

t- Intuitionistic Fuzzy Subalgebra of *BG*-Algebras

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Abstract. The aim of this paper is to introduced the notion of t-intuitionistic fuzzy subalgebra and t-intuitionistic fuzzy normal subalgebra of *BG*-algebras. We state and prove some theorems in t-intuitionistic fuzzy subalgebra and t-intuitionistic fuzzy normal subalgebra in *BG*-algebras. The homomorphic image and inverse image are investigated in both t-intuitionistic fuzzy subalgebra and normal subalgebras.

Introduction

In 1966, Imai and Iseki [6] introduced the two classes of abstract algebras, viz., *BCK*-algebras and *BCI*-algebras. It is known that the class of *BCK*-algebra is a proper subclass of the class of *BCI*-algebras. Neggers and Kim [8] introduced a new concept, called *B*-algebras, which are related to several classes of algebras such as *BCI/BCK*-algebras. Kim and Kim [7] introduced the notion of *BG*-algebra which is a generalization of *B*-algebra. The concept of intuitionistic fuzzy subset (IFS) was introduced by Atanassov [5] in 1983, which is a generalization of the notion of fuzzy sets. The concept of fuzzy subalgebras of *BG*-algebras was introduced by Ahn and Lee in [1]. The study of intuitionistic fuzzification of subalgebras and ideals of *BG*-algebras is done by Senapati et. al in [9]. The idea of t-intuitionistic fuzzy sets in fuzzy subgroups and fuzzy subrings is introduced by Sharma in [10, 11]. Here in this paper, we introduced the notion of t-intuitionistic fuzzy sets in fuzzy subalgebra and fuzzy normal subalgebras of *BG*-algebras and study their properties.

Preliminaries

Definition 0.1 ([1]) A *BG*-algebra is a non-empty set X with a constant '0' and a binary operation '*' satisfying the following axioms:

- (i) $x * x = 0$,
- (ii) $x * 0 = x$,
- (iii) $(x * y) * (0 * y) = x, \forall x, y \in X$.

For brevity, we also call X a *BG*-algebra. We can define a partial ordering " \leq " on X by $x \leq y$ iff $x * y = 0$

Definition 0.2 ([1]) A non-empty subset S of a *BG*-algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Definition 0.3 Let X and Y be two non empty sets and $f : X \rightarrow Y$ be a mapping. Let A and B be IFS's of X and Y respectively. Then the image of A under the map f is denoted by $f(A)$ and is

$$\text{defined by } f(A)(y) = (\mu_{f(A)}y, \nu_{f(A)}y), \text{ where } \mu_{f(A)}(y) = \begin{cases} \bigvee \{ \mu_A(x) : x \in f^{-1}(y) \} \\ 0 \end{cases} \quad \nu_{f(A)}(y) = \begin{cases} \bigwedge \{ \nu_A(x) : x \in f^{-1}(y) \} \\ 1 \end{cases} \text{ otherwise}$$

also pre image of B under f is denoted by $f^{-1}(B)$ and is defined as $f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)) = (\mu_B(f(x)), \nu_B(f(x))) ; \forall x \in X$

Remark Note that $\mu_A(x) \leq \mu_{f(A)}(f(x))$ and $\nu_A(x) \geq \nu_{f(A)}(f(x)) \quad \forall x \in X$ however equality hold when the map f is bijective.

Definition 0.4 ([2, 3]) An intuitionistic fuzzy set (IFS) A in a non empty set X is an object of the form $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non membership of the element x in the set A . For the sake of simplicity we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$. The function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$. is called the degree of uncertainty of $x \in A$. The class of IFSs on a universe X is denoted by $IFS(X)$.

