

## A proof of Pillai's conjecture

Jamel Ghanouchi

6 Rue Khansa 2070 Marsa Tunisia

**Keywords :** Diophantine equations; Catalan equation; Pillai conjecture; Algebraic resolution

**Abstract** Catalan theorem has been proved in 2002 by Preda Mihalescu. In 2004, it became officially Catalan-Mihalescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1,  $X > 1$  and  $Y > 1$ , for which with exponents strictly greater than 1,  $p > 1$  and  $q > 1$ ,  $Y^p = X^q + 1$  but for  $(X, Y, p, q) = (2, 3, 2, 3)$ . We can verify that  $3^2 = 2^3 + 1$  Euler has proved that the equation  $Y^2 = X^3 + 1$  has this only solution. We propose in this study a general solution. The particular cases already solved concern  $p=2$ , solved by Ko Choin 1965, and  $q=3$  which has been solved in 2002. The case  $q=2$  has been solved by Lebesgue in 1850. We solve here the equation for the general case. We generalize the proof to Pillai's conjecture  $Y^p = X^q + a$  and prove that it has always a finite number of solutions for a fixed  $a$

### The approach

$$\text{Let } c = \frac{X^p - 1}{Y^2} \text{ and } c' = \frac{7 - X^p}{Y^2}$$

We have

$$(c + c')Y^2 = X^p - 1 + 7 - X^p = 6 \Rightarrow Y^2 = \frac{6}{c + c'}$$

$$X^p = cY^2 + 1 = \frac{7c + c'}{c + c'}$$

And

$$X^q = Y^p - 1 = \frac{36 - (c + c')^2}{(c + c')^2}$$

We have  $c + c' > 0$  and  $7c + c' > 0$  and  $c' < 0$  (else  $c'Y^2 = 7 - X^p > 0 \Rightarrow X^p = 4$ ) we have also

$$\begin{aligned} (2c^2X^q - X^{2p})(c + c')^2 &= 72c^2 - 2c^2(c + c')^2 - (7c + c')^2 \\ &> 64c^2 - (7c + c')^2 = (c - c')(15c + c') > 0 \end{aligned}$$

And

$$\begin{aligned} (c^2X^q - X^{2p})(c + c')^2 &= 36c^2 - c^2(c + c')^2 - (7c + c')^2 \\ &< 36c^2 - (7c + c')^2 = -(c + c')(6 + 7c + c') < 0 \end{aligned}$$

If  $c > 1$  then

$$(X^q - X^{2p})(c + c')^2 = 36 - (c + c')^2 - (7c + c')^2 < 36 - (7c + c')^2 = (6 - 6c - c - c')(6 + 7c + c') < 0$$

Thus  $q+1 \leq 2p$  and  $2c^2 > X^{2p-q} \geq X$ , let  $uc = u'c' = 1$  and

$$X^p = \frac{7c + c'}{c + c'} = \frac{7u' + u}{u + u'}$$

$$Z = \frac{7c' + c}{c + c'} = \frac{7u + u'}{u + u'}$$

$$8X^q - (Z + X^p)^2 = \frac{288 - 8(c + c')^2 - 64(c + c')^2}{(c + c')^2} > 0$$

Thus  $8X^{2p-q} > (ZX^{-p+1})^2$  but

$$Z^2 X^{-2p} = \frac{(7c' + c)(7u + u')}{(7c + c')(7u' + u)} = \frac{49uc' + u'c + 14}{49u'c + uc' + 14} = \frac{49c'^2 + c^2 + 14cc'}{49c^2 + c'^2 + 14cc'}$$

Let  $v$  verifying

$$vZ^2 X^{-2p} - 1 = \frac{49vc'^2 + vc^2 + 14vcc'}{49c^2 + c'^2 + 14cc'} - 1 = \frac{(49v - 1)c'^2 + (v - 49)c^2 + 14cc'(v - 1)}{14c^2 + c'^2 + 14cc'} > 0$$

We have 
$$v = 1 + \frac{1}{n} = \frac{X^p}{2X^p - 16} > 0 \Rightarrow n = \frac{12X^p - 96}{-6X^p + 96}$$

