RELATIONS BETWEEN SCHMIDT'S, MILLER- MORENO'S GROUPS AND GROUPS WITH BASIS PROPERTY

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Abstract In this paper we prove that a Schmidt's group forms a group with basis property if and only if it is a Miller- Moreno's group of type $G(p,q)$ such that $p \neq q$ (p,q are primes). Therefore doesn't group with basis property if it is a Miller- Moreno's group of type $G(p, q^v), v \in \mathbb{N}, \ v > 1$.

1. Introduction

In our previous study of finite groups with basis property, [1] we note, that the structure of these groups is very close to some famous classes of groups, like Schmidt's and Miller- Moreno's groups.

Schmidt and Miller- Moreno groups had been studied at the beginning of the twentieth century by many mathematicians as Golfand [2], Jotrov [5], Schmidt [11], Redei [8], and others. Since these groups had been studied in details and had been classified in [3], and [11], while the class of groups with basis property had been classified by Al Khalaf [1]. In this paper we wil invlestigate a relations between these classes and the intersection among them.

2. Preliminaries

Definition 2.1
An inverse semigroup (a group) S is called an inverse semigroup (a group) with basis property, if any two minimal (irreducible) generating sets (with respect to inclusion) of any subsemigroup (subgroup) H of S are equivalent (i.e. they have the same cardinality) [4]

Definition 2.2
A finite nonabelian group, in which every proper subgroup is abelian, is called a Miller-Moreno's group [7]

Definition 2.3
A finite nonnilpotent group, in which every proper subgroup is nilpotent, is called a Schmidt's group, [10]

Example 2.4
Let G be the quaternion group Q_8

$$Q_8 = \langle x, y : \ x^4 = 1 \ , \ x^2 = y^2 \ , xy = yx^3 \rangle.$$

The order of Q_8 is 8 and its nonabelian group, but all proper subgroups of Q8 are abelian.
First, we like to know which Schmidt's and Miller- Moreno's groups are groups with basis property. Redei, [8] had classified the Miller- Moreno's group as follows: Lemma 2.5 (theorem (6.5.6)[8]
Let G be a finite Miller Moreno's group, then G is one of the followings:

1. G is the quaternion group
2. $G = \langle x, y : \ y^{p^u} = x^{p^{v+1}} = 1 \ , \ b^{-1}ab = a^{1+p^v} \rangle$, where p is a prime.

3. $G = M(p, m, n)$ which is given by:
$G = \langle a, b, c ; (a^{p^m} = b^{p^n} = c^p = 1), (, ac = ca, bc = cb, [a,b] = c) \rangle$, where p is a prime and
$m \geq n \geq 1$, and clearly the order G of is p^{m+n+1}.

4. $G = G(p, q^v)$ and G is a Schmidt's group.

Corollary 2.6[*]

Let p and q be distinct primes, $k \in \mathbb{N}$ and
$$t = min \{ k \in \mathbb{N} : p^k \equiv 1 \ (mod \ q) \}.$$

So t is the order of p − element in the multiplicative group $\mathbb{Z}_q^*$

Let $f(x)$ be an irreducible polynomial in $\mathbb{Z}_p[x]$ dividing:
$$x^{q-1} + x^{q-2} + \cdots + x + 1$$

Then by [8] the degree of $f(x)$ is t and it is the smallest positive integer such that
$$f(x) | x^{q-1} + x^{q-2} + \cdots + x + 1$$

Corollary 2.7

Consider the set of all pairs $(i, g(x))$ where $i \in \mathbb{Z}_{q^v}$, $g(x) \in \mathbb{Z}_p[x]$.

Define on the set:
$$C = \{ (i, g(x)) ; i \in \mathbb{Z}_{q^v}, g(x) \in \mathbb{Z}_p[x] \} \tag{1}$$

an operation:
$$(i, h(x)) \cdot (j, g(x)) = (i + j, h(x) + w^i g(x)), w \in \mathbb{Z}_p[x] / f(x).$$

Then by [6] $(C, \cdot)$ forms a Miller-Moreno's group denoted by $G(p, q^v)$
Therefore $G(p, q^v)$ has p^t Sylow q − subgroups and its commutator is:
$$(G(p, q^v))' = \{ (0, h(x)) : h(x) \in \mathbb{Z}_p[x] \}$$

Lemma 2.8 (Theorem (6.5.7) [9]

Let G be a finite nonnilpotent group, but all its proper subgroups are nilpotent. Then
$\exists P \in Syl_p(G)$ and $\exists Q \in Syl_q(G)$ for some distinct primes p and q such that the followings hold:

1. $G = PQ$.
2. $P \triangleleft G$.
3. P is cyclic group.
4. G is solvable.

Lemma 2.9 (Theorem 1), [1]

Let G be a finite semidirect product of a p-group $P = Fit(G)$ (Fitting subgroup) of G by a cyclic q-group $\langle y \rangle$ of order q^β where $p \neq q$ are primes, and $\beta \in \mathbb{N}$.

