RELATIONS BETWEEN SCHMIDT'S, MILLER- MORENO'S GROUPS AND GROUPS WITH BASIS PROPERTY

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Abstract In this paper we prove that a Schmidt's group forms a group with basis property if and only if it is a Miller- Moreno's group of type \( G(p, q) \) such that \( p \neq q (p, q \text{ are primes}) \). Therefore doesn’t group with basis property if it is a Miller- Moreno's group of type \( G(p, q^v), v \in \mathbb{N}, v > 1 \).

1. Introduction
In our previous study of finite groups with basis property, [1] we note, that the structure of these groups is very close to some famous classes of groups, like Schmidt's and Miller- Moreno's groups.

Schmidt and Miller- Moreno groups had been studied at the beginning of the twentieth century by many mathematicians as Golfand [2], Jotrov [5], Schmidt [11], Redei [8], and others. Since these groups had been studied in details and had been classified in [3], and [11], while the class of groups with basis property had been classified by Al Khalaf [1]. In this paper we will investigate a relations between these classes and the intersection among them.

2. Preliminaries

Definition 2.1
An inverse semigroup (a group) \( S \) is called an inverse semigroup (a group) with basis property, if any two minimal (irreducible) generating sets (with respect to inclusion) of any subsemigroup (subgroup) \( H \) of \( S \) are equivalent (i.e. they have the same cardinality) [4].

Definition 2.2
A finite nonabelian group, in which every proper subgroup is abelian, is called a Miller- Moreno's group [7].

Definition 2.3
A finite non-nilpotent group, in which every proper subgroup is nilpotent, is called a Schmidt's group, [10].

Example 2.4
Let \( G \) be the quaternion group \( Q_8 \)

\[
Q_8 = \langle x, y : x^4 = 1, x^2 = y^2, xy = yx^3 \rangle.
\]

The order of \( Q_8 \) is 8 and its nonabelian group, but all proper subgroups of \( Q_8 \) are abelian. First, we like to know which Schmidt's and Miller- Moreno's groups are groups with basis property. Redei, [8] had classified the Miller- Moreno's group as follows: Lemma 2.5 (theorem (6.5.6) [8]

Let \( G \) be a finite Miller Moreno's group, then \( G \) is one of the followings:

1. \( G \) is the quaternion group
2. \( G = \langle x, y : y^{p^x} = x^{p^{x+1}} = 1, b^{-1}ab = a^{1+p^x} \rangle \), where \( p \) is a prime.
3. \( G = M(p, m, n) \) which is given by:
\[
G = \langle a, b, c \mid (a^p)^m = (b^p)^n = c^d = 1, \, (a, c) = (a, b) = (a, c) \rangle,
\]
where \( p \) is a prime and \( m, n \geq 1 \), and clearly the order \( G \) of is \( p^{m+n+1} \).

4. \( G = G(p, q^v) \) and \( G \) is a Schmidt's group.

**Corollary 2.6**

Let \( p \) and \( q \) be distinct primes, \( k \in \mathbb{N} \) and
\[
t = \text{min} \{ k \in \mathbb{N} : p^k \equiv 1 \pmod{q} \}.
\]
So \( t \) is the order of \( p - \) element in the multiplicative group \( \mathbb{Z}_q^* \).

Let \( f(x) \) be an irreducible polynomial in \( \mathbb{Z}_p[x] \) dividing:
\[
x^{q-1} + x^{q-2} + \cdots + x + 1
\]
Then by [8] the degree of \( f(x) \) is \( t \) and it is the smallest positive integer such that
\[
f(x) \mid x^{q-1} + x^{q-2} + \cdots + x + 1
\]

**Corollary 2.7**

Consider the set of all pairs \( (i, g(x)) \) where \( i \in \mathbb{Z}_{q^v}, \, g(x) \in \mathbb{Z}_p[x] \).

Define on the set:
\[
C = \{ (i, g(x)) \mid i \in \mathbb{Z}_{q^v}, \, g(x) \in \mathbb{Z}_p[x] \}
\]

an operation:
\[
(i, h(g(x)) \cdot (j, g(x)) = (i + j, h(g(x)) + w^i g(x)) , \, w \in \mathbb{Z}_p[x]/f(x).
\]
Then by [6] \( (C, \cdot) \) forms a Miller-Moreno's group denoted by \( G(p, q^v) \).

Therefore \( G(p, q^v) \) has \( p^t \) Sylow \( q - \) subgroups and its commutator is:
\[
\{ (0, h(g(x))) \mid h(x) \in \mathbb{Z}_p[x] \}
\]

**Lemma 2.8** (Theorem (6.5.7) [9]

Let \( G \) be a finite nonnilpotent group, but all its proper subgroups are nilpotent. Then
\( \exists P \in \text{Syl}_p(G) \) and \( \exists Q \in \text{Syl}_q(G) \) for some distinct primes \( p \) and \( q \) such that the followings hold:
1. \( G = PQ \).
2. \( P \triangleleft G \).
3. \( P \) is cyclic group.
4. \( G \) is solvable.

