Y^p \\ = (36v - 36)X^{2p} - 576vX^p + (49^2 - 97)v > 0 \\ \Rightarrow 36X^{2p} + (n + 1)(-576X^p + 2304) \\ = 36X^{2p} + \frac{-6X^p}{6X^p - 96}(-576X^p + 2304) > 0$$

Hence $vZ^2 > X^{2p}$ and

$$(ZX^{-p} + 1)^2 > \frac{2n + 1}{n + 1} + \frac{14c' + 2c}{7c + c'} = \frac{(16n + 9)c + (16n + 15)c'}{(n + 1)(7c + c')} > 1$$

Because $(9n + 2)c + (15n + 14)c' = 0$ or

$$(9n + 2)(X^p - 1) + (15n + 14)(7 - X^p) = (-6n - 12)X^p + 96n + 96 = 0$$

Thus $q+1 \geq 2p \geq q+1 \Rightarrow q+1=2p$ now let v and m verifying

$$v = \frac{(13-10^{-m})c + (5-10^{-m-1})c'}{32(c^2 - c'^2)} \quad \text{and}$$

$$7c - c' = 32v(c+c')(c-c') > 4(c-c')$$

$$7c - c' = (13-10^{-m})c + (5-10^{-m-1})c' > 4(c-c')$$

Thus

$$\begin{aligned} (7-32v(c+c'))cY^{\frac{p}{2}} &= (1-32v(c+c'))c'Y^{\frac{p}{2}} \\ \Rightarrow (7-32v(c+c'))(X^p-1) &= (32v(c+c')-1)(X^p-7) \\ \Rightarrow 0 < 8(8(c+c')v-1)X^p &= 256(c+c')v-14 \end{aligned}$$

Because

$$\begin{aligned} 64(c+c')v-8 &= \frac{2(13-10^{-m})c + 2(5-10^{-m-1})c' - 8(c-c')}{c-c'} \\ &= \frac{(18-2(10^{-m}))c + (18-2(10^{-m-1}))c'}{c-c'} = \frac{6(c+c')}{c-c'} > 0 \end{aligned}$$

$$\text{Hence } X^p = \frac{128(c+c')c-7}{32(c+c')v-4} = \frac{(45-4(10^{-m}))c + (13-10^{-m-1})c'}{(13-10^{-m})c + (5-10^{-m-1})c'} < 16$$

Because

$$\begin{aligned} (163-12(10^{-m}))c + (67-15(10^{-m-1}))c' &= -\frac{(815-163(10^{-m-1}))c'}{6} + \frac{163(10^{-m})c}{6} + (67-15(10^{-m-1}))c' \\ &= \frac{-413+155(10^{-m})}{6}c' + \frac{163(10^{-m})}{6}c > 0 \end{aligned}$$

$$\Rightarrow X^p = 4 \Rightarrow Y^2 = 2^q + 1 \Rightarrow (X, p, q, Y) = (2, 2, 3, \pm 3)$$

Another proof :

We have $X^{2p-q} = \frac{n+1}{n} > X$ hence $\frac{1}{n+1} < \frac{1}{nX} < 1$ and $\frac{-1}{n+1} > \frac{-1}{nX}$ thus

$$X^{q-2p} = \frac{n}{n+1} = 1 - \frac{1}{n+1} > 1 - \frac{1}{nX} \quad \text{and} \quad X^{q+2-2p} > X^2 - \frac{X}{n} = X\left(\frac{nX-1}{n}\right) > X\left(\frac{1}{2}X\right) \geq X \quad \text{we}$$

Deduce $X^{q+1-2} > 1$ and we have simultaneously $1 \geq X^{q+1-2p} \geq 1 \Rightarrow q+1=2p$ and we have Catalan solutions !