Definition 0.5 ([2, 3]) If $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ and $B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle \mid x \in X \}$ be any two IFS of a set X then

$$\begin{aligned} A \subseteq B & \text{ iff for all } x \in X, \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \\ A = B & \text{ iff for all } x \in X, \mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x) \\ A \cap B & = \{ \langle x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x) \rangle \mid x \in X \} \text{ where} \\ & (\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\} \text{ and } (\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\} \\ A \cup B & = \{ \langle x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x) \rangle \mid x \in X \} \text{ where} \\ & (\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\} \text{ and } (\nu_A \cap \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\} \end{aligned}$$

Definition 0.6 ([4]) For any IFS $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ of X and $\alpha \in [0, 1]$, the operator $\square : IFS(X) \rightarrow IFS(X), \diamond : IFS(X) \rightarrow IFS(X), D_\alpha : IFS(X) \rightarrow IFS(X)$ are defined as

- (i) $\square(A) = \{ \langle x, \mu_A(x), 1 - \mu_A(x) \rangle \mid x \in X \}$ is called necessity operator
- (ii) $\diamond(A) = \{ \langle x, 1 - \nu_A(x), \nu_A(x) \rangle \mid x \in X \}$ is called possibility operator
- (iii) $D_\alpha(A) = \{ \langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1 - \alpha)\pi_A(x) \rangle \mid x \in X \}$ is called α - Model operator. Clearly $\square(A) \subseteq A \subseteq \diamond(A)$ and the equality hold, when A is a fuzzy set also $D_0(A) = \square(A)$ and $D_1(A) = \diamond(A)$. Therefore the α - Model operator $D_\alpha(A)$ is an extension of necessity operator $\square(A)$ and possibility operator $\diamond(A)$.

Definition 0.7 ([4]) For any IFS $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ of X and for any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, the (α, β) - model operator $F_{\alpha, \beta} : IFS(X) \rightarrow IFS(X)$ is defined as $F_{\alpha, \beta}(A) = \{ \langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x) \rangle \mid x \in X \}$, where $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$. Therefore we can write

$$F_{\alpha, \beta}(A) \text{ as } F_{\alpha, \beta}(A)(x) = (\mu_{F_{\alpha, \beta}(A)}(x), \nu_{F_{\alpha, \beta}(A)}(x))$$

where $\mu_{F_{\alpha, \beta}(A)}(x) = \mu_A(x) + \alpha\pi_A(x)$ and $\nu_{F_{\alpha, \beta}(A)}(x) = \nu_A(x) + \beta\pi_A(x)$.
Clearly, $F_{0, 1}(A) = \square(A), F_{1, 0}(A) = \diamond(A)$ and $F_{\alpha, 1-\alpha}(A) = D_\alpha(A)$

Definition 0.8 ([9]) An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of a BG-algebra X is said to be an intuitionistic fuzzy subalgebra of X if

- (i) $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$
- (ii) $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\} \quad \forall x, y \in X$.

Definition 0.9 ([7]) An IFS A of a BG-algebra X is said to be an IF normal subalgebra of X if

- (i) $\mu_A((x * a) * (y * b)) \geq \min\{\mu_A(x * y), \mu_A(a * b)\}$,
- (ii) $\nu_A((x * a) * (y * b)) \leq \max\{\nu_A(x * y), \nu_A(a * b)\}$, $\forall x, y \in X$.

Definition 0.10 ([11]) Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set of BG-algebra X . Let $t \in [0, 1]$. then the intuitionistic fuzzy set A^t of X is called t -intuitionistic fuzzy subset (t -IFS) of X w.r.t A and is defined by $A^t = \{ \langle x, \mu_{A^t}(x), \nu_{A^t}(x) \rangle \mid x \in X \} = \langle \mu_{A^t}, \nu_{A^t} \rangle$ where $\mu_{A^t}(x) = \min\{\mu_A(x), t\}$ and $\nu_{A^t}(x) = \max\{\nu_A(x), 1 - t\} \forall x \in X$

