

RELATIONS BETWEEN SCHMIDT'S, MILLER- MORENO'S GROUPS AND GROUPS WITH BASIS PROPERTY

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Abstract In this paper we prove that a Schmidt's group forms a group with basis property if and only if it is a Miller- Moreno's group of type $G(p, q)$ such that $p \neq q$ (p, q are primes). Therefore doesn't group with basis property if it is a Miller- Moreno's group of type $G(p, q^v), v \in \mathbb{N}, v > 1$.

1. Introduction

In our previous study of finite groups with basis property, [1] we note, that the structure of these groups is very close to some famous classes of groups, like Schmidt's and Miller- Moreno's groups.

Schmidt and Miller- Moreno groups had been studied at the beginning of the twentieth century by many mathematicians as Gelfand [2], Jotrov [5], Schmidt [11], Redei [8], and others. Since these groups had been studied in details and had been classified in [3], and [11], while the class of groups with basis property had been classified by Al Khalaf [1]. In this paper we wil invlestigate a relations between these classes and the intersection among them.

2. Preliminaries

Definition 2.1

An inverse semigroup (a group) S is called an inverse semigroup (a group) with basis property, if any two minimal (irreducible) generating sets (with respect to inclusion) of any subsemigroup (subgroup) H of S are equivalent (i.e. they have the same cardinality) [4]

Definition 2.2

A finite nonabelian group, in which every proper subgroup is abelian, is called a Miller-Moreno's group [7]

Definition 2.3

A finite nonnilpotent group, in which every proper subgroup is nilpotent, is called a Schmidt's group, [10]

Example 2.4

Let G be the quaternion group Q_8

$$Q_8 = \langle x, y : x^4 = 1, x^2 = y^2, xy = yx^3 \rangle.$$

The order of Q_8 is 8 and its nonabelian group, but all proper subgroups of Q_8 are abelian.

First, we like to know which Schmidt's and Miller- Moreno's groups are groups with basis property. Redei, [8] had classified the Miller- Moreno's group as follows: Lemma 2.5 (theorem (6.5.6))[8]

Let G be a finite Miller Moreno's group, then G is one of the followings:

1. G is the quaternion group
2. $G = \langle x, y : y^{p^u} = x^{p^{v+1}} = 1, b^{-1}ab = a^{1+p^v} \rangle$, where p is a prime.

3. $G = M(p, m, n)$ which is given by:

$G = \langle a, b, c; (a^{p^m} = b^{p^n} = c^p = 1), (ac = ca, bc = cb, [a, b] = c) \rangle$, where p is a prime and $m \geq n \geq 1$, and clearly the order G of is p^{m+n+1} .

4. $G = G(p, q^v)$ and G is a Schmidt's group.

Corollary 2.6*

Let p and q be distinct primes, $k \in \mathbb{N}$ and

$$t = \min \{ k \in \mathbb{N} : p^k \equiv 1 \pmod{q} \}.$$

So t is the order of p – element in the multiplicative group \mathbb{Z}_q^*

Let $f(x)$ be an irreducible polynomial in $\mathbb{Z}_p[x]$ dividing:

$$x^{q-1} + x^{q-2} + \dots + x + 1$$

Then by [8] the degree of $f(x)$ is t and it is the smallest positive integer such that

$$f(x) | x^{q-1} + x^{q-2} + \dots + x + 1$$

Corollary 2.7

Consider the set of all pairs $(i, g(x))$ where $i \in \mathbb{Z}_{q^v}$, $g(x) \in \mathbb{Z}_p[x]$.

Define on the set:

$$C = \{ (i, g(x)) ; i \in \mathbb{Z}_{q^v}, g(x) \in \mathbb{Z}_p[x] \} \quad (1)$$

an operation:

$$(i, h(x)) \cdot (j, g(x)) = (i + j, h(x) + w^i g(x)), w \in \mathbb{Z}_p[x]/f(x).$$

Then by [6] (C, \cdot) forms a Miller-Moreno's group denoted by $G(p, q^v)$

Therefore $G(p, q^v)$ has p^t Sylow q – subgroups and its commutator is:

$$(G(p, q^v))' = \{ (0, h(x)) : h(x) \in \mathbb{Z}_p[x] \}$$

Lemma 2.8 (Theorem (6.5.7) [9])

Let G be a finite nonnilpotent group, but all its proper subgroups are nilpotent. Then $\exists P \in Syl_p(G)$ and $\exists Q \in Syl_q(G)$ for some distinct primes p and q such that the followings hold:

1. $G = PQ$.
2. $P \triangleleft G$.
3. P is cyclic group.
4. G is solvable.