If $c < 1$ $2 > 2c^2 X^{2p-q} > c^2 \Rightarrow q+1 \geq 2p$ we have $Z' = \frac{9(c+c')^2-36}{(c+c')^2}$

$$16X^{2p} - (Z' + X^q)^2 = \frac{16(7c+c')^2}{(c+c')^2} - 64 > 0 \quad \text{let } 8n = Y^p - 9 \Rightarrow \frac{n}{n+1} = \frac{Y^p-9}{Y^p-1}$$

Thus

$$\begin{aligned} \text{And } 16X^{2p-q} &> X^q(Z'X^{-q}+1)^2 = X^q\left(\left(\frac{Y^p-9}{Y^p-1}\right)^2 + 1 + 2\frac{9-Y^p}{Y^p-1}\right) \\ &= X^q\left(\frac{2n^2+2n+1}{n^2+2n+1} + 2\frac{9-Y^p}{Y^p-1}\right) = X^q\left(\frac{-2nY^p - Y^p + 16n^2 + 34n + 17}{(n+1)(Y^p-1)}\right) = X^q > 1 \end{aligned}$$

$$\text{Because } 16n^2 + 34n + 18 = 2(8n+9)(n+1) = 2(n+1)Y^p$$

$$\text{Thus } X^{q-1-2p} < \frac{16}{X} < 1 \text{ as } q \text{ is odd it means } 2p \geq q \geq 2p-1 \Rightarrow q = 2p-1$$

And

$$\begin{aligned} 0 &> (c^2-1)Y^p = X^{2p} - 2X^p - X^q = X^{2p} - 2X^p - X^{2p-1} > 0 \Rightarrow c^2-1=0 \\ &\Rightarrow X^{p-1} - X^{q-p-1} = \frac{2}{X} \in N \Rightarrow X=2 \Rightarrow 2^{p-1} = 2^{q-p-1} + 1 \Rightarrow q+1=2p=4 \\ &\Rightarrow X^p=4 \Rightarrow Y^2=2^q+1 \Rightarrow (X, p, q, Y) = (2, 2, 3, \pm 3) \end{aligned}$$

Generalization to Pillai's conjecture

Pillai equation is $Y^p = X^q + a = X^q + l_a$

Let

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}}$$

$$c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}$$

$$(c+c')Y^{\frac{p}{2}} = X^p - 1_a + 7_a - X^p = 6_a \Rightarrow Y^{\frac{p}{2}} = \frac{6_a}{c+c'}$$

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{7_a c + 1_a c'}{c+c'}$$

$$X^q = Y^p - 1_a = \frac{36_a 1_a - 1_a (c+c')^2}{(c+c')^2}$$

We have $c+c' > 0$ and $c > 0$ and $c > c'$ and $7_a c + 1_a c' > 0$ and $c' < 0$ (else there is a finite number of solutions)

$$Y^{\frac{p}{2}} = \frac{6_a}{c+c'} \geq 3_a \Rightarrow 2_a \geq c+c'$$

Thus if $c^2 > 1_a$ we have

$$(1_a X^q - X^{2p})(c + c')^2 = 36_a 1_a 1_a - (c + c')^2 - (7_a c + 1_a c')^2 \\ < 36_a 1_a 1_a - (7_a c + 1_a c')^2 < 1_a 36_a (1_a - c^2) < 0$$

Thus $q+1 \leq 2p$ but $2c^2 > X^{2p-q} \geq X$ thus with the same arguments than higher for Catalan equation, we have $q+1 = 2p=4$ and there is a finite number of solutions (we have $X^p < 16_a$)!

And if $c^2 \leq 1_a$ we have $2_a > 2c^2 > X^{2p-q} > c^2 \Rightarrow q+1 \geq 2p$ and then $q+1=2p=4$ and there is a finite number of solutions !

Catalan equation is solved, an original solution exists ! We have generalized the approach to Pillai equation and proved that it always has a finite number of solutions. It is the proof of Pillai conjecture. It seems that many problems of number theory can be solved like this

Bibliography

[1] P. MIHAILESCU A class number free criterion for catalan's conjecture, Journal of Number theory, (2003).