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A proof of Pillai's conjecture

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Abstract Catalan theorem has been proved in 2002 by Preda Mihailescu. In 2004, it became officially Catalan-Mihailescu theorem. This theorem stipulates that there are not consecutive pure powers. There do not exist integers stricly greater than 1, X>1 and Y>1, for which with exponants strictly greater than 1, p>1 and q>1, $Y^P=X^q+1$ but for (X,Y,p,q) = (2,3,2,3) . We can verify that $3^2=2^3+1$ Euler has proved that the equation $Y^2=X^3+1$ has this only solution. We propose in this study a general solution. The particular cases already solved concern p=2, solved by Ko Chaoin 1965, and q=3 which has been solved in 2002. The case q=2 has been solved by Lebesgue in 1850. We solve here the equation for the general case. We generalize the proof to Pillai's conjecture $Y^P=X^q+a$ and prove that it has always a finite number of solutions for a fixed a

<u>The approach</u>

Let $c = \frac{X^P-1}{Y^{\frac{P}{2}}}$ and $c' = \frac{7-X^P}{Y^{\frac{P}{2}}}$

We have

$$(c+c')Y^{\frac{p}{2}} = X^p - 1 + 7 - X^p = 6 \Rightarrow Y^{\frac{p}{2}} = \frac{6}{c+c'}$$

$$X^p = cY^{\frac{p}{2}} + 1 = \frac{7c+c'}{c+c'}$$

And

$$X^q = Y^p - 1 = \frac{36-(c+c')^2}{(c+c')^2}$$

We have c+c' >0 and 7c + c' >0 and c' <0 (else $c'Y^{\frac{p}{2}} = 7 - X^p > 0 \Rightarrow X^p = 4$) we have also

$$(2c^2X^q - X^{2p})(c+c')^2 = 72c^2 - 2c^2(c+c')^2 - (7c+c')^2$$
$$> 64c^2 - (7c+c')^2 = (c-c')(15c+c') > 0$$

And

$$(c^2X^q - X^{2p})(c+c')^2 = 36c^2 - c^2(c+c')^2 - (7c+c')^2$$
$$< 36c^2 - (7c+c')^2 = -(c+c')(6+7c+c') < 0$$

If c>1 then

$$(X^q - X^{2p})(c+c')^2 = 36-(c+c')^2-(7c+c')^2$$
$$< 36-(7c+c')^2 = (6-6c-c-c')(6+7c+c') < 0$$

Thus $q+1 \le 2p$ and $2c^2 > X^{2p-q} \ge X$, let uc=u′c′=1 and

$$X^p = \frac{7c+c'}{c+c'} = \frac{7u'+u}{u+u'}$$
$$Z = \frac{7c'+c}{c+c'} = \frac{7u+u'}{u+u'}$$

$$8X^q - (Z+X^p)^2 = \frac{288-8(c+c')^2-64(c+c')^2}{(c+c')^2} > 0$$

Thus $8X^{2p-q} > (ZX^{-p}+1)^2$ but

$$Z^2X^{-2p} = \frac{(7c'+c)(7u+u')}{(7c+c')(7u'+u)} = \frac{49uc'+u'c+14}{49u'c+uc'+14} = \frac{49c'^2+c^2+14cc'}{49c^2+c'^2+14cc'}$$

Let v verifying

$$vZ^2X^{-2p} - 1 = \frac{49vc'^2+vc^2+14vcc'}{49c^2+c'^2+14cc'} - 1$$
$$= \frac{(49v-1)c'^2+(v-49)c^2+14cc'(v-1)}{14c^2+c'^2+14cc'} > 0$$

We have $$v = 1+\frac{1}{n} = \frac{X^p}{2X^p-16} > 0 \Rightarrow n = \frac{12X^p-96}{-6X^p+96}$$

Then

$$((49v-1)c'^2+(v-49)c^2+14cc'(v-1))Y^p$$
$$= (36v-36)X^{2p} - 576vX^p + (49^2-97)v > 0$$
$$\Rightarrow 36X^{2p} + (n+1)(-576X^p+2304)$$
$$= 36X^{2p} + \frac{-6X^p}{6X^p-96}(-576X^p+2304) > 0$$

Hence $vZ^2 > X^{2p}$ and

$$(ZX^{-p}+1)^2 > \frac{2n+1}{n+1} + \frac{14c'+2c}{7c+c'} = \frac{(16n+9)c+(16n+15)c'}{(n+1)(7c+c')} > 1$$

Because (9n+2)c+(15n+14)c' =0 or

$$(9n+2)(X^p-1)+(15n+14)(7-X^p) = (-6n-12)X^p+96n+96 = 0$$

Thus $q+1 \geq 2p \geq q+1 \Rightarrow q+1=2p$ now let v and m verifying

$$v=\frac{(13-10^{-m})c+(5-10^{-m-1})c'}{32(c^2-c'^2)}$$ and

$$7c-c'=32v(c+c')(c-c')>4(c-c')$$
$$7c-c'=(13-10^{-m})c+(5-10^{-m-1})c'>4(c-c')$$

Thus

$$(7-32v(c+c'))cY^{\frac{p}{2}}=(1-32v(c+c'))c'Y^{\frac{p}{2}}$$
$$\Rightarrow (7-32v(c+c'))(X^p-1)=(32v(c+c')-1)(X^p-7)$$
$$\Rightarrow 0<8(8(c+c')v-1)X^p=256(c+c')v-14$$

