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RELATIONS BETWEEN SCHMIDT'S, MILLER- MORENO'S GROUPS AND GROUPS WITH BASIS PROPERTY

Alkadhi M.

Al-Imam University KSA

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Abstract In this paper we prove that a Schmidt's group forms a group with basis property if and only if it is a Miller- Moreno's group of type $G(p,q)$ such that $p \neq q(\, p,q \;\; are \; primes)$. Therefore doesn't group with basis property if it is a Miller- Moreno's group of type $G(p,q^v), v \in \mathbb{N},\; v > 1$.

1. Introduction

In our previous study of finite groups with basis property, [1] we note, that the structure of these groups is very close to some famous classes of groups, like Schmidt's and Miller- Moreno's groups.

Schmidt and Miller- Moreno groups had been studied at the beginning of the twentieth century by many mathematicians as Golfand [2], Jotrov [5], Schmidt [11], Redei [8], and others. Since these groups had been studied in details and had been classified in [3], and [11], while the class of groups with basis property had been classified by Al Khalaf [1]. In this paper we wil invlestigate a relations between these classes and the intersection among them.

2. Preliminaries

Definition 2.1

An inverse semigroup (a group) S is called an inverse semigroup (a group) with basis property, if any two minimal (irreducible) generating sets (with respect to inclusion) of any subsemigroup (subgroup) H of S are equivalent (i.e. they have the same cardinality) [4]

Definition 2.2

A finite nonabelian group, in which every proper subgroup is abelian, is called a Miller- Moreno's group [7]

Definition 2.3

A finite nonnilpotent group, in which every proper subgroup is nilpotent, is called a Schmidt's group, [10]

Example 2.4

Let G be the quaternion group Q_8

$$Q_8 = \langle\, x, y : \;\; x^4 = 1 \;,\;\; x^2 = y^2 \;\;, xy = yx^3 \rangle.$$

The order of Q_8 is 8 and its nonabelian group, but all proper subgroups of Q8 are abelian.
First, we like to know which Schmidt's and Miller- Moreno's groups are groups with basis property.
Redei, [8] had classified the Miller- Moreno's group as follows: Lemma 2.5 (theorem (6.5.6)[8]
Let G be a finite Miller Moreno's group, then G is one of the followings:

1. G is the quaternion group
2. $G = \langle x, y \;:\;\; y^{p^u} = x^{p^{v+1}} = 1 \;,\; b^{-1}ab = a^{1+p^v} \rangle$, where p is a prime.

3. $G = M(p,m,n)$ which is given by:
$G = \langle a,b,c\,;\ (a^{p^m} = b^{p^n} = c^p = 1), (, ac = ca, bc = cb\ , [a,b] = c)\rangle$, where p is a prime and $m \geq n \geq 1$, and clearly the order G of is p^{m+n+1}.
4. $G = G(p,q^v)$ and G is a Schmidt's group.

Corollary 2.6[*]

Let p and q be distinct primes, $k \in \mathbb{N}$ and

$$t = min\{\, k \in \mathbb{N}\ :\ p^k \equiv 1\ (mod\ q)\}.$$

So t is the order of $p -$ element in the multiplicative group $\mathbb{Z}_q^*$

Let $f(x)$be an irreducible polynomial in $\mathbb{Z}_p[x]$ dividing:

$$x^{q-1} + x^{q-2} + \cdots + x + 1$$

Then by [8] the degree of $f(x)$ is t and it is the smallest positive integer such that

$$f(x)|x^{q-1} + x^{q-2} + \cdots + x + 1$$

Corollary 2.7

Consider the set of all pairs $(i, g(x))$ where $i \in \mathbb{Z}_{q^v}$, $g(x) \in \mathbb{Z}_p[x]$.
Define on the set:

$$C = \{(i, g(x))\,; i \in \mathbb{Z}_{q^v}, g(x) \in \mathbb{Z}_p[x]\} \tag{1}$$

an operation:
$(i, h(x)).(j, g(x)) = (i + j, h(x) + w^i g(x))\ , w \in \mathbb{Z}_p[x]/f(x)$.

Then by [6] $(C,\ \cdot)$ forms a Miller-Moreno's group denoted by $G(p,q^v)$
Therefore $G(p,q^v)$ has p^t Sylow $q -$ subgroups and its commutator is:

$$(G(p,q^v))\grave{} = \{(0, h(x)) : h(x) \in \mathbb{Z}_p[x]\}$$

Lemma 2.8 (Theorem (6.5.7) [9]

Let G be a finite nonnilpotent group, but all its proper subgroups are nilpotent. Then $\exists P \in Syl_p(G)$ and $\exists Q \in Syl_q(G)$ for some distinct primes p and q such that the followings hold:

1. $G = PQ$.
2. $P \lhd G$.
3. P is cyclic group.
4. G is solvable.