Remark 0.11 ([11]) Let $A^t = \langle \mu_{A^t}, \nu_{A^t} \rangle$ and $B^t = \langle \mu_{B^t}, \nu_{B^t} \rangle$ be two t -intuitionistic fuzzy subsets of BG-algebra X , then

$$(A \cap B)^t = A^t \cap B^t$$

Remark 0.12 ([11]) Let $f : X \rightarrow Y$ be a mapping. Let A and B are two IFS of X and Y respectively, then

$$(i) f^{-1}(B^t) = (f^{-1}(B))^t \quad (ii) f(A^t) = (f(A))^t \quad \forall t \in [0, 1]$$

Definition 0.13 Let $A^t = \langle \mu_{A^t}, \nu_{A^t} \rangle$ and $B^t = \langle \mu_{B^t}, \nu_{B^t} \rangle$ be two t -intuitionistic fuzzy subsets of BG-algebra X . Then their cartesian product $A^t \times B^t = \langle \mu_{A^t \times B^t}, \nu_{A^t \times B^t} \rangle$ is defined by

$$\begin{aligned} \mu_{A^t \times B^t}(x, y) &= \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \\ \nu_{A^t \times B^t}(x, y) &= \max\{\nu_{A^t}(x), \nu_{A^t}(y)\} \quad \forall x, y \in X. \end{aligned}$$

t-Intuitionistic Fuzzy Subalgebra BG-algebra

Now onwards, let X denote a BG-algebra unless otherwise stated.

Definition 0.14 Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set of BG-algebra X . Let $t \in [0, 1]$ then A is called t -intuitionistic fuzzy subalgebra (t -IFSA) of X if A^t is IFSA of X i.e. if A^t satisfies following conditions:

$$\begin{aligned} \mu_{A^t}(x * y) &\geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \\ \nu_{A^t}(x * y) &\leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\} \end{aligned}$$

Theorem 0.15 If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebra BG-algebra X , then A is also t -intuitionistic fuzzy subalgebra of X .

Proof. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebra BG-algebra X , therefore

$$\begin{aligned} \mu_A(x * y) &\geq \min\{\mu_A(x), \mu_A(y)\} \\ \nu_A(x * y) &\leq \max\{\nu_A(x), \nu_A(y)\}, \quad \forall x, y \in X. \end{aligned}$$

$$\begin{aligned} \text{Now, } \mu_{A^t}(x * y) &= \min\{\mu_A(x * y), t\} \\ &\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, t\} \\ &= \min\{\min(\mu_A(x), t), \min(\mu_A(y), t)\} \\ &= \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \\ \Rightarrow \mu_{A^t}(x * y) &\geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \end{aligned}$$

Similarly we can show

$$\nu_{A^t}(x * y) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\}$$

Hence A is also t -intuitionistic fuzzy subalgebra BG-algebra X .

Remark 0.16 *The converse of above Theorem is not true.*

Example 1. Consider a BG-algebra $X = \{0, 1, 2\}$ with the following cayley table:

Table 1: Example of intuitionistic fuzzy BG-subalgebra.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

The intuitionistic fuzzy subset $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ given by $\mu_A(0) = 0.4$, $\mu_A(1) = 0.5$, $\mu_A(2) = 0.3$ and $\nu_A(0) = 0.5$, $\nu_A(1) = 0.4$, $\nu_A(2) = 0.6$. Since $\mu_A(0) = 0.4 \not\geq \min\{\mu_A(1), \mu_A(2)\}$. Therefore A is not an intuitionistic fuzzy BG-subalgebra of X . Take $t = 0.2$. Then $\mu_A(x) > t$ for all $x \in X$ and also $\nu_A(x) < 1 - t$ for all $x \in X$. Therefore $\mu_{A^t}(x * y) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\}$ and $\nu_{A^t}(x * y) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\}$ for all $x \in X$. hold. Hence A is t-intuitionistic fuzzy subalgebra of X .

