*Bulletin of Mathematical Sciences and Applications* *Online: 2012-08-01*
*ISSN: 2278-9634, Vol. 1, pp 46-51*
*doi:10.18052/www.scipress.com/BMSA.1.46*


# $\Pi^*$ - Regular Semigroups

Xiaoqiang LUO

Department of mathematics Sichuan University of Arts and Science, Dazhou, Sichuan, 635000, China

**Keywords:** $\Pi$-regular semigroup, $\Pi^*$-regular semigroup, Completely, $\Pi^*$ -regular semigroup, Congruence

**Abstracts** $\Pi^*$-regulur semigroups are promotion of $\Pi$-regular semigroups. In the paper we show that a semigroup with some characterizations is $\Pi^*$ - regular, futhermore it is completely $\Pi^*$ -regular

## 1 Introduction

An element $a$ of semigroup $S$ is regular if there exists an element $x \in S$ with $a = axa$ The semiigroup $S$ is regular if its all elements are regular([1]). Obviously group is a regular semigroup,but we study that classes of regular semigroups are much wider than ones of groups . Therefore, it is necessary to explore deeply the regular semigroups. In fact that Clifford semigroups, regular order semigroups, quasi-regular semigroups, and so on are on the promotion of regular semigroups. In 1988 Hanurantha had attained beautiful conclusion congruences on regular semigroups problem, in addition, a lot of on regular semigroups problems had been studied (see[2-8]). Based on this, in this article we extends $\Pi$ -regular semigroups, this is the so-called $\Pi^*$ -regulur semigroup.Being a subclass of the impotant class of regular semigroup , $\Pi^*$ -regulur semigroup inherit all the concepts and machinery pertaining to regular semigroups.

**Definition 1** . An element $d$ of a semigroup $S$ is $\Pi^*$ -regular , if $S_1$ and $S_2$ are non empty regular semigroups,and $S$ is all non-empty subset of $S_1 \times S_2$ , for any $a \in S_1$ , $b \in S_2$ $a,b) = d \in S$ ,there exist $m \in Z^+$ , $(x, y) \in S$ such that $(a,b)^m = (a,b)^m (x,y)(a,b)^m$ . A semigroup $S$ is $\Pi^*$ -regulur, if every element of $S$ is $\Pi^*$ -regulur.

**Definition 2.** A semigroup $S$ is completely $\Pi^*$ -regular, if $S$ is $\Pi^*$ -regulur,for any $(a,b) \in S$ and element $(a,b)$ is a regulur, there exist $m \in Z^+$ , $(x, y) \in S$ such that $(a,b)^m$ $(x, y) = (x, y)(a,b)^m$ .

**Definition 3.** A semigroup $S$ is right (left) $\Pi^*$ -inverse if it is $\Pi^*$ -regular and

$$(a,b),(x,y),(u,v) \in S , (a,b) = (a,b)(x,y)(a,b) = (a,b)(u,v)(a,b)$$

implies $(x,y)(a,b) = (u,v)(a,b)$ $((a,b)(x,y) = (a,b)(u,v))$.

Now we begin to define a kind of Green relation on S, S at most has an idemptent certainly

For any $(a,b),(c,d)$ $\in S$ ,

$$(a,b)L^*(c,d) \Leftrightarrow S(a,b) = S(c,d);(a,b)R^*(c,d) \Leftrightarrow (a,b)S = (c,d)S;$$

$$(a,b)J^*(c,d) \Leftrightarrow S(a,b)S = S(c,d)S.$$

**Lemma 1** The following condition on element $(a,b)$ of $S$ are equivalent:

(1) $(a,b)$ is $\Pi^*$-regular;

(2) there is $m \in Z^+$ such that $R(a,b)^m \left(L(a,b)^m\right)$ has an idempotent generator;

(3) there is $m \in Z^+$ such that $R(a,b)^m \left(L(a,b)^m\right)$ has a left(or right) identity.

Proof. (1)$\Rightarrow$(2). If $(a,b) \in S$ is $\Pi^*$-regular, then $(a,b)^m = (a,b)^m (x,y)(a,b)^m$ for some $m \in Z^+$ and $(x,y) \in S$, and $(e,e) = (a,b)^m (x,y)$ is an idempotent for which $(e,e)(a,b)^m = (a,b)^m$. Hence, condition (2) holds.

(2)$\Rightarrow$(3). Let $(e,e)(a,b)^m = (a,b)^m$, for some $(x,y) \in S$ , and $m \in Z^+$. Then, for an arbitrary $(u,v) \in R(a,b)^m$, we have that $(u,v) = (a,b)^m (c,d)$ for some $(c,d) \in S$, so $(a,b)^m (x,y)(u,v) = (a,b)^m (x,y)(a,b)^m (c,d) = (a,b)^m (c,d) = (u,v)$ . Hence, $(a,b)^m (x,y)$ is a left identity for $R(a,b)^m$.