**Lemma 2.9** (Theorem 1), [1]

Let \( G \) be a finite semidirect product of a \( p \)-group \( P = \text{Fit}(G) \) (Fitting subgroup) of \( G \) by a cyclic \( q \)-group \( \langle y \rangle \) of order \( q^\beta \) where \( p \neq q \) are primes, and \( \beta \in \mathbb{N} \).

Then the group \( G \) has basis property if and only if for every element \( u \in \langle y \rangle, \, u \neq 1 \), and for any invariant subgroup \( H \) of \( P \) the automorphism \( \Phi \in \text{Aut}(G) \) defines an isotopic representation on every quotient Frattini subgroup \( H \).

In, [10] it had been shown that the order of the group \( G(p, q^v) \) is \( p^t q^v \) is and the \( q \)-Sylow subgroups of \( G(p, q^v) \) are cyclic groups of order \( q^v \) and \( P \)-subgroup of \( G(p, q^v) \) is elementary abelian group of order \( p^t \) which corresponds to the commutator \( \langle G(p, q^v) \rangle \). Then \( G(p, q^v) \) is a normal subgroup of \( G(p, q^v) \) and it is Sylow \( p \)-subgroup.

Let \( \overline{Q} \) be the cyclic group isomorphic to Sylow \( q \)-subgroup of order \( q^v \), hence the group \( G(p, q^v) \) is a semi direct product of the group \( \langle G(p, q^v) \rangle \) by \( \langle \overline{Q} \rangle \), i.e.
\[
G(p, q^v) = \langle G(p, q^v) \rangle \rtimes \langle \overline{Q} \rangle = \langle 0, h(x) \rangle \rtimes \langle (-1,0) \rangle, \, h(x) \in \mathbb{Z}_p[x].
\]
Theorem 2.10
Let $G$ be a Schmidt's group. Then $G$ is a group with basis property if and only if $G$ is a Miller-Moreno's group of type $G(p, q)$, where $p \neq q$ are primes.

Proof.
Let $G$ be a Schmidt's group satisfying basis property. Then by [1] and [11] $G$ is a group of order $p^\alpha q^\beta$, $\alpha, \beta \in \mathbb{N}$ where $p \neq q$ are primes, so $G$ is an extension of $p$-group $P = \text{Fit}(G)$ by the cyclic $q$-group $\langle y \rangle$ of order $q^\beta$. Thus

$$G = \langle P, y \rangle, \quad O(y) = q^\beta.$$  

By [9] $y^q$ is in the center of the subgroup $P$ since the group $G$ satisfies basis property, and every group with basis property is Quaziprimary [1] (Quaziprimary group: It is a finite group in which every element has prime power order), hence $y^q = 1$ and $\beta = 1$.

Let $M$ be a largest normal subgroup of $G$ properly contained in $P$. Then by [1] $y$ commutes with every element of $M$ and by Quaziprimarity of $G$ we get $M = \{1\}$. Thus $P$ is an elementary abelian group and every element of $\langle y \rangle$, induces an automorphism on a group $P$ denoted by $\varphi$. So $P$ can be viewed as a vector space $GF(p)$ with dimension $\alpha$ and $\varphi$ is a linear operator on the vector space $P$. Since $M = \{1\}$, then $P$ doesn’t contain any invariant subspaces with respect to $\varphi$, hence

$$\alpha = \min \{ k \in \mathbb{N} : p^k \equiv 1 \mod q \}.$$  

By [1] the matrix $A_f$ of the operator $\varphi$ in given basis of vector space $P$ has the form:

$$A_f = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{\alpha-1} \end{bmatrix}.$$  

And $A_f$ associated matrix with irreducible polynomial:

$$f(x) = x^\alpha + a_{\alpha-1}x^{\alpha-1} + \cdots + a_1x + a_0,$$  

over the field $GF(p)$ such that $\overline{f(x)}$ dividing the polynomial:

$$x^{q-1} + x^{q-2} + \cdots + x + 1 \in \mathbb{Z}_p[x].$$  

By [1], if $w \in P, w \neq 1$, then every element of $P$ can be represented as $g(\varphi)w$ where $g(x) \in GF(p)[x]$. Note that if $h(x) \in GF(p)[x]$ then

$$h(\varphi)w = g(\varphi)w \Rightarrow f(x)|\{h(x) - g(x)\},$$  

over $GF(p)[x]/f(x)$. Considering $P$ as an additive group of the field $GF(p)$, then for any elements $u, v \in G$, there exists a unique elements $i, j \in \mathbb{Z}_q$ and $w_1, w_2 \in P$ such that:

