

AN ANALYTIC APPROACH OF SOME CONJECTURES RELATED TO DIOPHANTINE EQUATIONS

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Abstract

Our purpose in this paper is to show how much diophantine equations are rich in analytic applications. Effectively, those equations allow to build sequences, series and numbers. The question of the analytic proof of some theorems remains of course, we will see it in this communication. We will make also an allusion to the Fermat numbers (x^{2^i}) and will see how this problem of the proof is actual and how it can be solved using the sequences and series.

The first sequences

Let the general equation $kU^n = \sum_{j=1}^{j=n} k_j x_j^{n_j}$ if we pose

$$u = k^2 U^{2n}$$

$$x = kU^n (kU^n - k_m X_m^{n_m})$$

$$y = kU^n (k_m X_m^{n_m})$$

$$z = k_m X_m^{n_m} (kU^n - k_m X_m^{n_m})$$

We have

LEMMA 1

$$u = x + y \tag{1}$$

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y} \tag{2}$$

Let now the following equation

$$\frac{a'}{n'} = \sum_{j=1}^{j=n} \left(\frac{1}{x_j}\right) \tag{3}$$

With

$$x_1 = x$$

$$x_2 = y$$

$$\text{We have } \frac{a'}{n'} - \sum_{j=3}^{j=n} \left(\frac{1}{x_j}\right) = \frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

Let us define the sequences whose first terms are

$$x_1 = x$$

$$y_1 = y$$

And $\forall x_1, y_1, \text{ integers}, \exists z_1$ verifying

$$\frac{1}{z_1} = \frac{1}{x_1} + \frac{1}{y_1}$$

And

$$z_1 = \frac{xy}{x+y} = z$$

Hence

$$(x_1 + y_1)z_1 = x_1y_1$$

And

$$x_1(y_1 - z_1) = z_1y_1$$

We pose

$$y_2 = y_1 - z_1 = \frac{z_1y_1}{x_1}$$

Also

$$y_1(x_1 - z_1) = \frac{z_1x_1}{y_1}$$

We pose

$$x_2 = x_1 - z_1 = \frac{z_1x_1}{y_1}$$

And

$$x_2y_2 = z_1^2$$

Which means that

$$x_1 = x_2 + z_1 = x_2 + \sqrt{x_2y_2}$$

$$y_1 = y_2 + z_1 = y_2 + \sqrt{x_2y_2}$$

$$u_1 = u = (x_1 + y_1) = (\sqrt{x_2} + \sqrt{y_2})^2 > x_2 + y_2 > 0$$

Or

$$x_1 = \sqrt{x_2}(\sqrt{x_2} + \sqrt{y_2}) > x_2 > 0$$

$$y_1 = \sqrt{y_2}(\sqrt{x_2} + \sqrt{y_2}) > y_2 > 0$$

$$z_1 = \frac{x_1y_1}{x_1 + y_1} = \sqrt{x_2y_2} > z_2 = \frac{x_2y_2}{x_2 + y_2} > 0$$

Because $\forall x_2, y_2, \exists z_2$ verifying

$$\frac{1}{z_2} = \frac{1}{x_2} + \frac{1}{y_2}$$

The process is available until infinity. For i

$$u_i = x_i + y_i = (\sqrt{x_{i+1}} + \sqrt{y_{i+1}})^2 > x_{i+1} + y_{i+1} > 0$$

$$x_i = \sqrt{x_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) > x_{i+1} > 0$$

$$y_i = \sqrt{y_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) > y_{i+1} > 0$$

$$z_i = \frac{x_iy_i}{x_i + y_i} = \sqrt{x_{i+1}y_{i+1}} > z_{i+1} = \frac{x_{i+1}y_{i+1}}{x_{i+1} + y_{i+1}} > 0$$

And of course

$$\frac{1}{z_{i+1}} = \frac{1}{x_{i+1}} + \frac{1}{y_{i+1}}$$

We have built the first the first sequences.

