

Π^* - Regular Semigroups

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Abstracts Π^* -regular semigroups are promotion of Π -regular semigroups. In the paper we show that a semigroup with some characterizations is Π^* -regular, furthermore it is completely Π^* -regular

1 Introduction

An element a of semigroup S is regular if there exists an element $x \in S$ with $a = axa$. The semigroup S is regular if its all elements are regular ([1]). Obviously group is a regular semigroup, but we study that classes of regular semigroups are much wider than ones of groups. Therefore, it is necessary to explore deeply the regular semigroups. In fact that Clifford semigroups, regular order semigroups, quasi-regular semigroups, and so on are on the promotion of regular semigroups. In 1988 Hanurantha had attained beautiful conclusion congruences on regular semigroups problem, in addition, a lot of on regular semigroups problems had been studied (see [2-8]). Based on this, in this article we extend Π -regular semigroups, this is the so-called Π^* -regular semigroup. Being a subclass of the important class of regular semigroup, Π^* -regular semigroup inherit all the concepts and machinery pertaining to regular semigroups.

Definition 1. An element d of a semigroup S is Π^* -regular, if S_1 and S_2 are non empty regular semigroups, and S is all non-empty subset of $S_1 \times S_2$, for any $a \in S_1, b \in S_2, a, b, d \in S$, there exist $m \in \mathbb{Z}^+, (x, y) \in S$ such that $(a, b)^m = (a, b)^m (x, y) (a, b)^m$. A semigroup S is Π^* -regular, if every element of S is Π^* -regular.

Definition 2. A semigroup S is completely Π^* -regular, if S is Π^* -regular, for any $(a, b) \in S$ and element (a, b) is a regular, there exist $m \in \mathbb{Z}^+, (x, y) \in S$ such that $(a, b)^m (x, y) = (x, y) (a, b)^m$.

Definition 3. A semigroup S is right (left) Π^* -inverse if it is Π^* -regular and

$$(a, b), (x, y), (u, v) \in S, (a, b) = (a, b)(x, y)(a, b) = (a, b)(u, v)(a, b)$$

$$\text{implies } (x, y)(a, b) = (u, v)(a, b) \text{ and } ((a, b)(x, y) = (a, b)(u, v)).$$

Now we begin to define a kind of Green relation on S , S at most has an idempotent certainly

For any $(a, b), (c, d) \in S$,

$$(a, b)L^*(c, d) \Leftrightarrow S(a, b) = S(c, d); (a, b)R^*(c, d) \Leftrightarrow (a, b)S = (c, d)S;$$

$$(a, b)J^*(c, d) \Leftrightarrow S(a, b)S = S(c, d)S.$$

Lemma 1 The following condition on element (a, b) of S are equivalent:

- (1) (a, b) is Π^* -regular;
- (2) there is $m \in \mathbb{Z}^+$ such that $R(a, b)^m (L(a, b)^m)$ has an idempotent generator;
- (3) there is $m \in \mathbb{Z}^+$ such that $R(a, b)^m (L(a, b)^m)$ has a left(or right) identity.

Proof. (1) \Rightarrow (2). If $(a, b) \in S$ is Π^* -regular, then $(a, b)^m = (a, b)^m (x, y)(a, b)^m$ for some $m \in \mathbb{Z}^+$ and $(x, y) \in S$, and $(e, e) = (a, b)^m (x, y)$ is an idempotent for which

$(e, e)(a, b)^m = (a, b)^m$. Hence, condition (2) holds.

(2) \Rightarrow (3). Let $(e, e)(a, b)^m = (a, b)^m$, for some $(x, y) \in S$, and $m \in \mathbb{Z}^+$. Then, for an arbitrary $(u, v) \in R(a, b)^m$, we have that $(u, v) = (a, b)^m (c, d)$ for some $(c, d) \in S$, so $(a, b)^m (x, y)(u, v) = (a, b)^m (x, y)(a, b)^m (c, d) = (a, b)^m (c, d) = (u, v)$. Hence, $(a, b)^m (x, y)$ is a left identity for $R(a, b)^m$.