Theorem 0.17 *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set of BG-algebra X and let $t < \min\{p, 1 - q\}$, where $p = \min\{\mu_A(x) \mid x \in X\}$ and $q = \max\{\nu_A(x) \mid x \in X\}$ then A is also t-intuitionistic fuzzy subalgebra BG-algebra X .*

Proof. Since $t < \min\{p, 1 - q\}$

$$\begin{aligned}
 t &< \min\{p, 1 - q\} \\
 \Rightarrow p &> t \quad \text{and} \quad 1 - q > t \\
 \Rightarrow p &> t \quad \text{and} \quad q < 1 - t \\
 \Rightarrow \min\{\mu_A(x) \mid x \in X\} &> t \quad \text{and} \quad \max\{\nu_A(x) \mid x \in X\} < 1 - t \\
 \Rightarrow \mu_A(x) &> t, \forall x \in X \quad \text{and} \quad \nu_A(x) < 1 - t, \forall x \in X
 \end{aligned}$$

Therefore $\mu_{A^t}(x * y) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\}$ and $\nu_{A^t}(x * y) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\}$ for all $x \in X$ hold. Hence A is t-intuitionistic fuzzy subalgebra of X .

Theorem 0.18 *Any IF set of BG-algebra X can be realised as t-intuitionistic fuzzy subalgebra X .*

Proof. It follows from Theorem 0.17 and Theorem 0.15.

Theorem 0.19 *The intersection of two t-intuitionistic fuzzy subalgebra BG-algebra X is also a t-intuitionistic fuzzy subalgebra of X .*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}
\mu_{(A \cap B)^t}(x * y) &= \min\{\mu_{(A \cap B)}(x * y), t\} \\
&\geq \min\{\min\{\mu_A(x * y), \mu_A(x * y)\}, t\} \\
&= \min\{\min(\mu_A(x * y), t), \min(\mu_B(x * y), t)\} \\
&= \min\{\mu_{A^t}(x * y), \mu_{B^t}(x * y)\} \\
&\geq \min\{\min\{\mu_{A^t}(x), \mu_{A^t}(y)\}, \min\{\mu_{B^t}(x), \mu_{B^t}(y)\}\} \\
&= \min\{\min\{\mu_{A^t}(x), \mu_{B^t}(x)\}, \min\{\mu_{A^t}(y), \mu_{B^t}(y)\}\} \\
&= \min\{\mu_{(A \cap B)^t}(x), \mu_{(A \cap B)^t}(y)\} \\
\Rightarrow \mu_{(A \cap B)^t}(x * y) &\geq \min\{\mu_{(A \cap B)^t}(x), \mu_{(A \cap B)^t}(y)\}
\end{aligned}$$

Similarly we can show that

$$\nu_{(A \cap B)^t}(x * y) \leq \max\{\nu_{(A \cap B)^t}(x), \nu_{(A \cap B)^t}(y)\}$$

Theorem 0.20 *The intersection of any number of t -intuitionistic fuzzy subalgebra BG-algebra X is also a t -intuitionistic fuzzy subalgebra of X .*

Theorem 0.21 *For every t -intuitionistic fuzzy subalgebra A^t of X , the following properties hold*

- (i) $\mu_{A^t}(0) \geq \mu_{A^t}(x)$
- (ii) $\nu_{A^t}(0) \leq \nu_{A^t}(x), \forall x \in X$.

Proof. We have $\mu_{A^t}(0) = \mu_{A^t}(x * x) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(x)\} = \mu_{A^t}(x)$
and $\nu_{A^t}(0) = \nu_{A^t}(x * x) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(x)\} = \nu_{A^t}(x)$

Theorem 0.22 *If A be IF subalgebra of BG-algebra X , then $\square A, \diamond A$ and $F_{\alpha, \beta}(A)$ are also t -intuitionistic fuzzy subalgebra of X .*

Proof. Here A be IF subalgebra of BG-algebra X , By Theorem 0.15 A is also t -intuitionistic fuzzy subalgebra of X .