LEMMA 2

x_i, y_i have an expression

$$x_i = x^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1} \tag{4}$$

$$y_i = y^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1} \tag{5}$$

Proof of lemma 2

By traditional induction, for i=2

$$x = \sqrt{x_2}(\sqrt{x_2} + \sqrt{y_2}) = \sqrt{x_2}(x + y)^{\frac{1}{2}}$$

$$x_2 = \frac{x^2}{x + y}$$

Also

$$y = \sqrt{y_2}(\sqrt{x_2} + \sqrt{y_2}) = \sqrt{y_2}(x + y)^{\frac{1}{2}}$$

$$y_2 = \frac{y^2}{x + y}$$

We suppose (4) and (5) true for i, then

$$x_i = \sqrt{x_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) = \sqrt{x_{i+1}}(x_i + y_i)^{\frac{1}{2}}$$

$$x_{i+1} = \frac{x_i^2}{x_i + y_i} = x^{2^i} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-2} (x^{2^{i-1}} + y^{2^{i-1}})^{-1} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j}) = x^{2^i} \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1}$$

Also

$$y_i = \sqrt{y_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) = \sqrt{y_{i+1}}(x_i + y_i)^{\frac{1}{2}}$$

$$y_{i+1} = \frac{y_i^2}{x_i + y_i} = y^{2^i} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-2} (x^{2^{i-1}} + y^{2^{i-1}})^{-1} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j}) = y^{2^i} \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1}$$

But $\forall x, y$

$$\prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j}) = \frac{x^{2^{i-1}} - y^{2^{i-1}}}{x - y}$$

Thus for $x \neq y$

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y)$$

$$y_i = \frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y)$$

LEMMA 4

There is the constant

$$x_l - y_l = x - y$$

And

$$x_l = \frac{x_l}{x_l - y_l} (x - y)$$

$$y_l = \frac{x_l}{x_l - y_l} (x - y)$$

$$x > y \Rightarrow \lim_{l \rightarrow \infty} (y_l) = \lim_{l \rightarrow \infty} \left(\frac{y^{2^{l-1}}}{x^{2^{l-1}} - y^{2^{l-1}}} (x - y) \right) = 0$$

$$\text{And } \lim_{l \rightarrow \infty} (x_l) = \lim_{l \rightarrow \infty} \left(\frac{x^{2^{l-1}}}{x^{2^{l-1}} - y^{2^{l-1}}} (x - y) \right) = x - y$$

$$x < y \Rightarrow \lim_{l \rightarrow \infty} (y_l) = \lim_{l \rightarrow \infty} \left(\frac{y^{2^{l-1}}}{x^{2^{l-1}} - y^{2^{l-1}}} (x - y) \right) = y - x$$

$$\lim_{l \rightarrow \infty} (x_l) = \lim_{l \rightarrow \infty} \left(\frac{x^{2^{l-1}}}{x^{2^{l-1}} - y^{2^{l-1}}} (x - y) \right) = 0$$

THEOREM

We have supposed $x-y$ different of zero. A priori nothing allows to say that x is different or equal to y . Nonetheless, our investigations leaded us to a strange result, which is that $x=y$, without any condition on x and y . Why this impossible result ? We think about Matyasavitch theorem. All diophantine equations do not have solutions and the conjectures linked to those equations are not all decidable. But, the sequences established here are available for all equations like (3). Nowadays, we do not know when there are solutions and when there are not.

But $u=x+y$

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$$

If there is an undecidability, those sequences should lead to an impossibility. The impossibility is $xy(x-y)=0$ for all (x,y) . We will prove that $xy(x-y)=0$ formally. Here are some proofs. The first utilizes series and particularly Fourier series.

Effectively, as

$$\sqrt{x_l y_l} = y_{l-1} - y_l = x_{l-1} - x_l$$

It implies the following sum

$$\sum_{j=2}^{j=l+1} (\sqrt{x_j y_j}) = x - x_2 + x_2 - x_3 + x_3 - x_4 + \dots + x_l - x_{l+1} = x - x_{l+1}$$

So

$$\sum_{j=2}^{l=\infty} (\sqrt{x_j y_j}) = \lim_{l \rightarrow \infty} (x - x_{l+1})$$

And the limits, if $x > y$

$$\lim_{l \rightarrow \infty} (y_l) = \lim_{l \rightarrow \infty} \left(\frac{y^{2^{l-1}}}{x^{2^{l-1}} - y^{2^{l-1}}} (x - y) \right) = 0$$

And

$$\lim_{l \rightarrow \infty} (x_l) = \lim_{l \rightarrow \infty} \left(\frac{x^{2^{l-1}}}{x^{2^{l-1}} - y^{2^{l-1}}} (x - y) \right) = x - y$$

