t- Intuitionistic Fuzzy Subalgebra of $BG$-Algebras

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Abstract. The aim of this paper is to introduced the notion of t-intuitionistic fuzzy subalgebra and t-intuitionistic fuzzy normal subalgebra of BG-algebras. We state and prove some theorems in t-intuitionistic fuzzy subalgebra and t-intuitionistic fuzzy normal subalgebra in BG-algebras. The homomorphic image and inverse image are investigated in both t-intuitionistic fuzzy subalgebra and normal subalgebras.

Introduction

In 1966, Imai and Iseki [6] introduced the two classes of abstract algebras, viz., $BCK$-algebras and $BCI$-algebras. It is known that the class of $BCK$-algebra is a proper subclass of the class of $BCI$-algebras. Neggers and Kim [8] introduced a new concept, called $B$-algebras, which are related to several classes of algebras such as $BCI/BCK$-algebras. Kim and Kim [7] introduced the notion of $BG$-algebra which is a generalization of $B$-algebra. The concept of intuitionistic fuzzy subset (IFS) was introduced by Atanassov [5] in 1983, which is a generalization of the notion of fuzzy sets. The concept of fuzzy subalgebras of BG-algebras was introduced by Ahn and Lee in [1]. The study of intuitionistic fuzzification of subalgebras and ideals of BG-algebras is done by Senapati et. al in [9]. The idea of t-intuitionistic fuzzy sets in fuzzy subgroups and fuzzy subrings is introduced by Sharma in [10, 11]. Here in this paper, we introduced the notion of t-intuitionistic fuzzy sets in fuzzy subalgebra and fuzzy normal subalgebras of BG-algebras and study their properties.

Preliminaries

Definition 0.1 ([1]) A $BG$-algebra is a non-empty set $X$ with a constant $0'$ and a binary operation $'*'$ satisfying the following axioms:

(i) $x * x = 0$,

(ii) $x * 0 = x$,

(iii) $(x * y) * (0 * y) = x$, $\forall x, y \in X$.

For brevity, we also call X a $BG$-algebra. We can define a partial ordering $'\leq'$ on $X$ by $x \leq y$ iff $x * y = 0$

Definition 0.2 ([1]) A non-empty subset $S$ of a $BG$-algebra $X$ is called a subalgebra of $X$ if $x * y \in S$, for all $x, y \in S$. 
Definition 0.3 Let $X$ and $Y$ be two non empty sets and $f : X \rightarrow Y$ be a mapping. Let $A$ and $B$ be IFS’s of $X$ and $Y$ respectively. Then the image of $A$ under the map $f$ is denoted by $f(A)$ and is defined by $f(A)(y) = (\mu_A(y), \nu_A(y))$, where $\mu_{f(A)}(y) = \begin{cases} \bigwedge \{\mu_A(x) : x \in f^{-1}(y)\} & \text{if } 0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X \\ 1 & \text{otherwise} \end{cases}$

Also, pre image of $B$ under $f$ is denoted by $f^{-1}(B)$ and is defined as $f^{-1}(B)(x) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)}) = (\mu_B(f(x)), \nu_B(f(x))) \forall x \in X$

Remark Note that

Definition 0.4 ([2, 3]) An intuitionistic fuzzy set (IFS) $A$ in a non empty set $X$ is an object of the form $A = \{< x, \mu_A(x), \nu_A(x) > \mid x \in X \}$ where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X.$ The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non membership of the element $x$ in the set $A$. For the sake of simplicity we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{< x, \mu_A(x), \nu_A(x) > \mid x \in X \}.$ The function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X.$ is called the degree of uncertainty of $x \in A.$ The class of IFSs on a universe $X$ is denoted by IFS($X$).

Definition 0.5 ([2, 3]) If $A = \{< x, \mu_A(x), \nu_A(x) > \mid x \in X \}$ and $B = \{< x, \mu_B(x), \nu_B(x) > \mid x \in X \}$ be any two IFS of a set $X$ then

- $A \subseteq B$ iff for all $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$
- $A = B$ iff for all $x \in X, \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$
- $A \cap B = \{< x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x) > \mid x \in X \}$ where $(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$
- $A \cup B = \{< x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x) > \mid x \in X \}$ where $(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cap \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\}$

Definition 0.6 ([4]) For any IFS $A = \{< x, \mu_A(x), \nu_A(x) > \mid x \in X \}$ of $X$ and $\alpha \in [0, 1]$, the operator $\square : IFS(X) \rightarrow IFS(X), \Diamond : IFS(X) \rightarrow IFS(X), D_\alpha : IFS(X) \rightarrow IFS(X)$ are defined as