$$\mu_{A^t}(x * y) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \quad (1)$$

$$\nu_{A^t}(x * y) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\} \quad \forall x, y \in X. \quad (2)$$

Now $\square A^t = \{ \langle x, \mu_{A^t}(x), 1 - \mu_{A^t}(x) \mid x \in X \rangle = \{ \langle x, \mu_{A^t}(x), \overline{\mu_{A^t}}(x) \mid x \in X \rangle$
 $\diamond A^t = \{ \langle x, 1 - \nu_{A^t}(x), \mu_{A^t}(x) \mid x \in X \rangle = \{ \langle x, \overline{\nu_{A^t}}(x), \mu_{A^t}(x) \mid x \in X \rangle$

Now

$$\begin{aligned}
\overline{\mu_{A^t}}(x * y) &= 1 - \mu_{A^t}(x * y) \\
&\leq 1 - \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \quad \text{By(1)} \\
&= \max\{1 - \mu_{A^t}(x), 1 - \mu_{A^t}(y)\} \\
&= \max\{\overline{\mu_{A^t}}(x), \overline{\mu_{A^t}}(y)\}
\end{aligned}$$

$$\Rightarrow \overline{\mu_{A^t}}(x * y) \leq \max\{\overline{\mu_{A^t}}(x), \overline{\mu_{A^t}}(y)\} \quad (3)$$

Hence by Eqⁿ (1) and (3) $\Rightarrow \square A^t = \{ \langle x, \mu_{A^t}(x), \overline{\mu_{A^t}}(x) \mid x \in X \rangle$ is t -intuitionistic fuzzy subalgebra of X .

Similarly we can show that

$\diamond A^t = \{ \langle x, \overline{\nu_{A^t}}(x), \mu_{A^t}(x) \mid x \in X \}$ is t-intuitionistic fuzzy subalgebra of X.

Again, we have $F_{\alpha,\beta}(A) = \langle \mu_{F_{\alpha,\beta}(A)}, \nu_{F_{\alpha,\beta}(A)} \rangle$ let $x, y \in X$, then $F_{\alpha,\beta}(x * y) = (\mu_{F_{\alpha,\beta}(A)}(x * y), \nu_{F_{\alpha,\beta}(A)}(x * y))$ where $\mu_{F_{\alpha,\beta}(A)}(x * y) = \mu_A(x * y) + \alpha\pi_A(x * y)$ and $\nu_{F_{\alpha,\beta}(A)}(x * y) = \nu_A(x * y) + \beta\pi_A(x * y)$

$$\begin{aligned} & \mu_{F_{\alpha,\beta}A^t}(x * y) \\ &= \mu_{A^t}(x * y) + \alpha\pi_{A^t}(x * y) \\ &= \mu_{A^t}(x * y) + \alpha(1 - \mu_{A^t}(x * y) - \nu_{A^t}(x * y)) \\ &\geq \alpha + (1 - \alpha)\mu_{A^t}(x * y) - \alpha\nu_{A^t}(x * y) \\ &\geq \alpha + (1 - \alpha)\min(\mu_{A^t}(x), \mu_{A^t}(y)) - \alpha\max(\nu_{A^t}(x), \nu_{A^t}(y)) \quad \text{By(1)} \\ &\geq \alpha\{1 - \max(\nu_{A^t}(x), \nu_{A^t}(y))\} + (1 - \alpha)\min(\mu_{A^t}(x), \mu_{A^t}(y)) \\ &\geq \alpha\min(1 - \nu_{A^t}(x), 1 - \nu_{A^t}(y))\} + (1 - \alpha)\min(\mu_{A^t}(x), \mu_{A^t}(y)) \\ &\geq \min\{\alpha(1 - \nu_{A^t}(x)) + (1 - \alpha)\mu_{A^t}(x), \alpha(1 - \nu_{A^t}(y)) + (1 - \alpha)\mu_{A^t}(y)\} \\ &\geq \min\{\mu_{A^t}(x) + \alpha(1 - \mu_{A^t}(x) - \nu_{A^t}(x)), \mu_{A^t}(y) + \alpha(1 - \mu_{A^t}(y) - \nu_{A^t}(y))\} \\ &\geq \min\{\mu_{F_{\alpha,\beta}A^t}(x), \mu_{F_{\alpha,\beta}A^t}(y)\} \end{aligned}$$