If $x < y$

$$\lim_{l \rightarrow \infty} (x_l) = \lim_{l \rightarrow \infty} \left(\frac{x^{2^{l-1}}}{x^{2^{l-1}} - y^{2^{l-1}}} (x - y) \right) = 0$$

$$\lim_{l \rightarrow \infty} (y_l) = \lim_{l \rightarrow \infty} \left(\frac{y^{2^{l-1}}}{x^{2^{l-1}} - y^{2^{l-1}}} (x - y) \right) = y - x$$

Let us study series. If $x > y$

$$\sum_{j=2}^{l=\infty} (\sqrt{x_j y_j}) = \lim_{l \rightarrow \infty} (x - x_{l+1}) = x - (x - y) = y$$

And if $x < y$

$$\sum_{j=2}^{l=\infty} (\sqrt{x_j y_j}) = \lim_{l \rightarrow \infty} (x - x_{l+1}) = x$$

The proof of the theorem or the application the sequences and series

We will consider firstly that $x > y$

$$\begin{aligned} \sum_{j=2}^{j=l} ((-1)^j \sqrt{x_j y_j}) &= x - x_2 - x_2 + x_3 + \dots + (-1)^l (x_{l-1} - x_l) \\ &= x - 2x_2 + 2x_3 - \dots + 2(-1)^l x_{l-1} + (-1)^{l+1} x_l \\ &= 2 \sum_{j=1}^{j=l} ((-1)^{j+1} x_j) - x - (-1)^{l+1} x_l \end{aligned}$$

Also

$$\begin{aligned} \sum_{j=2}^{j=l} ((-1)^j \sqrt{x_j y_j}) &= y - y_2 - y_2 + y_3 + \dots + (-1)^l (y_{l-1} - y_l) \\ &= y - 2y_2 + 2y_3 - \dots + 2(-1)^l y_{l-1} + (-1)^{l+1} y_l \\ &= 2 \sum_{j=1}^{j=l} ((-1)^{j+1} y_j) - y - (-1)^{l+1} y_l \end{aligned}$$

Then

$$2 \sum_{j=1}^{j=l} ((-1)^{j+1} x_j) = \sum_{j=2}^{j=l} ((-1)^j \sqrt{x_j y_j}) + x + (-1)^{l+1} x_l$$

And

$$2 \sum_{j=1}^{j=l} ((-1)^{j+1} y_j) = \sum_{j=2}^{j=l} ((-1)^j \sqrt{x_j y_j}) + y + (-1)^{l+1} y_l$$

We will study now the convergence of the series. As $\sum_{j=2}^{j=\infty} ((-1)^j \sqrt{x_j y_j})$ is convergent and

$$\lim_{l \rightarrow \infty} (y_l) = \lim_{l \rightarrow \infty} \left(\frac{y^{2^{l-1}}}{x^{2^{l-1}} - y^{2^{l-1}}} (x - y) \right) = 0 \text{ then}$$

$$\sum_{j=2}^{j=\infty} ((-1)^j \sqrt{x_j y_j}) = 2 \sum_{j=1}^{j=\infty} ((-1)^{j+1} x_j) - x - \lim_{l \rightarrow \infty} ((-1)^{l+1} x_l) = 2 \sum_{j=1}^{j=\infty} ((-1)^{j+1} y_j) - y - \lim_{l \rightarrow \infty} ((-1)^{l+1} y_l)$$

is convergent. It means that

$$2 \sum_{j=1}^{j=\infty} ((-1)^{j+1} x_j) - x - \lim_{l \rightarrow \infty} ((-1)^{l+1} x_l) \text{ exists. It means one thing : } \lim_{l \rightarrow \infty} ((-1)^{l+1} x_l) = 0, \text{ then}$$

$\lim_{l \rightarrow \infty} (x_l) = x - y = 0$. It is confirmed by the fact the limit of the general term of the series (here $x - y$)

is equal to zero, because $\sum_{j=1}^{j=\infty} ((-1)^{j+1} x_j)$ is convergent. And $x - y = 0$

The reasoning is the same for $x < y$.