(3) \Rightarrow (1). Let $(a, b)^m (x, y)$ is a left identity for $R(a, b)^m$, then

$(e, e) = (a, b)^m (x, y)$, condition (1) holds.

(1) \Rightarrow (3) and (2) \Rightarrow (1) are obvious.

In particular, we have the following corollary.

Corollary 2. Let S be a semigroup, the following conditions are equivalent:

- (1) S is Π^* -regular;
- (2) for every $(a, b) \in S$ there exists $m \in \mathbb{Z}^+$ such that $R((a, b)^m) = (e, e)S$
 $(L((a, b)^m) = S(e, e))$;
- (3) for every $(a, b) \in S$ there exists $m \in \mathbb{Z}^+$ such that $R(a, b)^m (L(a, b)^m)$ has a

left(or right) identity.

Corollary 3. An element (a, b) of a semigroup is Π^* -regular if and only if there is an idempotent $(e, e) \in S$ such that $(a, b)S^1 = (e, e)S$.

Lemma 4. Let σ be a congruence on a Π^* -regular semigroup S , and $I, \Lambda \in S/\sigma$, such that $I = I\Lambda$, $\Lambda = \Lambda I \in S/\sigma$ then, there exist element $(a, b), (c, d) \in S$ such that $(a, b) \in I$, $(c, d) \in \Lambda$, and $(a, b) = (a, b)(c, d)(a, b), (c, d) = (c, d)(a, b)(c, d)$ in S .

Proof. Let $(x, y) \in I, (u, v) \in \Lambda$ and let (h, l) be an inverse of $((x, y)(u, v))^{2m}$ for some $m \in \mathbb{Z}^+$.

Now we assume that $(a, b) = (x, y)(u, v)(h, l)((x, y)(u, v))^{2m-1}(x, y)$ and $(c, d) = (u, v)(h, l)((x, y)(u, v))^{2m-1}$

then $(a,b) = (a,b)(c,d)(a,b)$, $(c,d) = (c,d)(a,b)(c,d)$.

Since $I = I\Lambda I, \Lambda = \Lambda I \Lambda$

we can get $(x,y)\sigma(x,y)(u,v)(x,y)$, $(u,v)\sigma(u,v)(x,y)(u,v)$,

so we have $(x,y)(u,v)\sigma((x,y)(u,v))^k, k \in Z^+$,

from this it has that $(x,y)(u,v)(h,l)\sigma((x,y)(u,v))^{2m}(h,l)$,

$$((x,y)(u,v))^{2m-1}(x,y)\sigma((x,y)(u,v))^{2m}(x,y),$$

and so

$$(x,y)(u,v)(h,l)((x,y)(u,v))^{2m-1}(x,y)\sigma((x,y)(u,v))^{2m}(h,l) \sqcup$$

$$((x,y)(u,v))^{2m} = ((x,y)(u,v))^{2m} \sqcup (x,y).$$

In terms of $(x,y)(u,v)\sigma((x,y)(u,v))^k, k \in Z^+$, $((x,y)(u,v))^{2m}(x,y)\sigma(x,y)$, we have that $(a,b)\sigma(x,y)$ and similarly $(c,d)\sigma(u,v)$. Hence, $(a,b) \in I, (c,d) \in \Lambda$.

Corollary 5. Let σ be a congruence on a Π^* -regular semigroup S . Then every σ -class which is an idempotent of S/σ contains an idempotent of S .

Lemma 6. Let S be a completely Π^* -regular semigroup. Then $M(e,e) = \{(a,b) \in S \mid (x,y)(u,v) \in G(e,e)\}$, for some $(x,y) \in G(e,e)$ is a subsemigroup of S with right ideal $G(e,e)$.

Proof. By definition 2 and lemma 1, this result holds, we don't prove it here. Let S be a completely Π^* -regular semigroup. we define a relation K on S with $(x,y)K(u,v)$ if and only if there exists $(e,e) \in E(S)$, $(x,y), (u,v) \in K(e,e)$, we can attain following lemma([5]).