Lemma 2.9 (Theorem 1), [1]

Let G be a finite semidirect product of a p -group $P = Fit(G)$ (Fitting subgroup) of G by a cyclic q -group $\langle y \rangle$ of order q^β where $p \neq q$ are primes, and $\beta \in \mathbb{N}$.

Then the group G has basis property if and only if for every element $u \in \langle y \rangle$, $u \neq 1$, and for any invariant subgroup H of P the automorphism φ_u defines an isotopic representation on every quotient Frattini subgroup H .

In, [10] it had been shown that the order of the group $G(p, q^v)$ is $p^t q^v$ is and the q -Sylow subgroups of $G(p, q^v)$ are cyclic groups of order q^v and p -subgroup of $G(p, q^v)$ is elementary abelian group of order p^t which corresponds to the commutator $(G(p, q^v))'$. Then $(G(p, q^v))'$ is a normal subgroup of $G(p, q^v)$ and it is Sylow p -subgroup.

Let $\langle \bar{Q} \rangle$ be the cyclic group isomorphic to Sylow q - subgroup of order q^v hence the group $G(p, q^v)$ is a semi direct product of the group $(G(p, q^v))'$ by $\langle \bar{Q} \rangle$, i.e.

$$G(p, q^v) = (G(p, q^v))' \rtimes \langle \bar{Q} \rangle = (0, h(x)) \rtimes \langle (-1, 0) \rangle, h(x) \in \mathbb{Z}_p[x].$$

* The Fitting sub group $Fit(G)$ of a group is the maximum normal nilpotent subgroup

** Isotopic representation is the representation, which is writing as direct sum of equivalent irreducible representation.

Theorem 2.10

Let G be a Schmidt's group. Then G is a group with basis property if and only if G is a Miller-Moreno's group of type $G(p, q)$, where $p \neq q$ are primes.

Proof.

Let G be a Schmidt's group satisfying basis property. Then by [1] and [11] G is a group of order $p^\alpha q^\beta, \alpha, \beta \in \mathbb{N}$ where $p \neq q$ are primes, so G is an extension of p -group $P = \text{Fit}(G)$ by the cyclic q -group $\langle y \rangle$ of order q^β . Thus

$$G = \langle P, y \rangle, O(y) = q^\beta.$$

By [9] y^q is in the center of the subgroup P since the group G satisfies basis property, and every group with basis property is Quaziprimary [1] (Quaziprimary group: It is a finite group in which every element has prime power order), hence $y^q = 1$ and $\beta = 1$.

Let M be a largest normal subgroup of G properly contained in P . Then by [1] y commutes with every element of M and by Quaziprimarity of G we get $M = \{1\}$. Thus P is an elementary abelian group and every element of $\langle y \rangle$, induces an automorphism on a group P denoted by φ . So P can be viewed as a vector space $GF(p)$ with dimension α and φ is a linear operator on the vector space P . Since $M = \{1\}$, then P doesn't contain any invariant subspaces with respect to φ , hence

$$\alpha = \min \{ k \in \mathbb{N} : p^k \equiv 1 \pmod{q} \}.$$

By [1] the matrix A_φ of the operator φ in given basis of vector space P has the form:

$$A_\varphi = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{\alpha-1} \end{bmatrix},$$

And A_φ Associated matrix with irreducible polynomial:

$$f(x) = x^\alpha + a_{\alpha-1}x^{\alpha-1} + \dots + a_1x + a_0,$$

over the field $GF(p)$ such that $f(x)$ dividing the polynomial:

$$x^{q-1} + x^{q-2} + \dots + x + 1 \in \mathbb{Z}_p[x].$$

By [1], if $w \in P, w \neq 1$, then every element of P can be represented as $g(\varphi)w$ where $g(x) \in GF(p)[x]$. Note that if $h(x) \in GF(p)[x]$ then

$$h(\varphi)w = g(\varphi)w \Leftrightarrow f(x) \mid \frac{h(x) - g(x)}{f(x)}.$$