We have

$$\begin{aligned} & \sum_{k=1}^{k=2m} ((-1)^{k+1} x_k e^{\frac{-k}{\sqrt{2m}}}) \\ &= x e^{\frac{-2}{\sqrt{2m}}} + x e^{\frac{-2}{\sqrt{2m}}} (e^{\frac{1}{\sqrt{2m}}} - 1) - x_2 e^{\frac{-2}{\sqrt{2m}}} + x_3 e^{\frac{-4}{\sqrt{2m}}} + x_3 e^{\frac{-4}{\sqrt{2m}}} (e^{\frac{1}{\sqrt{2m}}} - 1) + \dots - x_{2m} e^{\frac{-2m}{\sqrt{2m}}} \\ &= (x - x_2) e^{\frac{-2}{\sqrt{2m}}} + (x_3 - x_4) e^{\frac{-4}{\sqrt{2m}}} + \dots + (x_{2m-1} - x_{2m}) e^{\frac{-2m}{\sqrt{2m}}} + (e^{\frac{1}{\sqrt{2m}}} - 1) (x e^{\frac{-2}{\sqrt{2m}}} + \dots + x_{2m-1} e^{\frac{-2m}{\sqrt{2m}}}) \\ &= (e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=1}^{k=m} (x_{2k-1} e^{\frac{-2k}{\sqrt{2m}}}) + \sum_{k=1}^{k=m} (\sqrt{x_{2k} y_{2k}} e^{\frac{-2k}{\sqrt{2m}}}) \end{aligned}$$

$$\text{And also } \sum_{k=1}^{k=2m} ((-1)^{k+1} y_k e^{\frac{-k}{\sqrt{2m}}}) = (e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=1}^{k=m} (y_{2k-1} e^{\frac{-2k}{\sqrt{2m}}}) + \sum_{k=1}^{k=m} (\sqrt{x_{2k} y_{2k}} e^{\frac{-2k}{\sqrt{2m}}})$$

$$\text{But } (e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=1}^{k=m} (y_{2k-1} e^{\frac{-2k-1}{\sqrt{2m}}}) < (e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=1}^{k=m} (y_{2k-1} e^{\frac{-2k}{\sqrt{2m}}}) = S < (e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=1}^{k=m} (y_{2k-1} e^{\frac{-2k+1}{\sqrt{2m}}})$$

And

$$(e^{\frac{1}{\sqrt{2m}}} - 1) e^{\frac{-3}{\sqrt{2m}}} \sum_{k=1}^{k=m} (y_{2k-1} e^{\frac{-2k+2}{\sqrt{2m}}}) < (e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=1}^{k=m} (y_{2k-1} e^{\frac{-2k}{\sqrt{2m}}}) = S < (e^{\frac{1}{\sqrt{2m}}} - 1) e^{\frac{-1}{\sqrt{2m}}} \sum_{k=1}^{k=m} (y_{2k-1} e^{\frac{-2k+2}{\sqrt{2m}}})$$

$$\text{Thus } \lim_{m \rightarrow \infty} ((e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=1}^{k=m} (y_{2k-1} e^{\frac{-2k+2}{\sqrt{2m}}})) = \lim_{m \rightarrow \infty} ((e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=2}^{k=m} (y_{2k-1} e^{\frac{-2k+2}{\sqrt{2m}}})) = \lim_{m \rightarrow \infty} (S)$$

$$\text{Let } (e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=p}^{k=m} (y_{2k-1} e^{\frac{-2k+p}{\sqrt{2m}}}) = A$$

$$\text{But } (e^{\frac{1}{\sqrt{2m}}} - 1) e^{\frac{-2}{\sqrt{2m}}} \sum_{k=p}^{k=m} (y_{2k-1} e^{\frac{-2k+p+1}{\sqrt{2m}}}) < A < (e^{\frac{1}{\sqrt{2m}}} - 1) \sum_{k=p}^{k=m} (y_{2k-1} e^{\frac{-2k+p+1}{\sqrt{2m}}})$$

$$\text{Thus } \lim_{m \rightarrow \infty} ((e^{\frac{1}{\sqrt{2^m}}} - 1) \sum_{k=p}^{k=m} (y_{2^{k-1}} e^{\frac{-2k+p+1}{\sqrt{2^m}}})) = \lim_{m \rightarrow \infty} ((e^{\frac{1}{\sqrt{2^m}}} - 1) \sum_{k=p+1}^{k=m} (y_{2^{k-1}} e^{\frac{-2k+p+1}{\sqrt{2^m}}})) = \lim_{m \rightarrow \infty} (A) = \lim_{m \rightarrow \infty} (S)$$

$$\text{Until } p = m - 1 \Rightarrow \lim_{m \rightarrow \infty} (A) = \lim_{m \rightarrow \infty} (S) = \lim_{m \rightarrow \infty} ((e^{\frac{1}{\sqrt{2^m}}} - 1) y_{2^{m-1}} e^{\frac{-m}{\sqrt{2^m}}}) = 0$$

$$\text{Also } \lim_{m \rightarrow \infty} ((e^{\frac{1}{\sqrt{2^m}}} - 1) x_{2^{m-1}} e^{\frac{-m}{\sqrt{2^m}}}) = 0$$