$$u = y^i w_1, \quad v = y^j w_2.$$  

By [1] the automorphism $\varphi$ given by:

$$\varphi(w) = y^{-1}wy; \quad \forall w \in P.$$  

Now using a multiplication on $G$

$$uv = y^i w_1 y^j w_2 = y^{i+j} \varphi^j(w_1)w_2$$  

(2)
Viewing \( u \) and \( v \) as pairs \((j, h(x)), (i, g(x))\) where \( g(x), h(x) \in GF(p)[x] \) and 
\[ w_1 = g(\varphi)w, w_2 = h(\varphi)w, \text{ for } w \in P \]
and considering the addition on \( P \), we have:
\[
(i, g(x)) = (j, h(x)) \iff i = j; g(x) \equiv h(x) (\text{mod } f(x))
\]
\[ i, j \in \mathbb{Z}_q. \tag{3} \]

Using (3) then the multiplication (2) becomes:
\[
(i, g(x))(j, h(x)) = (i + j, w^j g(x) + h(x))
\]
\[ \tag{4} \]
So by [7] \( G \) is a Miller-Moreno's group of type \( G(p, q) \).

Conversely, assuming that \( G \) a Schmidt's group, which is a Miller-Moreno's group of type \( G(p, q) \). Defining \( G \) as the group \((C, \cdot)\) and let \( f(x) \) be an irreducible polynomial over the field \( GF(p) \) of degree \( \alpha \) such that
\[ \alpha = \min\{k \in \mathbb{N}: p^k \equiv 1 (\text{mod } q)\}, \]
And \( f(x) \) divides
\[ x^{q-1} + x^{q-2} + \cdots + x + 1. \]
Applying the operation (4) on the set (1) and using the congruence in (3). Then we get that \( G \) is a semi direct product of the additive group of the field \( GF(p)[x] / f(x) \) by the cyclic group \( \langle (1,0) \rangle \) of the order \( q \). Now
\[
(1,0)^{-1}(0, h(x))(1,0) = (q - 1,0)(0, h(x))(1,0) = (q - 1, h(x))(1,0) = (0, wh(x)).
\]
Thus \( \langle (1,0) \rangle \) acts by conjugation, on the elements of
\[
\frac{GF(p)[x]}{f(x)}, \text{ so the resulting polynomial is a multiplication by } w. \text{ Thus the operator } \varphi \text{ in the basis }
\]
\[
\{(0,1), (0, w), \ldots, (0, w^{\alpha-1})\}
\]
has the above matrix \( A_f \). So by [1] \( G \) is a group with basis property.

\[ \blacksquare \]

**Theorem 2.11**

Let \( G \) be a Miller-Moreno's group, which is non-\( p \)- primary group. Then the group \( G \) has basis property if and only if it is a Miller-Moreno's group of type \( G(p, q) \) for distinct primes, \( q \).

**Proof.**

Let \( G \) be a group with basis property and it is not primary Miller-Moreno's group, then by Theorem (2.10) \( G \) is a semi direct product of \( p \)-group \( P \) by the cyclic \( q \)- group \( \langle y \rangle \), where \( y \) induces a regular automorphism on \( P \).
Then $G$ not nilpotent, and since all proper subgroups of $G$ are abelian, thus $G$ is Schmidt's group. Consequently by Theorem (2.10) $G$ is a group of type $G(p, q)$.

Conversely, let $G$ be a Miller Moreno's group of type $G(p, q)$. then by Theorem (2.10) it is a group of type $G(p, q)$, so it is a group with basis property.

Theorem 2.12

Let $G$ be a Miller-Moreno's group of type $G(p, q^v) \quad p \neq q$ are primes and $v \in \mathbb{N}$.

If $v > 1$, then $G$ does not have a basis property.

Proof.

By [9] in order that $G$ be a Miller-Moreno's group, then by Lemma (2.5) $G$ has one of the forms:

1. $G$ is the quaternion group $Q_8$, and it's a finite $p$- group.

2. The group $G$ of type:

   $G = \langle x, y : y^p = x^{p+1} = 1, b^{-1}ab = a^{1+p^v} \rangle$.

3. The group $G = M(p, m, n)$ is a primary group of order $p^m + n + 1$.

4. The group $G(p, q^v)$ and $G$ is a Schmidt's group.

In cases (1),(2) and (3) the group $G$ is a primary, so by [1] $G$ is a group with basis property.

Since in case (4) $G$ isn't primary group, and by Theorem (2.11) $G(p, q^v)$ is a group with basis property if $v=1$. Thus the group $G(p, q^v)$ hasn't basis property, when $v > 1$.

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References

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