(3)$\Rightarrow$(1). Let $(a,b)^m (x,y)$ is a left identity for $R(a,b)^m$, then $(e,e) = (a,b)^m (x,y)$, condition (1) holds.

(1)$\Rightarrow$(3) and (2)$\Rightarrow$(1) are obviouse.

In particular, we have the following corollary.

**Corollary 2.** Let S be a semigroup, the following conditions are equivalent:

(1) $S$ is $\Pi^*$-regular;

(2) for every $(a,b) \in S$ there exists $m \in Z^+$ such that $R\left((a,b)^m\right) = (e,e)S$ ($L\left((a,b)^m\right) = S(e,e)$);

(3) for every $(a,b) \in S$ there exists $m \in Z^+$ such that $R(a,b)^m \left(L(a,b)^m\right)$ has a left(or right) identity.

**Corollary 3**. An element $(a,b)$ of a semigroup is $\Pi^*$-regular if and only if there is an idempotent $(e,e) \in S$ such that $(a,b)S^1 = (e,e)S$.

**Lemma 4.** Let $\sigma$ be a congruence on a $\Pi^*$-regular semigroup $S$ ,and $\mathrm{I}, \Lambda \in S/\sigma$ , such that $\mathrm{I} = \mathrm{I}\Lambda\mathrm{I}$, $\Lambda = \Lambda\mathrm{I}\Lambda \in S/\sigma$ then , there exist element $(a,b),(c,d) \in S$ such that $(a,b) \in \mathrm{I}$ , $c,d) \in \Lambda$ , and $(a,b) = (a,b)(c,d)(a,b), (c,d) = (c,d)(a,b)(c,d)$ in $S$ .

Proof. Let $(x,y) \in \mathrm{I}, (u,v) \in \Lambda$ and let $(h,l)$ be an inverse of $((x,y)(u,v))^{2m}$ for some $m \in Z^+$.

Now we assume that $(a,b) = (x,y)(u,v)(h,l)((x,y)(u,v))^{2m-1}(x,y)$ and $(c,d) = (u,v)(h,l)((x,y)(u,v))^{2m-1}$

then $(a,b)=(a,b)(c,d)(a,b),\ (c,d)=(c,d)(a,b)(c,d)$.

Since $\mathrm{I}=\mathrm{I}\Lambda\mathrm{I},\Lambda=\Lambda\mathrm{I}\Lambda$

we can get $(x,y)\sigma(x,y)(u,v)(x,y),\ (u,v)\sigma(u,v)(x,y)(u,v)$,

so we have $(x,y)(u,v)\sigma((x,y)(u,v))^{k}, k\in Z^{+}$,

from this it has that $(x,y)(u,v)(h,l)\sigma((x,y)(u,v))^{2m}(h,l)$,

$$((x,y)(u,v))^{2m-1}(x,y)\sigma((x,y)(u,v))^{2m}(x,y),$$

and so

$$(x,y)(u,v)(h,l)((x,y)(u,v))^{2m-1}(x,y)\sigma((x,y)(u,v))^{2m}(h,l)\sqcup$$

$$((x,y)(u,v))^{2m}=((x,y)(u,v))^{2m}\sqcup(x,y).$$

In terms of $(x,y)(u,v)\sigma((x,y)(u,v))^{k}, k\in Z^{+}$, $((x,y)(u,v))^{2m}(x,y)\sigma(x,y)$, we have that $(a,b)\sigma$ $(x,y)$ and similarly $(c,d)\sigma(u,v)$.Hence , $(a,b)\in\mathrm{I}, (c,d)\in\Lambda$ .

**Corollary 5.** Let $\sigma$ be a congruence on a $\Pi^*$ -regular semigroup $S$ .Then every $\sigma$ – class which is an idempotent of $S/\sigma$ contains an idempotent of $S$ .

**Lemma 6.** Let $S$ be a completely $\Pi^*$ -regular semigroup.Then $M(e,e)=\{(a,b)\in S\mid(x,y)(u,v)\in G$ $(e,e)\}$, for some $(x,y)\in G(e,e)$ is a subsemigroup of $S$ with right ideal $G(e,e)$.

Proof. By definition 2 and lemma 1, this result holds, we don't prove it here. Let $S$ be a completely $\Pi^*$ -regular semigroup. we define a relation $K$ on $S$ with : $(x,y)K(u,v)$ if and only if there exists $(e,e)\in E(S)$, $(x,y),(u,v)\in K(e,e)$,we can attain following lemma([5]).