We have then

$$0 < \lim_{m \rightarrow \infty} (\sum_{k=1}^{k=2m} ((-1)^{k+1} x_k e^{\frac{-k}{\sqrt{2^m}}})) = \lim_{m \rightarrow \infty} (\sum_{k=1}^{k=2m} ((-1)^{k+1} x_k e^{\frac{-k}{\sqrt{2^m}}})) = \lim_{m \rightarrow \infty} (\sum_{k=1}^{k=m} (\sqrt{x_{2k} y_{2k}} e^{\frac{-2k}{\sqrt{2^m}}}))$$

$$< \lim_{m \rightarrow \infty} (\sum_{k=1}^{k=m} (\sqrt{x_{2k} y_{2k}})) < \lim_{m \rightarrow \infty} (\sum_{k=1}^{k=2m} (\sqrt{x_k y_k})) = y$$

$$\Rightarrow \lim_{m \rightarrow \infty} (\sum_{k=1}^{k=2m} ((-1)^{k+1} (x_k - y_k) e^{\frac{-k}{\sqrt{2^m}}})) = \lim_{m \rightarrow \infty} (\sum_{k=1}^{k=2m} ((-1)^{k+1} (x - y) e^{\frac{-k}{\sqrt{2^m}}})) = 0$$

$$= (x - y) \lim_{m \rightarrow \infty} (\sum_{k=1}^{k=2m} ((-1)^{k+1} e^{\frac{-k}{\sqrt{2^m}}})) = \frac{x - y}{2} = 0$$

Thus $x - y = 0$

Second sequences

$$\text{Let } \frac{a'}{n'} - \sum_{j=4}^{j=n} (\frac{1}{x_j}) = \frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

(6)

x, y, z, a and n are positive integers.

Thus

$$n(xy + xz + yz) = axyz$$

(7)

It is equivalent to three equations

$$x((az - n)y - nz) = nyz$$

$$y((ax - n) - nx) = nxz$$

$$z((ay - n) - ny) = nxy$$

The sequences

If

$$x_1 = x$$

$$y_1 = y$$

$$z_1 = z$$

Thus

$$ax_2 = ax - n < ax$$

$$ay_2 = ay - n < ay$$

$$az_2 = az - n < az$$

$$\forall (x_2, y_2, z_2), \exists n_2 \mid \frac{a}{n_2} = \frac{1}{x_2} + \frac{1}{y_2} + \frac{1}{z_2}$$

until infinity, for i

$$\frac{a}{n_i} = \frac{1}{x_i} + \frac{1}{y_i} + \frac{1}{z_i} \quad (8)$$

It is equivalent to the following equation

$$n_i(x_i y_i + x_i z_i + y_i z_i) = a x_i y_i z_i \quad (9)$$

Which is equivalent to three equations

$$x_i((a z_i - n_i) - n_i z_i) = n_i y_i z_i ; y_i((a x_i - n_i) - n_i x_i) = n_i x_i z_i ; z_i((a y_i - n_i) - n_i y_i) = n_i x_i y_i$$

Hence

$$a x_{i+1} = a x_i - n_i < a x_i$$

$$a y_{i+1} = a y_i - n_i < a y_i$$

$$a z_{i+1} = a z_i - n_i < a z_i$$

$$\text{And } \forall (x_{i+1}, y_{i+1}, z_{i+1}), \exists n_{i+1} \mid \frac{a}{n_{i+1}} = \frac{1}{x_{i+1}} + \frac{1}{y_{i+1}} + \frac{1}{z_{i+1}}$$

$$\text{And } 0 < n_{i+1} < n_i$$

Lemma

There are evidently three constants

$$x_i - y_i = x - y$$

$$x_i - z_i = x - z$$

$$y_i - z_i = y - z$$

The limits

$$\text{As } n_i = a(x_i - x_{i+1}) = a(y_i - y_{i+1}) = a(z_i - z_{i+1})$$

$$\text{Consequently } \lim_{i \rightarrow \infty} (n_i) = 0 \text{ and with } x > y > z \quad \lim_{i \rightarrow \infty} (z_i) = 0 \text{ and } \lim_{i \rightarrow \infty} (x_i - z_i) = \lim_{i \rightarrow \infty} (x_i) = x - z \text{ and}$$

$$\lim_{i \rightarrow \infty} (y_i - z_i) = \lim_{i \rightarrow \infty} (y_i) = y - z$$

