RELATIONS BETWEEN SCHMIDT'S, MILLER- MORENO'S GROUPS AND GROUPS WITH BASIS PROPERTY

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Abstract In this paper we prove that a Schmidt's group forms a group with basis property if and only if it is a Miller- Moreno's group of type $\mathcal{G}(p, q)$ such that $p \neq q (p, q \text{ are primes})$. Therefore doesn’t group with basis property if it is a Miller- Moreno's group of type $\mathcal{G}(p, q^v), v \in \mathbb{N}, v > 1$.

1. Introduction

In our previous study of finite groups with basis property, [1] we note, that the structure of these groups is very close to some famous classes of groups, like Schmidt's and Miller- Moreno's groups.

Schmidt and Miller- Moreno groups had been studied at the beginning of the twentieth century by many mathematicians as Golfand [2], Jotrov [5], Schmidt [11], Redei [8], and others. Since these groups had been studied in details and had been classified in [3], [11], while the class of groups with basis property had been classified by Al Khalaf [1]. In this paper we wil invetsigate a relations between these classes and the intersection among them.

2. Preliminaries

Definition 2.1
An inverse semigroup (a group) $S$ is called an inverse semigroup (a group) with basis property, if any two minimal (irreducible) generating sets (with respect to inclusion) of any subsemigroup (subgroup) $H$ of $S$ are equivalent (i.e. they have the same cardinality) [4]

Definition 2.2
A finite nonabelian group, in which every proper subgroup is abelian, is called a Miller- Moreno's group [7]

Definition 2.3
A finite nonnilpotent group, in which every proper subgroup is nilpotent, is called a Schmidt's group, [10]

Example 2.4
Let $G$ be the quaternion group $Q_8$

$Q_8 = \langle x, y : x^4 = 1, x^2 = y^2, xy = yx^3 \rangle$.

The order of $Q_8$ is 8 and its nonabelian group, but all proper subgroups of $Q_8$ are abelian.

First, we like to know which Schmidt's and Miller- Moreno's groups are groups with basis property. Redei, [8] had classified the Miller- Moreno's group as follows: Lemma 2.5 (theorem (6.5.6))[8]

Let $G$ be a finite Miller Moreno's group, then $G$ is one of the followings:

1. $G$ is the quaternion group $Q_8$
2. $G = \langle x, y : y^{p^x} = x^{p^{-1}} = 1, b^{-1}ab = a^{1+p^x} \rangle$, where $p$ is a prime.
3. \( G = M(p, m, n) \) which is given by:
\[
G = \{a, b, c ; (a^m) = b^p = c^p = 1, (a, c) = a^c, b^c = cb, [a, b] = c\},
\]
where \( p \) is a prime and \( m \geq n \geq 1 \), and clearly the order \( G \) of is \( p^{m+n+1} \).

4. \( G = G(p, q^v) \) and \( G \) is a Schmidt's group.

**Corollary 2.6**

Let \( p \) and \( q \) be distinct primes, \( k \in \mathbb{N} \) and
\[
t = \min \{ k \in \mathbb{N} : p^k \equiv 1 \pmod{q} \}.
\]

So \( t \) is the order of \( p - \) element in the multiplicative group \( \mathbb{Z}_q^* \).

Let \( f(x) \) be an irreducible polynomial in \( \mathbb{Z}_p[x] \) dividing:
\[
x^{q-1} + x^{q-2} + \cdots + x + 1
\]

Then by [8] the degree of \( f(x) \) is \( t \) and it is the smallest positive integer such that
\[
f(x)|x^{q-1} + x^{q-2} + \cdots + x + 1
\]

**Corollary 2.7**

Consider the set of all pairs \( (i, g(x)) \) where \( i \in \mathbb{Z}_{q^v} \), \( g(x) \in \mathbb{Z}_p[x] \).

Define on the set:
\[
\mathcal{C} = \{(i, g(x)) ; i \in \mathbb{Z}_{q^v}, g(x) \in \mathbb{Z}_p[x]\}
\]

an operation:
\[
(i, h(x)).(j, g(x)) = (i + j, h(x) + w^j g(x)), w \in \mathbb{Z}_p[x] / f(x).
\]

Then by [6] \( \mathcal{C}, \cdot \) forms a Miller-Moreno's group denoted by \( G(p, q^v) \).

Therefore \( G(p, q^v) \) has \( p \)-Sylow \( q \)-subgroups and its commutator is:
\[
(G(p, q^v)) = \{[0, h(x) : h(x) \in \mathbb{Z}_p[x]\}
\]

**Lemma 2.8** (Theorem (6.5.7) [9]

Let \( G \) be a finite nonnilpotent group, but all its proper subgroups are nilpotent. Then
\[
\exists P \in \text{Syl}_p(G) \quad \text{and} \quad \exists Q \in \text{Syl}_q(G)
\]

for some distinct primes \( p \) and \( q \) such that the followings hold:

1. \( G = PQ \).
2. \( P \trianglelefteq G \).
3. \( P \) is cyclic group.
4. \( G \) is solvable.