Then

$$((49v - 1)c'^2 + (v - 49)c^2 + 14cc'(v - 1))Y^p = (36v - 36)X^{2p} - 576vX^p + (49^2 - 97)v > 0$$

$$\Rightarrow 36X^{2p} + (n + 1)(-576X^p + 2304)$$

$$= 36X^{2p} + \frac{-6X^p}{6X^p - 96}(-576X^p + 2304) > 0$$

Hence  $vZ^2 > X^{2p}$  and

$$(ZX^{-p} + 1)^2 > \frac{2n + 1}{n + 1} + \frac{14c' + 2c}{7c + c'} = \frac{(16n + 9)c + (16n + 15)c'}{(n + 1)(7c + c')} > 1$$

Because  $(9n + 2)c + (15n + 14)c' = 0$  or

$$(9n + 2)(X^p - 1) + (15n + 14)(7 - X^p) = (-6n - 12)X^p + 96n + 96 = 0$$

Thus  $q+1 \geq 2p \geq q+1 \Rightarrow q+1 = 2p$  now let v and m verifying

$$v = \frac{(13-10^{-m})c + (5-10^{-m-1})c'}{32(c^2 - c'^2)} \quad \text{and}$$

$$7c - c' = 32v(c+c')(c-c') > 4(c-c')$$

$$7c - c' = (13-10^{-m})c + (5-10^{-m-1})c' > 4(c-c')$$

Thus

$$\begin{aligned} (7-32v(c+c'))cY^{\frac{p}{2}} &= (1-32v(c+c'))c'Y^{\frac{p}{2}} \\ \Rightarrow (7-32v(c+c'))(X^p-1) &= (32v(c+c')-1)(X^p-7) \\ \Rightarrow 0 < 8(8(c+c')v-1)X^p &= 256(c+c')v-14 \end{aligned}$$

Because

$$\begin{aligned} 64(c+c')v-8 &= \frac{2(13-10^{-m})c + 2(5-10^{-m-1})c' - 8(c-c')}{c-c'} \\ &= \frac{(18-2(10^{-m}))c + (18-2(10^{-m-1}))c'}{c-c'} = \frac{6(c+c')}{c-c'} > 0 \end{aligned}$$

$$\text{Hence } X^p = \frac{128(c+c')c-7}{32(c+c')v-4} = \frac{(45-4(10^{-m}))c + (13-10^{-m-1})c'}{(13-10^{-m})c + (5-10^{-m-1})c'} < 16$$

Because

$$\begin{aligned} (163-12(10^{-m}))c + (67-15(10^{-m-1}))c' &= -\frac{(815-163(10^{-m-1}))c'}{6} + \frac{163(10^{-m})c}{6} + (67-15(10^{-m-1}))c' \\ &= \frac{-413+155(10^{-m})}{6}c' + \frac{163(10^{-m})}{6}c > 0 \end{aligned}$$

$$\Rightarrow X^p = 4 \Rightarrow Y^2 = 2^q + 1 \Rightarrow (X, p, q, Y) = (2, 2, 3, \pm 3)$$

Another proof :

We have  $X^{2p-q} = \frac{n+1}{n} > X$  hence  $\frac{1}{n+1} < \frac{1}{nX} < 1$  and  $\frac{-1}{n+1} > \frac{-1}{nX}$  thus

$$X^{q-2p} = \frac{n}{n+1} = 1 - \frac{1}{n+1} > 1 - \frac{1}{nX} \quad \text{and} \quad X^{q+2-2p} > X^2 - \frac{X}{n} = X\left(\frac{nX-1}{n}\right) > X\left(\frac{1}{2}X\right) \geq X \quad \text{we}$$

Deduce  $X^{q+1-2} > 1$  and we have simultaneously  $1 \geq X^{q+1-2p} \geq 1 \Rightarrow q+1 = 2p$  and we have Catalan solutions !