$$\therefore \mu_{F_{\alpha,\beta}A^t}(x * y) \geq \min\{\mu_{F_{\alpha,\beta}A^t}(x), \mu_{F_{\alpha,\beta}A^t}(y)\}$$

Similarly we can prove that

$$\nu_{F_{\alpha,\beta}A^t}(x * y) \leq \max\{\nu_{F_{\alpha,\beta}A^t}(x), \nu_{F_{\alpha,\beta}A^t}(y)\}$$

Hence $F_{\alpha,\beta}(A)$ is t-intuitionistic fuzzy subalgebra of X.

Theorem 0.23 Cartesian product of two t-intuitionistic fuzzy subalgebra of X is again a t-intuitionistic fuzzy subalgebra of $X \times X$.

Proof. Let $A^t = \langle \mu_{A^t}, \nu_{A^t} \rangle$ and $B^t = \langle \mu_{B^t}, \nu_{B^t} \rangle$ be two t-intuitionistic fuzzy subalgebra of BG-algebra X

Then their cartesian product $A^t \times B^t = \langle \mu_{A^t \times B^t}, \nu_{A^t \times B^t} \rangle$, where

$$\begin{aligned} \mu_{A^t \times B^t}(x, y) &= \min\{\mu_{A^t}(x), \mu_{B^t}(y)\} \\ \nu_{A^t \times B^t}(x, y) &= \max\{\nu_{A^t}(x), \nu_{B^t}(y)\} \quad \forall x, y \in X. \end{aligned}$$

Also

$$\mu_{A^t}(x * y) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \tag{4}$$

$$\nu_{A^t}(x * y) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\} \quad \forall x, y \in X. \tag{5}$$

$$\begin{aligned} \mu_{A^t \times B^t}((x_1, y_1) * (x_2, y_2)) &= \mu_{A^t \times B^t}(x_1 * x_2, y_1 * y_2) \\ &= \min\{\mu_{A^t}(x_1 * x_2), \mu_{B^t}(y_1 * y_2)\} \\ &\geq \min\{\min\{\mu_{A^t}(x_1), \mu_{A^t}(x_2)\}, \min\{\mu_{B^t}(y_1), \mu_{B^t}(y_2)\}\} \\ &= \min\{\min\{\mu_{A^t}(x_1), \mu_{B^t}(y_1)\}, \min\{\mu_{A^t}(x_2), \mu_{B^t}(y_2)\}\} \\ &= \min\{\mu_{A^t \times B^t}((x_1, y_1), \mu_{A^t \times B^t}((x_2, y_2))\} \\ \Rightarrow \mu_{A^t \times B^t}((x_1, y_1) * (x_2, y_2)) &\geq \min\{\mu_{A^t \times B^t}((x_1, y_1), \mu_{A^t \times B^t}((x_2, y_2))\} \end{aligned}$$

Similarly we can show

$$\nu_{A^t \times B^t}((x_1, y_1) * (x_2, y_2)) \leq \max\{\nu_{A^t \times B^t}((x_1, y_1), \nu_{A^t \times B^t}((x_2, y_2))\}$$

Corollary 0.24 If $A^t = \langle \mu_{A^t}, \nu_{A^t} \rangle$ and $B^t = \langle \mu_{B^t}, \nu_{B^t} \rangle$ be two t -intuitionistic fuzzy subalgebra of BG-algebra X . Then $\square(A^t \times B^t)$, $\diamond(A^t \times B^t)$, $F_{\alpha, \beta}(A^t \times B^t)$ are also t -intuitionistic fuzzy subalgebra of $X \times X$.