**Lemma 7.** Let $S$ be a completely $\Pi^*$ -regular semigroup. If $J\subseteq K$ , then
(1)for every $(a,b),(c,d)\in S$ , $(e,e)\in E(S)$, $(a,b)(c,d)\in K(e,e)$ if and only if $(c,d)(a,b)\in K(e,e)$;
(2) $(a,b)(c,d)K(a,b)^2\ (c,d),(a,b)(c,d)K(a,b)(c,d)^2$ .
Proof. (1)Since $J\subseteq K$ , $(a,b),(c,d)\in S$ , $(e,e)\in E(S)$,

$$(a,b)J^*(c,d)\Leftrightarrow S(a,b)S=S(c,d)S,$$

Then $(a,b)(c,d)\in K(e,e)$ if and only if $(c,d)(a,b)\in K(e,e)$.

(2) By(1), $(a,b)(c,d)K(a,b)^2\ (c,d),(a,b)(c,d)K(a,b)(c,d)^2$ ,this is obviouse.

## 2 Main Results

**Theorem 1.** Let $\sigma$ be a congruence on a $\Pi^*$ -regular semigroup $S$ and $n\in Z^+$ . If $A, S_1, S_2,\ldots,$ $S_n\in S/\sigma$ and $A=AS_i A, S_i=S_i AS_i, i=1,2,\ldots,n$ , then there exist $(a_{01},a_{02}),(s_{11},s_{12}),(s_{21},$ $s_{22}),\ldots,(s_{n1},s_{n2})\in S$ such that $(a_{01},a_{02})\in A,(s_{i1},s_{i2})\in S_i$ , $(a_{01},a_{02})=(a_{01},a_{02})(s_{i1},s_{i2})$ $(a_{01},a_{02})$, $(s_{i1},s_{i2})=(s_{i1},s_{i2})(a_{01},a_{02})(s_{i1},s_{i2}), i=1,2,\ldots,n$.

Proof. ( to use mathematical induction to prove). When $n=1$ the theorem is true, in terms of lemma 4. Now to suppose that the theorem holds for positive integer $k\langle n$ . Then there exist elements

$$(x_{01},x_{02}),(y_{11},y_{12}),(y_{21},y_{22})...,(y_{k1},y_{k2})\in S$$

such that $$(x_{01},x_{02})\in A,(y_{i1},y_{i2})\in S_i,$$

$$(x_{01},x_{02})=(x_{01},x_{02})(y_{i1},y_{i2})(x_{01},x_{02}),$$

$$(y_{i1},y_{i2})=(y_{i1},y_{i2})(x_{01},x_{02})(y_{i1},y_{i2}),i=1,2,...,k.$$

Take any element $(y_{k+11},y_{k+12})\in S_{k+1}$ . Because of $S$ is $\Pi$ -regular semigroup, we obtain that there exists $n\in Z^+$ such that $((x_{01},x_{02})(y_{k+11},y_{k+12}))^{2n}$ is $\Pi^*$ -aregular element and let $(h,l)$ be any inverse of it. Let us make

$$\alpha=(x_{01},x_{02})(y_{k+11},y_{k+12})(h,l)((x_{01},x_{02})(y_{k+11},y_{k+12}))^{2n-1}(x_{01},x_{02})$$

$$\beta=(y_{k+11},y_{k+12})(h,l)((x_{01},x_{02})(y_{k+11},y_{k+12}))^{2n-1}$$

$$\gamma_i=(y_{i1},y_{i2})(x_{01},x_{02})(y_{k+11},y_{k+12})(h,l)\cdot$$

$$((x_{01},x_{02})(y_{k+11},y_{k+12}))^{2n-1}(x_{01},x_{02})(y_{i1},y_{i2}),i=1,2,...,k.$$

It is quite routine to show that $\alpha\in A,\gamma_i\in S_i$ ,and $\alpha=\alpha\gamma_i\alpha$ , $\gamma_i=\gamma_i\alpha\gamma_i$ ,for $i=1,2,...,k$ , $k+1$.

In addition , from the theorem, we can also define a completely $\Pi^*$ -regular semigroup $S$ if for any $(a,b)\in S$ , there exist $m\in Z^+$ $(x,y)\in S$ such that

$$(a,b)^m(x,y)=(a,b)^m(x,y)(a,b)^m,$$

And $$(a,b)^m(x,y)=(x,y)(a,b)^m.$$

**Theorem 2.** Let $S$ be a right (left) $\Pi^*$ -inverse. Then the following conditions are equivalent on a semigroup $S$.

