A proof of Pillai’s conjecture
Jamel Ghanouchi
6 Rue Khansa 2070 Marsa Tunisia

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Abstract Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers strictly greater than 1, \(X>1\) and \(Y>1\), for which with exponents strictly greater than 1, \(p>1\) and \(q>1\), \(Y^p=X^q+1\) but for \((X,Y,p,q) = (2,3,2,3)\). We can verify that \(3^2=2^3+1\) Euler has proved that the equation \(Y^2=3^3+1\) has this only solution. We propose in this study a general solution. The particular cases already solved concern \(p=2\), solved by Ko Chaoin 1965, and \(q=3\) which has been solved in 2002. The case \(q=2\) has been solved by Lebesgue in 1850. We solve here the equation for the general case. We generalize the proof to Pillai’s conjecture \(Y^p=X^q+a\) and prove that it has always a finite number of solutions for a fixed \(a\)

The approach

Let \(c = \frac{X^p-1}{Y^p}\) and \(c' = \frac{7-X^p}{Y^p}\)

We have

\[(c + c')Y^p = X^p - 1 + 7 - X^p = 6 \Rightarrow Y^p = \frac{6}{c + c'}\]

\[X^p = cY^2 + 1 = \frac{7c + c'}{c + c'}\]

And

\[X^q = Y^p - 1 = \frac{36 - (c+c')^2}{(c+c')^2}\]

We have \(c+c' > 0\) and \(7c + c' > 0\) and \(c' < 0\) (else \(cY^p = 7 - X^p > 0 \Rightarrow X^p = 4\)) we have also

\[(2c^2X^q - X^{2p})(c+c')^2 = 72c^2 - 2c^2(c+c')^2 - (7c+c')^2\]

\[> 64c^2 - (7c + c')^2 = (c-c')(15c+c') > 0\]

And

\[(c^2X^q - X^{2p})(c+c')^2 = 36c^2 - c^2(c+c')^2 - (7c+c')^2\]

\[< 36c^2 - (7c + c')^2 = -(c+c')(6 + 7c + c') < 0\]
If $c > 1$ then 

\[(X^q - X^{2p})(c + c')^2 = 36 - (c + c')^2 - (7c + c')^2 < 36 - (7c + c')^2 = (6 - 6c - c - c')(6 + 7c + c') < 0\]

Thus $q + 1 \leq 2p$ and $2c^2 > X^{2p-q} \geq X$, let $u_c = u'c' = 1$ and 

\[X^p = \frac{7c + c'}{c + c'} = \frac{7u' + u}{u + u'} \]

\[Z = \frac{7c' + c}{c + c'} = \frac{7u + u'}{u + u'} \]

\[8X^q - (Z + X^p)^2 = \frac{288 - 8(c + c')^2 - 64(c + c')^2}{(c + c')^2} > 0 \]

Thus $8X^{2p-q} > (ZX^{n+1})^2$ but 

\[Z^2 X^{-2p} = \frac{(7c' + c)(7u + u')}{(7c + c')(7u' + u)} = \frac{49uc' + uc + 14}{49u'c + uc' + 14} = \frac{49c'^2 + c'^2 + 14cc'}{49c^2 + c'^2 + 14cc'} \]

Let $v$ verifying 

\[vZ^2 X^{-2p} - 1 = \frac{49vc'^2 + vc^2 + 14vcc'}{49c^2 + c'^2 + 14cc'} - 1 \]

\[= \frac{(49v - 1)c'^2 + (v - 49)c^2 + 14cc'(v - 1)}{14c^2 + c'^2 + 14cc'} > 0 \]

\[v = 1 + \frac{1}{n} = \frac{X^p}{2X^p - 16} > 0 \Rightarrow n = \frac{12X^p - 96}{-6X^p + 96} \]

We have 

\[((49v - 1)c'^2 + (v - 49)c^2 + 14cc'(v - 1))Y^p \]

\[= (36v - 36)X^{2p} - 576vX^p + (49^2 - 97)v > 0 \]

\[\Rightarrow 36X^{2p} + (n + 1)(-576X^p + 2304) \]

\[= 36X^{2p} + \frac{-6X^p}{-576X^p + 2304} > 0 \]

Hence $vZ^2 > X^{2p}$ and 

\[(ZX^{-p} + 1)^2 > \frac{2n + 1}{n + 1} + \frac{14c' + 2c}{7c + c'} = \frac{(16n + 9)c + (16n + 15)c'}{(n + 1)(7c + c')} > 1 \]

Because $(9n+2)c+(15n+14)c' = 0$ or 

\[(9n + 2)(X^p - 1) + (15n + 14)(7 - X^p) = (-6n - 12)X^p + 96n + 96 = 0 \]
Thus \( q + 1 \geq 2p \geq q + 1 \Rightarrow q + 1 = 2p \) now let \( v \) and \( m \) verifying

\[
v = \frac{(13 - 10^{-m})c + (5 - 10^{-m-1})c'}{32\left(c^2 - c'^2\right)}
\]

and

\[
7c - c' = 32v(c + c')(c - c') > 4(c - c')
\]

\[
7c - c' = (13 - 10^{-m})c + (5 - 10^{-m-1})c' > 4(c - c')
\]

Thus

\[
(7 - 32v(c+c'))cY^2 = (1 - 32v(c+c'))c'Y^2
\]

\[
\Rightarrow (7 - 32v(c+c'))(X^p-1) = (32v(c+c')-1)(X^p-7)
\]

\[
\Rightarrow 0 < 8(8(c+c')v-1)X^p = 256(c+c')v-14
\]