Then the group G has basis property if and only if for every element $u \in \langle y \rangle$, $u \neq 1$, and for any invariant subgroup H of P the automorphism φ_u defines an isotopic representation on every quotient Frattini subgroup H.

In, [10] it had been shown that the order of the group $G(p, q^v)$ is $p^t q^v$ is and the q-Sylow subgroups of $G(p, q^v)$ are cyclic groups of order q^v and p-subgroup of $G(p, q^v)$ is elementary abelian group of order p^t which corresponds to the commutator $(G(p, q^v))'$. Then $(G(p, q^v))'$ is a normal subgroup of $G(p, q^v)$ and it is Sylow p-subgroup.

Let $\langle \bar{Q} \rangle$ be the cyclic group isomorphic to Sylow q- subgroup of order q^v hence the group $G(p, q^v)$ is a semi direct product of the group $(G(p, q^v))'$ by $\langle \bar{Q} \rangle$ \$, i.e.
$$G(p, q^v) = (G(p, q^v))' \rtimes \langle \bar{Q} \rangle = (0, h(x)) \rtimes \langle (-1, 0) \rangle, h(x) \in \mathbb{Z}_p[x].$$

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[*] The Fitting sub group Fit (G) of a group is the maximum normal nilpotent subgroup
[**] Isotopic representation is the representation, which is writing as direct sum of equivalent irreducible representation.

Theorem 2.10

Let G be a Schmidt's group. Then G s a group with basis property if and only if G is a Miller-Moreno's group of type $G(p, q)$, where $p \neq q$ are primes.

Proof.

Let G be a Schmidt's group satisfying basis property. Then by [1] and [11] G is a group of order $p^\alpha q^\beta, \alpha, \beta \in \mathbb{N}$ where $p \neq q$ are primes, so G is an extension of p-group $P = Fit\,(G)$ by the cyclic q-group $\langle y \rangle$ of order q^β. Thus

$$G = \langle P, y \rangle, O(y) = q^\beta.$$

By [9] y^q is in the center of the subgroup P since the group G satisfices basis property, and every group with basis property is Quaziprimary [1] (Quaziprimary group: It is a finite group in which every element has prime power order), hence $y^q = 1$ and $\beta = 1$.

Let M be a largest normal subgroup of G properly contained in P Then by [1] y commutate with every element of M and by Quaziprimarity of G we get $M=\{1\}$ .Thus P is an elementary abelian group and every element of $\langle y \rangle$, induces an automorphism on a group P denoted by φ . So P can be viewed as a vector space $GF(p)$ with dimension α and φ is a linear operator on the vector space P. Since $M=\{1\}$, then P doesn't contain any invariant subspaces with respect to φ,hence

$$\alpha = min\,\{\,k \in \mathbb{N}\,:\,p^k \equiv 1\,\,(mod\,\,q)\}.$$

By [1] the matrix A_f of the operator φ in given basis of vector space P has the form:

$$A_f = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{\alpha-1} \end{bmatrix},$$

And A_f Associated matrix with irreducible polynomial:

$$f(x) = x^\alpha + a_{\alpha-1}x^{\alpha-1} + \cdots + a_1 x + a_0 \,,$$

over the field $GF(p)$ such that $f(x)$ dividing the polynomial:

$$x^{q-1} + x^{q-2} + \cdots + x + 1 \in \mathbb{Z}_p[x]\,.$$

By [1], if $w \in P, w \neq 1$, then every element of P can be represented as $g(\varphi)w$ where $g(x) \in GF(p)[x]$. Note that if $h(x) \in GF(p)[x]$ then

$$h(\varphi)w = g(\varphi)w \Leftrightarrow f(x)\big|\big(h(x) - g(x)\big)\,.$$

Considering P as an additive group of the field $\left. GF(p)[x] \middle/ f(x) \right.$ Then for any elements $u, v \in G$, there exists a unique elements $i, j \in \mathbb{Z}_q$ and $w_1, w_2 \in P$ such that:

$$u = y^i w_1\,,$$
$$v = y^j w_2.$$

By [1] the automorphism φ given by:

$$\varphi(w) = y^{-1}wy\,;\,\forall w \in P\,.$$

Now using a multiplication on G

$$uv = y^i w_1 y^j w_2 = y^{i+j}\varphi^j(w_1)w_2 \tag{2}$$

Viewing u and v as pairs $(j, h(x)), (i, g(x))$ where $g(x), h(x) \in GF(p)[x]$ and

$$w_1 = g(\varphi)w, \quad w_2 = h(\varphi)w, \text{ for } w \in P$$

and considering the addition on P, we have:

$$(i, g(x)) = (j, h(x)) \Leftrightarrow$$

$$i = j; \; g(x) \equiv h(x)(mod \; f(x))$$

$$i, j \in \mathbb{Z}_q .$$

(3)

Using (3) then the multiplication (2) becomes:

$$(i, g(x))(j, h(x)) = \left(i + j, w^j g(x) + h(x)\right)$$

(4)

So by [7] G is a Miller-Moreno's group of type $G(p, q)$.