Lemma 7. Let S be a completely Π^* -regular semigroup. If $J \subseteq K$, then

(1) for every $(a,b), (c,d) \in S$, $(e,e) \in E(S)$, $(a,b)(c,d) \in K(e,e)$ if and only if $(c,d)(a,b) \in K(e,e)$;

(2) $(a,b)(c,d)K(a,b)^2(c,d), (a,b)(c,d)K(a,b)(c,d)^2$.

Proof. (1) Since $J \subseteq K$, $(a,b), (c,d) \in S, (e,e) \in E(S)$,

$$(a,b)J^*(c,d) \Leftrightarrow S(a,b)S = S(c,d)S,$$

Then $(a,b)(c,d) \in K(e,e)$ if and only if $(c,d)(a,b) \in K(e,e)$.

(2) By(1), $(a,b)(c,d)K(a,b)^2(c,d), (a,b)(c,d)K(a,b)(c,d)^2$, this is obvious.

2 Main Results

Theorem 1. Let σ be a congruence on a Π^* -regular semigroup S and $n \in Z^+$. If $A, S_1, S_2, \dots, S_n \in S/\sigma$ and $A = AS_i A, S_i = S_i AS_i, i = 1, 2, \dots, n$, then there exist $(a_{01}, a_{02}), (s_{11}, s_{12}), (s_{21}, s_{22}), \dots, (s_{n1}, s_{n2}) \in S$ such that $(a_{01}, a_{02}) \in A, (s_{i1}, s_{i2}) \in S_i, (a_{01}, a_{02}) = (a_{01}, a_{02})(s_{i1}, s_{i2})(a_{01}, a_{02}), (s_{i1}, s_{i2}) = (s_{i1}, s_{i2})(a_{01}, a_{02})(s_{i1}, s_{i2}), i = 1, 2, \dots, n$.

Proof. (to use mathematical induction to prove). When $n = 1$ the theorem is true, in terms of lemma 4. Now to suppose that the theorem holds for positive integer $k < n$. Then there exist elements

$$(x_{01}, x_{02}), (y_{11}, y_{12}), (y_{21}, y_{22}), \dots, (y_{k1}, y_{k2}) \in S$$

such that

$$(x_{01}, x_{02}) \in A, (y_{i1}, y_{i2}) \in S_i,$$

$$(x_{01}, x_{02}) = (x_{01}, x_{02})(y_{i1}, y_{i2})(x_{01}, x_{02}),$$

$$(y_{i1}, y_{i2}) = (y_{i1}, y_{i2})(x_{01}, x_{02})(y_{i1}, y_{i2}), i = 1, 2, \dots, k.$$

Take any element $(y_{k+11}, y_{k+12}) \in S_{k+1}$. Because of S is Π -regular semigroup, we obtain

that there exists $n \in \mathbb{Z}^+$ such that $((x_{01}, x_{02})(y_{k+11}, y_{k+12}))^{2n}$ is Π^* -regular element and let (h, l) be any inverse of it. Let us make

$$\alpha = (x_{01}, x_{02})(y_{k+11}, y_{k+12})(h, l)((x_{01}, x_{02})(y_{k+11}, y_{k+12}))^{2n-1}(x_{01}, x_{02})$$

$$\beta = (y_{k+11}, y_{k+12})(h, l)((x_{01}, x_{02})(y_{k+11}, y_{k+12}))^{2n-1}$$

$$\gamma_i = (y_{i1}, y_{i2})(x_{01}, x_{02})(y_{k+11}, y_{k+12})(h, l).$$

$$((x_{01}, x_{02})(y_{k+11}, y_{k+12}))^{2n-1}(x_{01}, x_{02})(y_{i1}, y_{i2}), i = 1, 2, \dots, k.$$

It is quite routine to show that $\alpha \in A, \gamma_i \in S_i$, and $\alpha = \alpha\gamma_i\alpha, \gamma_i = \gamma_i\alpha\gamma_i$, for $i = 1, 2, \dots, k, k+1$.