The series

Let

$$\sum_{j=1}^{j=i} n_j = a(x - x_2 + x_2 - x_3 + \dots + x_{i-1} - x_i) = a(x - x_i) = a(y - y_i) = a(z - z_i)$$

$$\text{In the infinity } \sum_{j=1}^{j=\infty} n_j = a z \text{ And with the same proof that we saw we have } x=y=z$$

Other sequences

Now, let the general following equation

$$Y^n = X_1^{n_1} + X_2^{n_2} + \dots + X_i^{n_i}$$

$$\text{GCD}(X_k) = 1$$

We pose

$$u = Y^{2^n}$$

$$x = Y^n (Y^n - X_k^{n_k})$$

$$y = Y^n X_k^{n_k}$$

$$z = X_k^{n_k} (Y^n - X_k^{n_k})$$

With $k=1,2,\dots,i$

Then

$$u = x + y \tag{9}$$

$$\frac{1}{z} = \frac{1}{X_k^{n_k} (Y^n - X_k^{n_k})} = \frac{Y^{2n}}{Y^n X_k^{n_k} Y^n (Y^n - X_k^{n_k})} = \frac{u}{xy} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y} \tag{10}$$

(9) and (10) are the new equations, generalized equation does not have solution for $n > i(i-1)$ and $n_k > i(i-1)$ but

$$Y^n = 2X_k^{n_k} = 2X_m^{n_m}; m \neq k$$

First conclusion

The new equations allow to build sequences and series that leads to test the impossibility of the resolution of an equation. If they are a consequence of some Diophantine equations, they remain an intellectual building.

The generalized sequences

Now, we will generalize the results. Let the following equation

$$Y^n = X_1^{n_1} + X_2^{n_2} + \dots + X_i^{n_i} \tag{E}$$

We will prove that this equation has not solution for $n > i(i-1), n_j > i(i-1), \forall j \in \{1, 2, \dots, i\}$

When $n \leq i(i-1), n_k \leq i(i-1)$, there are solutions, for example

$$i=2 \text{ has } 3^2 + 4^2 = 5^2$$

$$i=3 \text{ has } 3^3 + 4^3 + 5^3 = 6^3$$

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

$$i=4 \text{ has } 27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

Let

$$x_k = Y^{(i-1)n} X_k^{n_k}, \forall k \in \{1, 2, \dots, i\}$$

$$u = Y^{in}$$

$$v = X_1^{n_1} X_2^{n_2} \dots X_i^{n_i}$$

LEMMA 6

$$x_1 + x_2 + \dots + x_i = Y^{(i-1)n} (X_1^{n_1} + X_2^{n_2} + \dots + X_i^{n_i}) = Y^{in} = u \tag{11}$$

$$\frac{1}{v} = \frac{1}{X_1^{n_1} X_2^{n_2} \dots X_i^{n_i}} = \frac{Y^{i(i-1)n}}{Y^{(i-1)n} X_1^{n_1} Y^{(i-1)n} X_2^{n_2} \dots Y^{(i-1)n} X_i^{n_i}} = \frac{u^{i-1}}{x_1 x_2 \dots x_i} \tag{12}$$

We will define the sequences

$$x_{k,0} = x_k$$

$$u_0 = u$$

$$v_0 = v$$

$$x_{k,1} = x_k^i (x_1 + x_2 + \dots + x_i)^{-(i-1)}, \forall k \in \{1, 2, \dots, i\}$$

Which implies

$$u = x_1 + x_2 + \dots + x_i = (x_{1,1}^{\frac{1}{i}} + x_{2,1}^{\frac{1}{i}} + \dots + x_{i,1}^{\frac{1}{i}})^i > u_1 > 0$$

$$x_{k,0} = x_k = x_{k,1}^{\frac{1}{i}} (x_1 + x_2 + \dots + x_i)^{\frac{i-1}{i}} = x_{k,1}^{\frac{1}{i}} (x_{1,1}^{\frac{1}{i}} + x_{2,1}^{\frac{1}{i}} + \dots + x_{i,1}^{\frac{1}{i}})^{i-1} > x_{k,1} > 0$$

$$v = \frac{x_{1,0} x_{2,0} \dots x_{i,0}}{u^{i-1}} = x_{1,1}^{\frac{1}{i}} x_{2,1}^{\frac{1}{i}} \dots x_{i,1}^{\frac{1}{i}} > v_1 = \frac{x_{1,1} x_{2,1} \dots x_{i,1}}{u_1^{i-1}} > 0$$