Conversely, assuming that G a Schmidt's group, which is a Miller-Moreno's group of type $G(p, q)$. Defining G as the group (C,) and let $f(x)$ be an irreducible polynomial over the field $GF(p)$ of degree α such that

$$\alpha = min\{k \in \mathbb{N} : p^k \equiv 1(mod \; q)\},$$

And $f(x)$ divides

$$x^{q-1} + x^{q-2} + \cdots + x + 1.$$

Applying the operation (4) on the set (1) and using the congruence in (3) .Then we get that G is a semi direct product of the additive group of the field $GF(p)[x]/f(x)$ by the cyclic group $\langle(1,0)\rangle$ of the order q . Now

$$(1,0)^{-1}(0, h(x))(1,0) =$$

$$(q - 1,0)(0, h(x))(1,0) =$$

$$(q - 1, h(x))(1,0) =$$

$$(0, wh(x)).$$

Thus $\langle(1,0)\rangle$ acts by conjugation, on the elements of $GF(p)[x]/f(x)$, so the resulting polynomial is a multiplication by w. Thus the operator φ in the basis

$$\{(0,1), (0, w), \dots, (0, w^{\alpha-1})\}$$

has the above matrix A_f. So by [1] G is a group with basis property. ∎

Theorem 2.11

Let G be a Miller-Moreno's group, which is non p- primary group. Then the group G has basis property if and only if it is a Miller-Moreno's group of type $G(p, q)$. for distinct primes, q

Proof.

Let G be a group with basis property and it is not primary Miller-Moreno's group, then by Theorem (2.10) G is a semi direct product of p-group P by the cyclic q- group $\langle y \rangle$, where y induces a regular automorphism on P.

Then G not nilpotent, and since all proper subgroups of G are abelian, thus G is Schmidt's group. Consequently by Theorem (2.10) G is a group of type $G(p,q)$.

Conversely, let G be a Miller Moreno's group of type $G(p,q)$. then by Theorem (2.10) it is a group of type $G(p,q)$. so it is a group with basis property.   ∎

Theorem 2.12

Let G be a Miller-Moreno's group of type $G(p,q^v)$ $p \neq q$ are primes and $v \in \mathbb{N}$.
If $v > 1$, then G does not have a basis property.

Proof.

By [9] in order that G be a Miller-Moreno's group, then by Lemma (2.5) G has one of the forms:
1. G is the quaternion group Q_8, and it's a finite $p-$ group.
2. The group G of type:
$$G = \langle x, y \: : \: y^{p^u} = x^{p^{v+1}} = 1 \: , \: b^{-1}ab = a^{1+p^v} \rangle,$$
3. The group $G = M(p,m,n)$ is a primary group of order p^{m+n+1}.
4. The group $G(p,q^v)$ and G is a Schmidt's group.

In cases (1),(2) and (3) the group G is a primary, so by [1] G is a group with basis property.

Since in case (4) G isn't primary group, and by Theorem (2.11) $G(p,q^v)$ is a group with basis property if $v=1$. Thus the group $G(p,q^v)$ hasn't basis property, when $v > 1$ ∎

References

[1] AlKhalaf A. Finite groups with the basis property , Dokl, Akad. Nauk BSSR 33 (1980), no. 11, 1989. (Russian)
[2] Golfand U.A. About groups, all subgroups are nilpotent .DAN SSSR 1984 T.60 No 8 p.1313-1315. (Russian)
[3] Gorenstein D., Finite Groups, Second Edition (Chelsea, New York, 1980).
[4] Jones P. R. Basis properties for inversesemigroup, J. Algebra 50 (1978), 135-152.
[5] Jortov. A., Siken S. A. About Schmidt groups sib Math. J. 1987, T. 28 N2 p. 74 - 78. (Russian)
[6] Miller G. A. , Moreno H. C. Non-abelian groups in which every subgroup is abelian , Stanford Un. 1903.
[7] Redei L. Das Schiefe product in der Gruppentheorie mit An wendungen auf die endlichen nicht kommutativen Gruppen mit lauter kommutativen echten untergruppen and die ordnung zahlen , zu denen nur commutative Gruppen gehoren com . Math. Helv. 1947. B. 20 – s. 225 – 264.
[8] Redei L. Die endlichen einstufig nichtnilpotent Gruppen fubl. Math. 1956 B4. S.303-324.
[9] Scott W. R. Group theory, Dover publication, INC., New York 1987.
[10] Shmidt O. U. Groups, which all subgruops nilpotent. Math Gb 1943 t. 31 No 3 p.366-372 (Russian)
[11] Shmidt O. U. Selected works .M. AN SSSR 1959, 315 p. (Russian).