Because

$$64(c+c')v-8=\frac{2(13-10^{-m})c+2(5-10^{-m-1})c-8(c-c')}{c-c'}$$
$$=\frac{(18-2(10^{-m}))c+(18-2(10^{-m-1}))c'}{c-c'}=\frac{6(c+c')}{c-c'}>0$$

Hence $X^p=\frac{128(c+c')c-7}{32(c+c')v-4}=\frac{(45-4(10^{-m}))c+(13-10^{-m-1})c'}{(13-10^{-m})c+(5-10^{-m-1})c'}<16$

Because

$$(163-12(10^{-m}))c+(67-15(10^{-m-1}))c'=-\frac{(815-163(10^{-m-1}))c'}{6}+\frac{163(10^{-m})c}{6}+(67-15(10^{-m-1}))c'$$
$$=\frac{-413+155(10^{-m})}{6}c'+\frac{163(10^{-m})}{6}c>0$$

$$\Rightarrow X^p=4\Rightarrow Y^2=2^q+1\Rightarrow (X,p,q,Y)=(2,2,3,\pm 3)$$

Another proof :

We have $X^{2p-q}=\frac{n+1}{n}>X$ hence $\frac{1}{n+1}<\frac{1}{nX}<1$ and $\frac{-1}{n+1}>\frac{-1}{nX}$ thus

$X^{q-2p}=\frac{n}{n+1}=1-\frac{1}{n+1}>1-\frac{1}{nX}$ and $X^{q+2-2p}>X^2-\frac{X}{n}=X(\frac{nX-1}{n})>X(\frac{1}{2}X)\geq X$ we

Deduce $X^{q+1-2}>1$ and we have simultaneously $1\geq X^{q+1-2p}\geq 1\Rightarrow q+1=2p$ and we have Catalan solutions !

If c<1 $2>2c^2 X^{2p-q}>c^2 \Rightarrow q+1 \geq 2p$ we have $Z'=\frac{9(c+c')^2-36}{(c+c')^2}$

Thus $16X^{2p}-(Z'+X^q)^2=\frac{16(7c+c')^2}{(c+c')^2}-64>0$ let $8n=Y^p-9\Rightarrow \frac{n}{n+1}=\frac{Y^p-9}{Y^p-1}$

And $16X^{2p-q} > X^q(Z'X^{-q}+1)^2 = X^q((\frac{Y^p-9}{Y^p-1})^2+1+2\frac{9-Y^p}{Y^p-1})$

$$= X^q(\frac{2n^2+2n+1}{n^2+2n+1}+2\frac{9-Y^p}{Y^p-1}) = X^q(\frac{-2nY^p-Y^p+16n^2+34n+17}{(n+1)(Y^p-1)}) = X^q > 1$$

Because $16n^2+34n+18 = 2(8n+9)(n+1) = 2(n+1)Y^p$

Thus $X^{q-1-2p} < \frac{16}{X} < 1$ as q is odd it means $2p \geq q \geq 2p-1 \Rightarrow q = 2p-1$

And

$$0 > (c^2-1)Y^p = X^{2p}-2X^p-X^q = X^{2p}-2X^p-X^{2p-1} > 0 \Rightarrow c^2-1 = 0$$

$$\Rightarrow X^{p-1}-X^{q-p-1} = \frac{2}{X} \in N \Rightarrow X = 2 \Rightarrow 2^{p-1} = 2^{q-p-1}+1 \Rightarrow q+1 = 2p = 4$$

$$\Rightarrow X^p = 4 \Rightarrow Y^2 = 2^q+1 \Rightarrow (X,p,q,Y) = (2,2,3,\pm 3)$$

Generalization to Pillai's conjecture

Pillai equation is $Y^P=X^q+a=X^q+1_a$
Let

$$c = \frac{X^p-1_a}{Y^{\frac{p}{2}}}$$

$$c' = \frac{7_a-X^p}{Y^{\frac{p}{2}}}$$

$$(c+c')Y^{\frac{p}{2}} = X^p-1_a+7_a-X^p = 6_a \Rightarrow Y^{\frac{p}{2}} = \frac{6_a}{c+c'}$$

$$X^p = cY^{\frac{p}{2}}+1_a = \frac{7_ac+1_ac'}{c+c'}$$

$$X^q = Y^p-1_a = \frac{36_a1_a-1_a(c+c')^2}{(c+c')^2}$$

We have c+c' >0 and c>0 and c>c' and $7_ac+1_ac'>0$ and c'<0 (else there is a finite number of solutions)

$Y^{\frac{p}{2}} = \frac{6_a}{c+c'} \geq 3_a \Rightarrow 2_a \geq c+c'$

Thus if $c^2>1_a$ we have

$$(1_a X^q - X^{2p})(c+c')^2 = 36_a 1_a 1_a - (c+c')^2 - (7_a c + 1_a c')^2$$
$$< 36_a 1_a 1_a - (7_a c + 1_a c')^2 < 1_a 36_a (1_a - c^2) < 0$$

Thus $q+1 \leq 2p$ but $2c^2 > X^{2p-q} \geq X$ thus with the same arguments than higher for Catalan equation, we have $q+1 = 2p=4$ and there is a finite number of solutions (we have $X^P < 16_a$)!
And if $c^2 \leq 1_a$ we have $2_a > 2c^2 > X^{2p-q} > c^2 \Rightarrow q+ \geq 2p$ and then q+1=2p=4 and there is a finite number of solutions !

Catalan equation is solved, an original solution exists ! We have generalized the approach to Pillai equation and proved that it always has a finite number of solutions. It is the proof of Pillai conjecture. It seems that many problems of number theory can be solved like this

Bibliography

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