If  $c < 1 > 2 > 2c^2 X^{2p-q} > c^2 \Rightarrow q+1 \geq 2p$  we have  $Z' = \frac{9(c+c')^2-36}{(c+c')^2}$

$$\text{Thus } 16X^{2p} - (Z'+X^q)^2 = \frac{16(7c+c')^2}{(c+c')^2} - 64 > 0 \quad \text{let } 8n = Y^p - 9 \Rightarrow \frac{n}{n+1} = \frac{Y^p-9}{Y^p-1}$$

$$\begin{aligned} \text{And } 16X^{2p-q} > X^q(Z'X^{-q} + 1)^2 &= X^q \left( \left( \frac{Y^p - 9}{Y^p - 1} \right)^2 + 1 + 2 \frac{9 - Y^p}{Y^p - 1} \right) \\ &= X^q \left( \frac{2n^2 + 2n + 1}{n^2 + 2n + 1} + 2 \frac{9 - Y^p}{Y^p - 1} \right) = X^q \left( \frac{-2nY^p - Y^p + 16n^2 + 34n + 17}{(n+1)(Y^p - 1)} \right) = X^q > 1 \end{aligned}$$

Because  $16n^2 + 34n + 18 = 2(8n + 9)(n + 1) = 2(n + 1)Y^p$

Thus  $X^{q-1-2p} < \frac{16}{X} < 1$  as  $q$  is odd it means  $2p \geq q \geq 2p - 1 \Rightarrow q = 2p - 1$

And

$$\begin{aligned} 0 > (c^2 - 1)Y^p &= X^{2p} - 2X^p - X^q = X^{2p} - 2X^p - X^{2p-1} > 0 \Rightarrow c^2 - 1 = 0 \\ \Rightarrow X^{p-1} - X^{q-p-1} &= \frac{2}{X} \in N \Rightarrow X = 2 \Rightarrow 2^{p-1} = 2^{q-p-1} + 1 \Rightarrow q + 1 = 2p = 4 \\ \Rightarrow X^p = 4 \Rightarrow Y^2 = 2^q + 1 &\Rightarrow (X, p, q, Y) = (2, 2, 3, \pm 3) \end{aligned}$$

Generalization to Pillai’s conjecture

Pillai equation is  $Y^p = X^q + a = X^q + 1_a$

Let

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}}$$

$$c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}}$$

$$(c + c')Y^{\frac{p}{2}} = X^p - 1_a + 7_a - X^p = 6_a \Rightarrow Y^{\frac{p}{2}} = \frac{6_a}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{7_a c + 1_a c'}{c + c'}$$

$$X^q = Y^p - 1_a = \frac{36_a 1_a - 1_a (c + c')^2}{(c + c')^2}$$

We have  $c + c' > 0$  and  $c > 0$  and  $c > c'$  and  $7_a c + 1_a c' > 0$  and  $c' < 0$  (else there is a finite number of solutions)

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'} \geq 3_a \Rightarrow 2_a \geq c + c'$$

Thus if  $c^2 > 1_a$  we have

$$(1_a X^q - X^{2p})(c + c')^2 = 36_a 1_a 1_a - (c + c')^2 - (7_a c + 1_a c')^2$$

$$< 36_a 1_a 1_a - (7_a c + 1_a c')^2 < 1_a 36_a (1_a - c^2) < 0$$

Thus  $q+1 \leq 2p$  but  $2c^2 > X^{2p-q} \geq X$  thus with the same arguments than higher for Catalan equation, we have  $q+1 = 2p=4$  and there is a finite number of solutions (we have  $X^p < 16_a$ )!

And if  $c^2 \leq 1_a$  we have  $2_a > 2c^2 > X^{2p-q} > c^2 \implies q+1 \geq 2p$  and then  $q+1=2p=4$  and there is a finite number of solutions !

Catalan equation is solved, an original solution exists ! We have generalized the approach to Pillai equation and proved that it always has a finite number of solutions. It is the proof of Pillai conjecture. It seems that many problems of number theory can be solved like this

### Bibliography

[1] P. MIHAILESCU A class number free criterion for catalan's conjecture, Journal of Number theory, (2003).