**Lemma 2.9** (Theorem 1), [1]

Let \( G \) be a finite semidirect product of a \( p \)-group \( P = \text{Fit}(G) \) (Fitting subgroup) of \( G \) by a cyclic \( q \)-group \( \langle y \rangle \) of order \( q^\beta \) where \( p \neq q \) are primes, and \( \beta \in \mathbb{N} \).

Then the group \( G \) has basis property if and only if for every element \( u \in \langle y \rangle \cdot u \neq 1 \), and for any invariant subgroup \( H \) of \( P \) the automorphism \( \Phi_u \) defines an isotopic representation on every quotient Frattini subgroup \( H \).

In, [10] it had been shown that the order of the group \( G(p, q^v) \) is \( p^\ell q^v \) is and the \( q \)-Sylow subgroups of \( G(p, q^v) \) are cyclic groups of order \( q^v \) and \( p \)-subgroup of \( G(p, q^v) \) is elementary abelian group of order \( p^\ell \) which corresponds to the commutator \( (G(p, q^v)) \). Then \( \varphi(q^v) \) is a normal subgroup of \( G(p, q^v) \) and it is Sylow \( p \)-subgroup.

Let \( \langle \bar{Q} \rangle \) be the cyclic group isomorphic to Sylow \( q \)-subgroup of order \( q^v \) hence the group \( G(p, q^v) \) is a semi direct product of the group \( G(p, q^v) \) by \( \langle \bar{Q} \rangle \) s, i.e.
\[
G(p, q^v) = (G(p, q^v)) \rtimes \langle \bar{Q} \rangle = (0, h(x)) \rtimes \langle (-1, 0) \rangle, h(x) \in \mathbb{Z}_p[x].
\]

\* The Fitting subgroup \( \text{Fit}(G) \) of a group is the maximum normal nilpotent subgroup.

\* Isotopic representation is the representation, which is writing as direct sum of equivalent irreducible representation.
Theorem 2.10
Let $G$ be a Schmidt's group. Then $G$ is a group with basis property if and only if $G$ is a Miller-Moreno's group of type $G(p, q)$ where $p \neq q$ are primes.

Proof.
Let $G$ be a Schmidt's group satisfying basis property. Then by [1] and [11] $G$ is a group of order $p^\alpha q^\beta, \alpha, \beta \in \mathbb{N}$ where $p \neq q$ are primes, so $G$ is an extension of $p$-group $P = \text{Fit} (G)$ by the cyclic $q$-group $\langle y \rangle$ of order $q^\beta$. Thus

$$G = \langle P, y \rangle. O(y) = q^\beta.$$  

By [9] $y^q$ is in the center of the subgroup $P$ since the group $G$ satisfies basis property, and every group with basis property is Quaziprimary [1] (Quaziprimary group: It is a finite group in which every element has prime power order), hence $y^q = 1$ and $\beta = 1$.

Let $M$ be a largest normal subgroup of $G$ properly contained in $P$. Then by [1] $y$ commutes with every element of $M$ and by Quaziprimarity of $G$ we get $M = \{1\}$. Thus $P$ is an elementary abelian group and every element of $\langle y \rangle$, induces an automorphism on a group $P$ denoted by $\varphi$. So $P$ can be viewed as a vector space $GF(p)$ with dimension $\alpha$ and $\varphi$ is a linear operator on the vector space $P$. Since $M = \{1\}$, then $P$ doesn’t contain any invariant subspaces with respect to $\varphi$, hence

$$\alpha = \min \{ k \in \mathbb{N} : p^k \equiv 1 \pmod{q} \}.$$  

By [1] the matrix $A_f$ of the operator $\varphi$ in given basis of vector space $P$ has the form:

$$A_f = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{\alpha-1} \end{bmatrix}.$$  