Because

\[
64(c+c')v-8 = 2(13 - 10^{-m})c + 2(5 - 10^{-m-1})c - 8(c - c')
\]

\[
= \frac{(18 - 2(10^{-m}))c + (18 - 2(10^{-m-1}))c'}{c - c'} \geq \frac{6(c + c')}{c - c'} > 0
\]

Hence

\[
X^p = \frac{128(c + c')c - 7}{32(c + c')v - 4} = \frac{(45 - 4(10^{-m}))c + (13 - 10^{-m-1})c'}{13 - 10^{-m})c + (5 - 10^{-m-1})c'} < 16
\]

Because

\[
(163 - 12(10^{-m}))c + (67 - 15(10^{-m-1}))c' = \frac{(815 - 163(10^{-m-1}))c'}{6} + \frac{163(10^{-m})c'}{6} + (67 - 15(10^{-m-1}))c'
\]

\[
= \frac{-413 + 155(10^{-m})}{6}c' + \frac{163(10^{-m})}{6}c > 0
\]

\[
\Rightarrow X^p = 4 \Rightarrow Y^2 = 2^4 + 1 \Rightarrow (X, p, q, Y) = (2, 2, 3, \pm 3)
\]

Another proof:

We have \( X^{2p-q} = \frac{n+1}{n} > X \) hence \( \frac{1}{n+1} < \frac{1}{nX} < 1 \) and \( \frac{-1}{n+1} > \frac{-1}{nX} \) \( \Rightarrow X^{q+2-2p} > X^2 - \frac{X}{n} = X^2 - \frac{X}{n} > X^2 - \frac{X}{n} \geq X \) \( \Rightarrow \) we

Deduce \( X^{q+1-2} > 1 \) and we have simultaneously \( 1 \geq X^{q+1-2p} \geq 1 \Rightarrow q + 1 = 2p \) and we have Catalan solutions!

If \( c < 1 \) \( \Rightarrow 2c^2 X^{2p-q} > c^2 \Rightarrow q + 1 \geq 2p \) we have \( Z = \frac{9(c+c')^2 - 36}{(c+c')^2} \)

\[
16X^{2p - (Z+X^p)} = \frac{16(7c + c')^2}{(c + c')^2} > 64 > 0 \text{ let } 8u = Y^p - 9 \Rightarrow \frac{n}{n+1} = \frac{Y^p - 9}{Y^p - 1}
\]

Thus
And \(16X^{2p-q} > X^q(Z^2X^{-q}+1)^2 = X^q(\left(\frac{Y^p - 9}{Y^p - 1}\right)^2 + 1 + 2\frac{9-Y^p}{Y^p - 1})\)

\[= X^q\left(\frac{2n^2 + 2n + 1}{n^2 + 2n + 1} + 2\frac{9-Y^p}{Y^p - 1}\right) = X^q\frac{(-2nY^p - Y^p + 16n^2 + 34n + 17)}{(n+1)(Y^p - 1)} = X^q > 1\]

Because \(16n^2 + 34n + 18 = 2(8n + 9)(n + 1) = 2(n+1)Y^p\)

Thus \(X^{q-1-p} < \frac{16}{X} < 1\) as \(q\) is odd it means \(2p \geq q \geq 2p - 1 \Rightarrow q = 2p - 1\)

And

\[0 > (c^2 - 1)Y^p = X^{2p} - 2X^p - X^q = X^{2p} - X^{2p-1} > 0 \Rightarrow c^2 - 1 = 0\]

\(\Rightarrow X^{p-1} - X^{q-p-1} = \frac{2}{X} \in N \Rightarrow X = 2 \Rightarrow 2^{p-1} = 2^{q-p-1} + 1 \Rightarrow q + 1 = 2p = 4\)

\(\Rightarrow X^p = 4 \Rightarrow Y^2 = 2^q + 1 \Rightarrow (X, p, q, Y) = (2, 2, 3, 3)\)

**Generalization to Pillai’s conjecture**

Pillai equation is \(Y^p = X^q + a = X^q + I_a\)

Let

\[c = \frac{X^p - 1_a}{Y^2}\]

\[c' = \frac{7_a - X^p}{Y^2}\]

\[(c + c')Y^2 = X^p - 1_a + 7_a - X^p = 6_a \Rightarrow Y^2 = \frac{6_a}{c + c'}\]

\[X^p = cY^2 + 1_a = \frac{7_a c + 1_a c'}{c + c'}\]

\[X^q = Y^p - 1_a = \frac{36_a 1_a - 1_a (c + c')^2}{(c + c')^2}\]

We have \(c+c' > 0\) and \(c>0\) and \(c>c'\) and \(7_a c + 1_a c' > 0\) and \(c<0\) (else there is a finite number of solutions)

\[Y^2 = \frac{6_a}{c + c'} \geq 3_a \Rightarrow 2_a \geq c + c'\]

Thus if \(c^2 > 1_a\) we have
Thus $q+1 \leq 2p$ but with the same arguments than higher for Catalan equation, we have $q+1 = 2p = 4$ and there is a finite number of solutions (we have $X < 16^a$).

And if $c^2 \leq 1_a$ we have $2c^2 > X^{2p-q} \geq X$ thus with the same arguments than higher for Catalan equation, we have $q+1 = 2p = 4$ and there is a finite number of solutions!

Catalan equation is solved, an original solution exists! We have generalized the approach to Pillai equation and proved that it always has a finite number of solutions. It is the proof of Pillai conjecture. It seems that many problems of number theory can be solved like this.

Bibliography