The reasoning is available until infinity. Then

$$x_{k,j} = x_{k,j+1}^{\frac{1}{i}} (x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + \dots + x_{l,j+1}^{\frac{1}{i}})^{i-1} > x_{k,j+1} > 0$$

$$u_j = x_{1,j} + x_{2,j} + \dots + x_{l,j} = (x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + \dots + x_{l,j+1}^{\frac{1}{i}})^i > u_{j+1} > 0$$

$$v_j = \frac{x_{1,j} x_{2,j} \dots x_{l,j}}{u_j^{i-1}} = x_{1,j+1}^{\frac{1}{i}} x_{2,j+1}^{\frac{1}{i}} \dots x_{l,j+1}^{\frac{1}{i}} > v_{j+1} = \frac{x_{1,j+1} x_{2,j+1} \dots x_{l,j+1}}{u_{j+1}^{i-1}} > 0$$

LEMMA 7

(P) is the following expression

$$x_{k,j} = x_k^{i^j} \left(\prod_{l=0}^{i=j-1} x_1^{i^l} + x_2^{i^l} + \dots + x_l^{i^l} \right)^{-(i-1)}$$

Proof of lemma 7

By traditional induction, it is verified for $j=1$, we suppose that (P) is true for j , so

$$x_{k,j+1}^{\frac{1}{i}} = x_{k,j} (x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + \dots + x_{l,j+1}^{\frac{1}{i}})^{-(i-1)}$$

$$x_{k,j+1} = x_{k,j}^i (x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + \dots + x_{l,j+1}^{\frac{1}{i}})^{-i(i-1)} = x_{k,j}^i (x_{1,j}^{\frac{1}{i}} + x_{2,j}^{\frac{1}{i}} + \dots + x_{l,j}^{\frac{1}{i}})^{-i(i-1)}$$

$$= x_k^{i^{j+1}} \left(\prod_{l=0}^{i=j-1} x_1^{i^l} + x_2^{i^l} + \dots + x_l^{i^l} \right)^{-i(i-1)} (x_1^{i^j} + x_2^{i^j} + \dots + x_l^{i^j})^{-(i-1)} \prod_{l=0}^{i=j-1} x_1^{i^l} + x_2^{i^l} + \dots + x_l^{i^l})^{(i-1)^2}$$

$$= x_k^{i^{j+1}} \left(\prod_{l=0}^{i=j} x_1^{i^l} + x_2^{i^l} + \dots + x_l^{i^l} \right)^{-(i-1)}$$

And it is true for $j+1$.

LEMMA 8

The equation (E) leads to an impossibility, effectively, if we pose

$$u = Y^{2^n}$$

$$x = Y^n X_k^{n_k}$$

$$y = Y^n (Y^n - X_k^{n_k})$$

$$z = X_k^{n_k} (Y^n - X_k^{n_k})$$

u, x, y, z verify the lemma 1

$$u = x + y$$

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$$

Which leads, we will see it, to $x=y$

Because they are coprime. Now, the question is : why did we propose solutions for $n \leq i(i-1), n_k \leq i(i-1)$?