And $A_f$ Associated matrix with irreducible polynomial:

$$f(x) = x^\alpha + a_{\alpha-1}x^{\alpha-1} + \cdots + a_1x + a_0,$$  

over the field $GF(p)$ such that $\tilde{f}(x)$ dividing the polynomial:

$$x^{q-1} + x^{q-2} + \cdots + x + 1 \in \mathbb{Z}_p[x].$$  

By [1], if $w \in P, w \neq 1$, then every element of $P$ can be represented as $g(\varphi)w$ where $g(x) \in GF(p)[x]$. Note that if $h(x) \in GF(p)[x]$ then

$$h(\varphi)w = g(\varphi)w \Leftrightarrow f(x) \mid (h(x) - g(x)).$$  

Considering $P$ as an additive group of the field $\mathbb{Z}_q$. Then for any elements $u, v \in G$, there exists a unique elements $i, j \in \mathbb{Z}_q$ and $w_1, w_2 \in P$ such that:

$$u = y^iw_1,$$  

$$v = y^jw_2.$$  

By [1] the automorphism $\varphi$ given by:

$$\varphi(w) = y^{-1}wy; \forall w \in P.$$  

Now using a multiplication on $G$

$$uv = y^iw_1y^jw_2 = y^{i+j}\varphi^j(w_1)w_2$$  

(2)
Viewing $u$ and $v$ as pairs $(i, h(x)), (i, g(x))$ where $g(x), h(x) \in GF(p)[x]$ and $w_1 = g(\varphi)w, w_2 = h(\varphi)w$, for $w \in P$
and considering the addition on $P$, we have:

$$
(i, g(x)) = (j, h(x)) \iff 

i = j; g(x) \equiv h(x) (mod \ f(x))
$$

$$
i, j \in \mathbb{Z}_q.
$$

(3)

Using (3) then the multiplication (2) becomes:

$$
(i, g(x))(j, h(x)) = (i + j, w^j g(x) + h(x))
$$

(4)

So by [7] $G$ is a Miller-Moreno's group of type $G(p, q)$.

Conversely, assuming that $G$ a Schmidt's group, which is a Miller-Moreno's group of type $G(p, q)$. Defining $G$ as the group $(C, \cdot)$ and let $f(x)$ be an irreducible polynomial over the field $GF(p)$ of degree $\alpha$ such that

$$
\alpha = \min \{ k \in \mathbb{N} : p^k \equiv 1 (mod \ q) \},
$$

And $f(x)$ divides

$$
x^{q-1} + x^{q-2} + \cdots + x + 1.
$$

Applying the operation (4) on the set (1) and using the congruence in (3). Then we get that $G$ is a semi-direct product of the additive group of the field $GF(p)[x]/f(x)$ by the cyclic group $\langle \langle 1, 0 \rangle \rangle$ of the order $q$. Now

$$
(1,0)^{-1}(0, h(x))(1,0) =

(q - 1,0)(0, h(x))(1,0) =

(q - 1, h(x))(1,0) =

(0, wh(x)).
$$

Thus $\langle (1,0) \rangle$ acts by conjugation, on the elements of

$$
\{ (0, 1), (0, w), \ldots, (0, w^{q-1}) \}
$$

has the above matrix $A_f$. So by [1] $G$ is a group with basis property.

\textbf{Theorem 2.11}

Let $G$ be a Miller-Moreno's group, which is non $p$- primary group. Then the group $G$ has basis property if and only if it is a Miller-Moreno's group of type $G(p, q)$, for distinct primes, $q$

\textbf{Proof.}

Let $G$ be a group with basis property and it is not primary Miller-Moreno's group, then by Theorem (2.10) $G$ is a semi direct product of $p$-group $P$ by the cyclic $q$- group $\langle y \rangle$, where $y$ induces a regular automorphism on $P$. 
Then $G$ not nilpotent, and since all proper subgroups of $G$ are abelian, thus $G$ is Schmidt's group. Consequently by Theorem (2.10) $G$ is a group of type $G(p, q)$.

Conversely, let $G$ be a Miller Moreno's group of type $G(p, q)$, then by Theorem (2.10) it is a group of type $G(p, q)$, so it is a group with basis property.

**Theorem 2.12**

Let $G$ be a Miller-Moreno's group of type $G(p, q^v)$ $p \neq q$ are primes and $v \in \mathbb{N}$. If $v > 1$, then $G$ does not have a basis property.

**Proof.**

By [9] in order that $G$ be a Miller-Moreno's group, then by Lemma (2.5) $G$ has one of the forms:

1. $G$ is the quaternion group $Q_8$, and it's a finite $p$-group.
2. The group $G$ of type:
   $$G = \langle x, y : y^p \cdot x = x^p \cdot y^{p+1} = 1, b^{-1} \cdot ab = a^{1+p^v} \rangle.$$
3. The group $G = M(p, m, n)$ is a primary group of order $p^{m+n+1}$.
4. The group $G(p, q^v)$ and $G$ is a Schmidt's group.

In cases (1), (2) and (3) the group $G$ is a primary, so by [1] $G$ is a group with basis property.

Since in case (4) $G$ isn't primary group, and by Theorem (2.11) $G(p, q^v)$ is a group with basis property if $v=1$. Thus the group $G(p, q^v)$ hasn't basis property, when $v > 1$.

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