Considering P as an additive group of the field $GF(p)[x]/f(x)$ Then for any elements $u, v \in G$, there exists a unique elements $i, j \in \mathbb{Z}_q$ and $w_1, w_2 \in P$ such that:

$$\begin{aligned} u &= y^i w_1, \\ v &= y^j w_2. \end{aligned}$$

By [1] the automorphism φ given by:

$$\varphi(w) = y^{-1}wy ; \forall w \in P.$$

Now using a multiplication on G

$$uv = y^i w_1 y^j w_2 = y^{i+j} \varphi^j(w_1) w_2 \tag{2}$$

Viewing u and v as pairs $(j, h(x)), (i, g(x))$ where $g(x), h(x) \in GF(p)[x]$ and $w_1 = g(\varphi)w, w_2 = h(\varphi)w, \text{ for } w \in P$

and considering the addition on P , we have:

$$\begin{aligned} (i, g(x)) &= (j, h(x)) \Leftrightarrow \\ i = j; g(x) &\equiv h(x) \pmod{f(x)} \\ i, j &\in \mathbb{Z}_q. \end{aligned} \tag{3}$$

Using (3) then the multiplication (2) becomes:

$$(i, g(x))(j, h(x)) = (i + j, w^j g(x) + h(x)) \tag{4}$$

So by [7] G is a Miller-Moreno's group of type $G(p, q)$.

Conversely, assuming that G a Schmidt's group, which is a Miller-Moreno's group of type $G(p, q)$. Defining G as the group (C, \cdot) and let $f(x)$ be an irreducible polynomial over the field $GF(p)$ of degree α such that

$$\alpha = \min\{k \in \mathbb{N} : p^k \equiv 1 \pmod{q}\},$$

And $f(x)$ divides

$$x^{q-1} + x^{q-2} + \dots + x + 1.$$

Applying the operation (4) on the set (1) and using the congruence in (3). Then we get that G is a

semi direct product of the additive group of the field $GF(p)[x]/f(x)$ by the cyclic group $\langle (1, 0) \rangle$ of the order q . Now

$$\begin{aligned} (1,0)^{-1}(0, h(x))(1,0) &= \\ (q - 1,0)(0, h(x))(1,0) &= \\ (q - 1, h(x))(1,0) &= \\ (0, wh(x)). \end{aligned}$$

Thus $\langle (1,0) \rangle$ acts by conjugation, on the elements of $GF(p)[x]/f(x)$, so the resulting polynomial is a multiplication by w . Thus the operator φ in the basis $\{(0,1), (0, w), \dots, (0, w^{\alpha-1})\}$

has the above matrix A_f . So by [1] G is a group with basis property. ■

Theorem 2.11

Let G be a Miller-Moreno's group, which is non p - primary group. Then the group G has basis property if and only if it is a Miller-Moreno's group of type $G(p, q)$. for distinct primes, q

Proof.

Let G be a group with basis property and it is not primary Miller-Moreno's group, then by Theorem (2.10) G is a semi direct product of p -group P by the cyclic q - group $\langle y \rangle$, where y induces a regular automorphism on P .

Then G not nilpotent, and since all proper subgroups of G are abelian, thus G is Schmidt's group. Consequently by Theorem (2.10) G is a group of type $G(p, q)$.

Conversely, let G be a Miller Moreno's group of type $G(p, q)$. then by Theorem (2.10) it is a group of type $G(p, q)$. so it is a group with basis property. ■

Theorem 2.12

Let G be a Miller-Moreno's group of type $G(p, q^v)$ $p \neq q$ are primes and $v \in \mathbb{N}$.

If $v > 1$, then G does not have a basis property.

Proof.

By [9] in order that G be a Miller-Moreno's group, then by Lemma (2.5) G has one of the forms:

1. G is the quaternion group Q_8 , and it's a finite p – group.

2. The group G of type:

$$G = \langle x, y : y^{p^u} = x^{p^{v+1}} = 1, b^{-1}ab = a^{1+p^v} \rangle,$$

3. The group $G = M(p, m, n)$ is a primary group of order p^{m+n+1} .

4. The group $G(p, q^v)$ and G is a Schmidt's group.

In cases (1),(2) and (3) the group G is a primary, so by [1] G is a group with basis property.

Since in case (4) G isn't primary group, and by Theorem (2.11) $G(p, q^v)$ is a group with basis property if $v=1$. Thus the group $G(p, q^v)$ hasn't basis property, when $v > 1$ ■

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