Let us pose

$$n = i(i-1), n_k = i(i-1), \forall k \in \{1, 2, \dots, i\}$$

The expression (P) becomes

$$\begin{aligned} x_{k,j} &= x_k^{i^j} \left(\prod_{l=0}^{j-1} x_1^{i^l} + x_2^{i^l} + \dots + x_i^{i^l} \right)^{-(i-1)} \\ &= Y^{i(i-1)^j} X_k^{i(i-1)^j} \left(\left(\prod_{l=0}^{j-1} Y^{i(i-1)^l} X_1^{i(i-1)^l} + Y^{i(i-1)^l} X_2^{i(i-1)^l} + \dots + Y^{i(i-1)^l} X_i^{i(i-1)^l} \right) \right)^{-(i-1)} \\ &= Y^{(i-1)^{j+1}} X_k^{(i-1)^{j+1}} \left(\left(\prod_{l=0}^{j-1} Y^{i(i-1)^l} X_1^{i(i-1)^l} + Y^{i(i-1)^l} X_2^{i(i-1)^l} + \dots + Y^{i(i-1)^l} X_i^{i(i-1)^l} \right) \right)^{-(i-1)} \\ &= Y^{(i-1)^{j+1}} X_k^{(i-1)^{j+1}} \left(\left(\prod_{l=0}^{j-1} Y^{(i-1)^{l+1}} X_1^{(i-1)^{l+1}} + Y^{(i-1)^{l+1}} X_2^{(i-1)^{l+1}} + \dots + Y^{(i-1)^{l+1}} X_i^{(i-1)^{l+1}} \right) \right)^{-(i-1)} \\ &= Y^{(i-1)^{j+1}} X_k^{(i-1)^{j+1}} \left(\left(\prod_{l=1}^j Y^{(i-1)^l} X_1^{(i-1)^l} + Y^{(i-1)^l} X_2^{(i-1)^l} + \dots + Y^{(i-1)^l} X_i^{(i-1)^l} \right) \right)^{-(i-1)} \end{aligned}$$

It is the expression for the exponent (i-1). If there are solutions for the exponent (i-1), there will be solutions for the exponent i(i-1). It is not true for i, because of the exponent -(i-1) in the expression (P).

Second conclusion

The sequences and the series as we defined them have several applications in several diophantine equations, for example Fermat, Beal, Erdos, we saw the generalized equation (1), but there are many others like Pillai, Smarandache, Catalan...

Other sequences

Now, let the equation

$$U^n = X^n + Y^n = X^n + i(-iY^n)$$

$$i^2 = -1$$

We pose

$$x' = U^n X^n$$

$$y' = -iU^n Y^n$$

$$u' = U^{2n} = U^n (X^n + i(-iY^n)) = x + iy \tag{3}$$

$$z' = X^n (-iY^n) = \frac{x' y'}{u'}$$

$$\frac{1}{z'} = \frac{1}{x'} + \frac{i}{y'} \tag{4}$$

We will build sequences

$$x'_1 = x'$$

$$y'_1 = y'$$

$$u'_1 = u'$$

$$z'_1 = z'$$

And

$$(y'_1 - z'_1)x'_1 = y'_2 x'_1 = iy'_1 z'_1$$

$$(x'_1 - iz'_1)y'_1 = x'_1 z'_1 = x'_2 y'_1$$

$$iz'_1{}^2 = x'_2 y'_2$$

$$y'_1 = y'_2 + z'_1 = y'_2 + \sqrt{\frac{x'_2 y'_2}{i}}$$

$$x'_1 = x'_2 + iz'_1 = x'_2 + i\sqrt{\frac{x'_2 y'_2}{i}}$$

And

$$\frac{1}{z'_2} = \frac{1}{x'_2} + \frac{i}{y'_2}$$

The process is available until infinity, for j

$$y'_j = y'_{j+1} + z'_j = y'_{j+1} + \sqrt{\frac{x'_{j+1} y'_{j+1}}{i}}$$

$$x'_j = x'_{j+1} + iz'_j = x'_{j+1} + i\sqrt{\frac{x'_{j+1} y'_{j+1}}{i}}$$

$$x'_j + iy'_j = (\sqrt{x'_{j+1}} + \sqrt{iy'_{j+1}})^2$$

And

$$\frac{1}{z'_{j+1}} = \frac{1}{x'_{j+1}} + \frac{i}{y'_{j+1}}$$

The expressions are

$$x'_j = x'^{2^{j-1}} \prod_{m=0}^{m=j-2} (x'^{2^m} + (iy')^{2^m})^{-1}$$

$$y'_j = i^{2^{j-1}-1} y'^{2^{j-1}} \prod_{m=0}^{m=j-2} (x'^{2^m} + (iy')^{2^m})^{-1}$$

We prove it by induction, like we did for rational sequences

So

$$x'_j = x_j$$

$$y'_j = i^{-1} y_j$$

So the only solution is

$$x=y=0$$

Conclusion

It appeared since the beginning, before the change of the data, that the equations contain a symmetry between x and y. effectively, we found $u=x+y$. We broke the symmetry by changing the

$$u = x + y \text{ and } \frac{1}{z} = \frac{1}{x} + \frac{1}{y}$$

equation in two equations

The conclusion is that the equation (1) leads always to an impossibility which is $x=y$. It is the case of Fermat-Catalan or Erdos equations. The cause is the undecidability of some conjectures related to Diophantine equations.