Theorem 0.25 If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy normal subalgebra BG-algebra X , then A is also t -intuitionistic fuzzy normal subalgebra of X .

Proof. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy normal subalgebra BG-algebra X , therefore

$$\begin{aligned} (i) \mu_A((x * a) * (y * b)) &\geq \min\{\mu_A(x * y), \mu_A(a * b)\} \\ (ii) \nu_A((x * a) * (y * b)) &\leq \max\{\nu_A(x * y), \nu_A(a * b)\}, \forall x, y \in X. \end{aligned}$$

$$\begin{aligned} \text{Now, } \mu_{A^t}((x * a) * (y * b)) &= \min\{\mu_A((x * a) * (y * b)), t\} \\ &\geq \min\{\min\{\mu_A(x * y), \mu_A(a * b)\}, t\} \\ &= \min\{\min(\mu_A(x * y), t), \min(\mu_A(a * b), t)\} \\ &= \min\{\mu_{A^t}(x * y), \mu_{A^t}(a * b)\} \\ \Rightarrow \mu_{A^t}((x * a) * (y * b)) &\geq \min\{\mu_{A^t}(x * y), \mu_{A^t}(a * b)\} \end{aligned}$$

Similarly we can show that

$$\nu_{A^t}((x * a) * (y * b)) \leq \max\{\nu_{A^t}(x * y), \nu_{A^t}(a * b)\}$$

Hence A is also t -intuitionistic fuzzy normal subalgebra BG-algebra X .

Remark 0.26 The converse of above Theorem is not true.

Theorem 0.27 If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set of BG-algebra X and let $t < \min\{p, 1 - q\}$, where $p = \min\{\mu_A(x) | x \in X\}$ and $q = \max\{\nu_A(x) | x \in X\}$ then A is also t -intuitionistic fuzzy normal subalgebra BG-algebra X .

Proof. Same as Theorem 0.17.

Theorem 0.28 The intersection of two t -intuitionistic fuzzy normal subalgebra BG-algebra X is also a t -intuitionistic fuzzy normal subalgebra of X .

Proof. Same as Theorem 0.19.

Theorem 0.29 If A be IF normal subalgebra of BG-algebra X , then $\square A$, $\diamond A$ and $F_{\alpha, \beta}(A)$ are also t -intuitionistic fuzzy normal subalgebra of X .

Proof. Same as Theorem 0.22.

Theorem 0.30 Cartesian product of two t -intuitionistic fuzzy normal subalgebra of X is again a t -intuitionistic fuzzy normal subalgebra of $X \times X$.

Proof. Same as Theorem 0.23.

Corollary 0.31 If $A^t = \langle \mu_{A^t}, \nu_{A^t} \rangle$ and $B^t = \langle \mu_{B^t}, \nu_{B^t} \rangle$ be two t -intuitionistic fuzzy normal subalgebra of BG-algebra X . Then $\square(A^t \times B^t)$, $\diamond(A^t \times B^t)$, $F_{\alpha, \beta}(A^t \times B^t)$ are also t -intuitionistic fuzzy normal subalgebra of $X \times X$.

Proof. Same as Corollary 0.24.

Homomorphism of t-intuitionistic fuzzy subalgebra BG-algebra

Definition 0.32 Let X and Y be two BG-algebras, then a mapping $f : X \rightarrow Y$ is said to be homomorphism if $f(x * y) = f(x) * f(y), \forall x, y \in X$.

Theorem 0.33 Let $f : X \rightarrow Y$ be a homomorphism of BG-algebras, If A be a t-intuitionistic fuzzy subalgebra of Y , then $f^{-1}(A)$ is t-intuitionistic fuzzy subalgebra X .