In addition, from the theorem, we can also define a completely Π^* -regular semigroup S if for any $(a, b) \in S$, there exist $m \in \mathbb{Z}^+$ $(x, y) \in S$ such that

$$(a, b)^m(x, y) = (a, b)^m(x, y)(a, b)^m,$$

And

$$(a, b)^m(x, y) = (x, y)(a, b)^m.$$

Theorem 2. Let S be a right (left) Π^* -inverse. Then the following conditions are equivalent on a semigroup S .

(1) S is a right Π^* -inverse;

(2) S is Π^* -regular and for any $(e, e), (f, f) \in E(S)$ there exists $m \in \mathbb{Z}^+$ such that

$$((e, e)(f, f))^m = ((f, f)(e, e)(f, f))^m;$$

(3) S is Π^* -regular and $(e, e), (f, f) \in E(S)$ there exists $m \in \mathbb{Z}^+$ such that $((e, e)(f, f))^m$

$$R^* ((f, f)(e, e))^m;$$

(4) I and S/I are right Π^* -inverse for every ideal I of S .

Proof. By definition 3 and lemma 6, 7, the results holds.

Theorem 3. The following conditions are equivalent on a semigroup S :

- (1) S is a completely Π^* -regular;
- (2) S is left and right Π^* -regular;
- (3) every left (right) ideal of S is Π^* -regular;
- (4) some power of each element of S lies in a subsemigroup of S ;
- (5) for some $(a,b) \in S$ there exists $m \in \mathbb{Z}^+$ such that $(a,b)^m \in (a,b)^m S (a,b)^{m+1}$.

Proof. (1) \Rightarrow (2). This implication follows by lemma 1 .

(2) \Leftrightarrow (3). This is obviose.

(3) \Leftrightarrow (4). By lemma 6,7, This is also obviose.

(4) \Rightarrow (5). This is also obviose.

(5) \Rightarrow (4). If (5) is true ,then for $(a,b) \in S$ there exists $m \in \mathbb{Z}^+$ and $(x,y) \in S$ such that

$$(a) \quad (a,b)^m = (a,b)^m (x,y)(a,b)^{2m} .$$

From this it follows that

$$(b) (a,b)^m = (a,b)^m \left((x,y)(a,b)^m \right)^2 (a,b)^{2m} = \dots = (a,b)^m \left((x,y)(a,b)^m \right)^n (a,b)^{nm} = \dots$$

By the hypothesis,we find that there exists $k \in \mathbb{Z}^+$ and $(u,v) \in S$ such that

$$(c) \quad \left((x,y)(a,b)^m \right)^k = \left((x,y)(a,b)^m \right)^k (u,v) \left((x,y)(a,b)^m \right)^{3k}$$

and by (a) we still obtain

$$(d) \quad \left((x,y)(a,b)^m \right)^{3k} (a,b)^m = \left((x,y)(a,b)^m \right)^{3k-1} .$$

Again we have

$$(a,b)^m = (a,b)^m \left((x,y)(a,b)^m \right)^k (a,b)^{km} \quad (\text{by (b)})$$

$$= (a,b)^m \left((x,y)(a,b)^m \right)^k (u,v) \left((x,y)(a,b)^m \right)^{3k} (a,b)^{km} \quad (\text{by (c)})$$

$$= (a,b)^m \left((x,y)(a,b)^m \right)^k (u,v) \left((x,y)(a,b)^m \right)^{2k} \quad (\text{by (d)})$$

$$= (a,b)^m \left((x,y)(a,b)^m \right)^k (u,v) \left((x,y)(a,b)^m \right)^k (u,v) \left((x,y)(a,b)^m \right)^{4k}$$

So,at that time,there exist $(h,l) \in S$ such that

$$(a,b)^m = (a,b)^m (h,l) \left((x,y)(a,b)^m \right)^{4k} ,$$

$$(h,l) = \left((x,y)(a,b)^m \right)^k (u,v) \left((x,y)(a,b)^m \right)^k (u,v) .$$

Furthermore,

$$(e) \quad (a,b)^{2m} (x,y)(a,b)^m = (a,b)^m \left((a,b)^m (x,y)(a,b)^m \right) = (a,b)^m$$

By (a) and (e) ,this implication follows.

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