Lemma 2.9 (Theorem 1), [1]

Let G be a finite semidirect product of a p-group $P = Fit\,(\,G\,)$ (Fitting subgroup) of G by a cyclic q-group ⟨y⟩ of order q^β where $p \neq q$ are primes, and $\beta \in \mathbb{N}$. .

Then the group G has basis property if and only if for every element $u \in \langle y\rangle$, $u \neq 1$, and for any invariant subgroup H of P the automorphism φ_u defines an isotopic representation on every quotient Frattini subgroup H.

In, [10] it had been shown that the order of the group $G(p,q^v)$ is $p^t q^v$ is and the q-Sylow subgroups of $G(p,q^v)$ are cyclic groups of order q^v and p-subgroup of $G(p,q^v)$ is elementary abelian group of order p^t which corresponds to the commutator $(G(p,q^v))\grave{}$. Then $(G(p,q^v))\grave{}$ is a normal subgroup of $G(p,q^v)$ and it is Sylow p-subgroup.

Let $\langle\bar{Q}\rangle$ be the cyclic group isomorphic to Sylow q- subgroup of order q^v hence the group $G(p,q^v)$ is a semi direct product of the group $(G(p,q^v))\grave{}$ by $\langle\bar{Q}\rangle$ $, i.e.

$$G(p,q^v) = (G(p,q^v))\grave{} \rtimes \langle\bar{Q}\rangle = (0, h(x)) \rtimes \langle(-1,0)\rangle\, , h(x) \in \mathbb{Z}_p[x].$$

[*] The Fitting sub group Fit (G) of a group is the maximum normal nilpotent subgroup
[**] Isotopic representation is the representation, which is writing as direct sum of equivalent irreducible representation.

Theorem 2.10

Let G be a Schmidt's group. Then G s a group with basis property if and only if G is a Miller-Moreno's group of type $G(p,q)$, where $p \neq q$ are primes.

Proof.

Let G be a Schmidt's group satisfying basis property. Then by [1] and [11] G is a group of order $p^{\alpha}q^{\beta}, \alpha, \beta \in \mathbb{N}$ where $p \neq q$ are primes, so G is an extension of p-group $P = Fit\,(G)$ by the cyclic q-group $\langle y\rangle$ of order q^{β}. Thus

$$G = \langle P, y\rangle \,,\, O(y) = q^{\beta} \,.$$

By [9] y^q is in the center of the subgroup P since the group G satisfices basis property, and every group with basis property is Quaziprimary [1] (Quaziprimary group: It is a finite group in which every element has prime power order), hence $y^q = 1$ and $\beta = 1$.

Let M be a largest normal subgroup of G properly contained in P Then by [1] y commutate with every element of M and by Quaziprimarity of G we get M={1} .Thus P is an elementary abelian group and every element of $\langle y\rangle$, induces an automorphism on a group P denoted by φ . So P can be viewed as a vector space $GF(p)$ with dimension α and φ is a linear operator on the vector space P. Since M={1}, then P doesn't contain any invariant subspaces with respect to φ,hence

$$\alpha = min\,\{\, k \in \mathbb{N} \,:\, p^k \equiv 1 \pmod q\}.$$

By [1] the matrix A_f of the operator φ in given basis of vector space P has the form:

$$A_f = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{\alpha-1} \end{bmatrix},$$

And A_f Associated matrix with irreducible polynomial:

$$f(x) = x^{\alpha} + a_{\alpha-1}x^{\alpha-1} + \cdots + a_1 x + a_0 \,,$$

over the field $GF(p)$ such that $f(x)$ dividing the polynomial:

$$x^{q-1} + x^{q-2} + \cdots + x + 1 \in \mathbb{Z}_p[x] \,.$$

By [1], if $w \in P, w \neq 1$, then every element of P can be represented as $g(\varphi)w$ where $g(x) \in GF(p)[x]$. Note that if $h(x) \in GF(p)[x]$ then

$$h(\varphi)w = g(\varphi)w \;\Leftrightarrow f(x)\big|\big(h(x) - g(x)\big)\,.$$