Proof. A be a t-intuitionistic fuzzy subalgebra of Y . Let $x, y \in X$ be any elements, then $f^{-1}(A^t)(x * y) = (\mu_{f^{-1}(A^t)}(x * y), \nu_{f^{-1}(A^t)}(x * y))$

$$\begin{aligned} &\text{Now, } \mu_{f^{-1}(A^t)}(x * y) \\ &= \mu_{A^t} f(x * y) \\ &= \mu_{A^t} [f(x) * f(y)] \\ &\geq \min\{\mu_{A^t}(f(x)), \mu_{A^t}(f(y))\} \quad [\text{Since } A \text{ is t-IF subalgebra of } Y] \\ &= \min\{\mu_{f^{-1}A^t}(x), \mu_{f^{-1}A^t}(y)\} \end{aligned}$$

Therefore $\mu_{f^{-1}A^t}(x * y) \geq \min\{\mu_{f^{-1}A^t}(x), \mu_{f^{-1}A^t}(y)\}$

Similarly we can show that

$$\nu_{f^{-1}(A^t)}(x * y) \leq \max\{\nu_{f^{-1}A^t}(x), \nu_{f^{-1}A^t}(y)\}$$

Hence, $f^{-1}(A^t) = (f^{-1}(A))^t$ is t-intuitionistic fuzzy subalgebra X .

Theorem 0.34 Let $f : X \rightarrow Y$ be a homomorphism of BG-algebras, If A be a t-intuitionistic fuzzy normal subalgebra of Y , then $f^{-1}(A)$ is t-intuitionistic fuzzy normal subalgebra X .

Theorem 0.35 Let $f : X \rightarrow Y$ be a onto homomorphism of BG-algebras, If A be t-intuitionistic fuzzy subalgebra X . Then $f(A)$ is t-intuitionistic fuzzy subalgebra of Y .

Proof. Let $y_1, y_2 \in Y$ Since f is onto, therefore there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$,

$$\begin{aligned} &f(A)(y_1 * y_2) = (\mu_{f(A)}(y_1 * y_2), \nu_{f(A)}(y_1 * y_2)), \text{ Now} \\ &\mu_{f(A)}(y_1 * y_2) = \mu_A(t) \text{ where } f(t) = y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2) \end{aligned}$$

$$\begin{aligned} &\mu_{f(A^t)}(y_1 * y_2) \\ &= \mu_{(f(A))^t}(y_1 * y_2) \\ &= \min\{\mu_{f(A)}(y_1 * y_2), t\} \\ &= \min\{\mu_{f(A)}(f(x_1) * f(x_2)), t\} \\ &= \min\{\mu_{f(A)}f(x_1 * x_2), t\} \\ &= \min\{\mu_A(x_1 * x_2), t\} \\ &= \mu_{A^t}(x_1 * x_2) \\ &\geq \min\{\mu_{A^t}(x_1), \mu_{A^t}(x_2)\}, \text{ for all } x_1, x_2 \in X \text{ such that } f(x_1) = y_1 \text{ and } f(x_2) = y_2 \\ &= \min\{\bigvee\{\mu_{A^t}(x_1) | f(x_1) = y_1\}, \bigvee\{\mu_{A^t}(x_2) | f(x_2) = y_2\}\} \\ &= \min\{\mu_{f(A^t)}(y_1), \mu_{f(A^t)}(y_2)\} \end{aligned}$$

Therefore $\mu_{f(A^t)}(y_1 * y_2) \geq \min\{\mu_{f(A^t)}(y_1), \mu_{f(A^t)}(y_2)\}$

Similarly we can show that

$$\nu_{f(A^t)}(y_1 * y_2) \leq \max\{\nu_{f(A^t)}(y_1), \nu_{f(A^t)}(y_2)\}$$

Hence $f(A)$ is t-intuitionistic fuzzy subalgebra of Y .

Theorem 0.36 *Let $f : X \longrightarrow Y$ be a onto homomorphism of BG-algebras, If A be t -intuitionistic fuzzy normal subalgebra X , then $f(A)$ is t -intuitionistic fuzzy normal subalgebra of Y .*

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