(1) $S$ is a right $\Pi^*$-inverse;

(2) $S$ is $\Pi^*$- regular and for any $(e,e),(f,f)\in E(S)$ there exists $m\in Z^+$ such that

$$((e,e)(f,f))^m=((f,f)(e,e)(f,f))^m;$$

(3) $S$ is $\Pi^*$ - regular and $(e,e),(f,f)\in E(S)$ there exists $m\in Z^+$ such that $((e,e)(f,f))^m$ $R^*$ $((f,f)(e,e))^m$ ;

(4)I and $S/I$ are right $\Pi^*$ -inverse for every ideal $I$ of $S$.

Proof. By definition 3 and lemma 6 , 7, the results holds.

**Theorem 3.** The following conditions are equivalent on a semigroup $S$:

(1) $S$ is a completely $\Pi^*$ -regular;

(2) $S$ is left and right $\Pi^*$ -regular;

(3) every left (right) ideal of $S$ is $\Pi^*$ -regular;

(4) some power of each element of $S$ lies in a subsemigroup of $S$;

(5) for some $(a,b)\in S$ there exists $m\in Z^+$ such that $(a,b)^m \in (a,b)^m S(a,b)^{m+1}$.

Proof. $(1)\Rightarrow(2)$. This implication follows by lemma 1 .

(2) $\Leftrightarrow$ (3). This is obviose.

(3) $\Leftrightarrow$ (4). By lemma6,7, This is also obviose.

$(4)\Rightarrow(5)$. This is also obviose.

(5) $\Rightarrow$(4). If (5) is true ,then for $(a,b)\in S$ there exists $m\in Z^+$ and $(x,y)\in S$ such that

(a) $$(a,b)^m = (a,b)^m (x,y)(a,b)^{2m}.$$

From this it follows that

(b) $$(a,b)^m = (a,b)^m\left((x,y)(a,b)^m\right)^2(a,b)^{2m} = \dots = (a,b)^m\left((x,y)(a,b)^m\right)^n(a,b)^{nm} = \dots$$

By the hypothesis,we find that there exists $k\in Z^+$ and $(u,v)\in S$ such that

(c) $$\left((x,y)(a,b)^m\right)^k = \left((x,y)(a,b)^m\right)^k(u,v)\left((x,y)(a,b)^m\right)^{3k}$$

and by (a) we still obtain

(d) $$\left((x,y)(a,b)^m\right)^{3k}(a,b)^m = \left((x,y)(a,b)^m\right)^{3k-1}.$$

Again we have

$$(a,b)^m = (a,b)^m\left((x,y)(a,b)^m\right)^k(a,b)^{km} \qquad \text{( by (b) )}$$

$$= (a,b)^m\left((x,y)(a,b)^m\right)^k(u,v)\left((x,y)(a,b)^m\right)^{3k}(a,b)^{km} \qquad \text{(by (c) )}$$

$$= (a,b)^m\left((x,y)(a,b)^m\right)^k(u,v)\left((x,y)(a,b)^m\right)^{2k} \qquad \text{(by (d) )}$$

$$= (a,b)^m\left((x,y)(a,b)^m\right)^k(u,v)\left((x,y)(a,b)^m\right)^k(u,v)\left((x,y)(a,b)^m\right)^{4k}$$

So,at that time,there exist $(h,l)\in S$ such that

$$(a,b)^m = (a,b)^m(h,l)\left((x,y)(a,b)^m\right)^{4k},$$

$$(h,l) = \left((x,y)(a,b)^m\right)^k(u,v)\left((x,y)(a,b)^m\right)^k(u,v).$$

Furthermore,

(e) $$(a,b)^{2m}(x,y)(a,b)^m = (a,b)^m\left((a,b)^m(x,y)(a,b)^m\right) = (a,b)^m$$

By (a) and (e) ,this implication follows.

**References**

[1] Mario Petrich and Norman R .Reilly., Completely Regular Semigroups, New York , JohN Wiley & Sons , Inc. Press, 1999.

[2] Howie J M ., An Introduction to Semigroup Theory,New York ,Academic Press,1976.

[3] Bogdanovic`. S`., Power Regular Semigrops , Zbornik Radova PMF Novi Sad, 1982 (12), 418-428.

[4] Bogdanovic`. S`., Right ∏-Inverse Semigroups, Zbornik Radova PMF Novi Sad, 1985 (18), 312-316.

[5] Pastijn F , Petrich M., Congruences on Regular Semigroups , Trans.Amer.Math.Soc, 1986(295), 607-633.

[6] Hall T E., On Regular Semigroups , J. Algebra, 1973(24), 1-24.

[7] Xiaoqiang LUO, Another Characterition of Congruences on Completely Simple Semigroups,Journal of Sichuan University of Arts and Science, 2010,20(2),20-22.

[8] Xiaoqiang LUO, Another Property of Congruence-free Regular Semigroups, South Journal of Mathematics, 2012, 2(3), 238-241.