Considering P as an additive group of the field $GF(p)[x]\big/_{f(x)}$ Then for any elements $u, v \in G$, there exists a unique elements $i, j \in \mathbb{Z}_q$ and $w_1, w_2 \in P$ such that:

$$u = y^i w_1 \,,$$
$$v = y^j w_2 .$$

By [1] the automorphism φ given by:

$$\varphi(w) = y^{-1}wy \;;\; \forall w \in P \,.$$

Now using a multiplication on G

$$uv = y^i w_1 y^j w_2 = y^{i+j}\varphi^j(w_1)w_2 \tag{2}$$

Viewing *u* and *v* as pairs $(j,h(x)),(i,g(x))$ where $g(x),h(x)\in GF(p)[x]$ and
$$w_1=g(\varphi)w,\ w_2=h(\varphi)w, \text{for } w\in P$$
and considering the addition on *P,* we have:
$$(i,g(x))=(j,h(x)) \Leftrightarrow$$
$$i=j\,;\ g(x)\equiv h(x)(mod\ \ f(x)) \tag{3}$$
$$i,j\in\mathbb{Z}_q\,.$$
Using (3) then the multiplication (2) becomes:
$$(i,g(x))(j,h(x))=\left(i+j,w^j g(x)+h(x)\right) \tag{4}$$
So by [7] *G* is a Miller-Moreno's group of type $G(p,q)$.

Conversely, assuming that *G* a Schmidt's group, which is a Miller-Moreno's group of type $G(p,q)$. Defining *G* as the group (C,) and let $f(x)$ be an irreducible polynomial over the field $GF(p)$ of degree α such that
$$\alpha=min\{k\in\mathbb{N}\colon p^k\equiv 1(mod\ q)\},$$
And $f(x)$ divides
$$x^{q-1}+x^{q-2}+\cdots+x+1.$$
Applying the operation (4) on the set (1) and using the congruence in (3) .Then we get that G is a semi direct product of the additive group of the field $GF(p)[x]/f(x)$ by the cyclic group ⟨(1,0)⟩ of the order q . Now
$$(1{,}0)^{-1}(0,h(x))(1{,}0)=$$
$$(q-1{,}0)(0,h(x))(1{,}0)=$$
$$(q-1,h(x))(1{,}0)=$$
$$(0,wh(x)).$$
Thus $\langle(1{,}0)\rangle$ acts by conjugation, on the elements of $GF(p)[x]/f(x)$, so the resulting polynomial is a multiplication by *w*. Thus the operator φ in the basis
$$\{(0{,}1),(0,w),\dots,(0,w^{\alpha-1})\}$$
has the above matrix A_f. So by [1] G is a group with basis property. ∎

Theorem 2.11

Let *G* be a Miller-Moreno's group, which is non *p*- primary group. Then the group *G* has basis property if and only if it is a Miller-Moreno's group of type $G(p,q)$. for distinct primes, *q*

Proof.

Let *G* be a group with basis property and it is not primary Miller-Moreno's group, then by Theorem (2.10) *G* is a semi direct product of *p*-group *P* by the cyclic *q*- group ⟨y⟩, where *y* induces a regular automorphism on *P*.

Then G not nilpotent, and since all proper subgroups of G are abelian, thus G is Schmidt's group. Consequently by Theorem (2.10) G is a group of type $G(p,q)$.

Conversely, let G be a Miller Moreno's group of type $G(p,q)$. then by Theorem (2.10) it is a group of type $G(p,q)$. so it is a group with basis property. ■

Theorem 2.12

Let G be a Miller-Moreno's group of type $G(p,q^v)$ $p \neq q$ are primes and $v \in \mathbb{N}$.
If $v > 1$, then G does not have a basis property.

Proof.

By [9] in order that G be a Miller-Moreno's group, then by Lemma (2.5) G has one of the forms:

1. G is the quaternion group Q_8, and it's a finite $p-$ group.
2. The group G of type:

$$G = \langle x, y \ : \ y^{p^u} = x^{p^{v+1}} = 1 \ , \ b^{-1}ab = a^{1+p^v} \rangle,$$

3. The group $G = M(p,m,n)$is a primary group of order p^{m+n+1}.
4. The group $G(p,q^v)$ and G is a Schmidt's group.

In cases (1),(2) and (3) the group G is a primary, so by [1] G is a group with basis property.

Since in case (4) G isn't primary group, and by Theorem (2.11) $G(p,q^v)$ is a group with basis property if *v=1*. Thus the group $G(p,q^v)$ hasn't basis property, when $v > 1$ ■

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