- (i) $\square(A) = \{< x, \mu_A(x), 1 - \mu_A(x) > \mid x \in X \}$ is called necessity operator
- (ii) $\Diamond(A) = \{< x, 1 - \nu_A(x), \nu_A(x) > \mid x \in X \}$ is called possibility operator
- (iii) $D_\alpha(A) = \{< x, \mu_A(x) + \alpha \pi_A(x), \nu_A(x) + (1 - \alpha) \pi_A(x) > \mid x \in X \}$ is called $\alpha -$ Model operator.

Clearly $\square(A) \subseteq A \subseteq \Diamond(A)$ and the equality hold, when $A$ is a fuzzy set also $D_0(A) = \square(A)$ and $D_1(A) = \Diamond(A)$. Therefore the $\alpha -$ Model operator $D_\alpha(A)$ is an extension of necessity operator $\square(A)$ and possibility operator $\Diamond(A)$.

Definition 0.7 ([4]) For any IFS $A = \{< x, \mu_A(x), \nu_A(x) > \mid x \in X \}$ of $X$ and for any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1,$ the $(\alpha, \beta)$- model operator $F_{\alpha,\beta} : IFS(X) \rightarrow IFS(X)$ is defined as $F_{\alpha,\beta}(A) = \{< x, \mu_A(x) + \alpha \pi_A(x), \nu_A(x) + \beta \pi_A(x) > \mid x \in X \}$, where $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X.$ Therefore we can write $F_{\alpha,\beta}(A)$ as $F_{\alpha,\beta}(A)(x) = (\mu_{F_{\alpha,\beta}(A)}(x), \nu_{F_{\alpha,\beta}(A)}(x))$

where $\mu_{F_{\alpha,\beta}}(x) = \mu_A(x) + \alpha \pi_A(x)$ and $\nu_{F_{\alpha,\beta}}(A)(x) = \nu_A(x) + \beta \pi_A(x)$.

Clearly, $F_{0,1}(A) = \square(A), F_{1,0}(A) = \Diamond(A)$ and $F_{1,1-\alpha}(A) = D_\alpha(A)$.

Definition 0.8 ([9]) An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of a BG-algebra $X$ is said to be an intuitionistic fuzzy subalgebra of $X$ if

- (i) $\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}$
- (ii) $\nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}$ \quad \forall x, y \in X.
Definition 0.9 ([7]) An IFS $A$ of a BG-algebra $X$ is said to be an IF normal subalgebra of $X$ if
(i) $\mu_A((x \ast a) \ast (y \ast b)) \geq \min\{\mu_A(x \ast y), \mu_A(a \ast b)\}$,
(ii) $\nu_A((x \ast a) \ast (y \ast b)) \leq \max\{\nu_A(x \ast y), \nu_A(a \ast b)\}, \forall x, y \in X.$

Definition 0.10 ([11]) Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set of BG-algebra $X$. Let $t \in [0, 1]$, then the intuitionistic fuzzy set $A^t$ of $X$ is called t-intuitionistic fuzzy subset(t-IFS) of $X$ w.r.t $A$ and is defined by $A^t = \{< x, \mu_A^t(x), \nu_A^t(x) > | x \in X \} = < \mu_A^t, \nu_A^t >$ where $\mu_A^t(x) = \min\{\mu_A(x), t\}$ and $\nu_A^t = \max\{\nu_A(x), 1 - t\}, \forall x \in X$.

Remark 0.11 ([11]) Let $A^t = < \mu_A^t, \nu_A^t >$ and $B^t = < \mu_B^t, \nu_B^t >$ be two t-intuitionistic fuzzy subsets of BG-algebra $X$, then $(A \cap B)^t = A^t \cap B^t$.

Remark 0.12 ([11]) Let $f : X \rightarrow Y$ be a mapping. Let $A$ and $B$ are two IFS of $X$ and $Y$ respectively, then
(i) $f^{-1}(B^t) = (f^{-1}(B))^t$, (ii) $f(A^t) = (f(A))^t$, $\forall t \in [0, 1]$.

Definition 0.13 Let $A^t = < \mu_A^t, \nu_A^t >$ and $B^t = < \mu_B^t, \nu_B^t >$ be two t-intuitionistic fuzzy subsets of BG-algebra $X$. Then their cartesian product $A^t \times B^t = < \mu_A^t \times B^t, \nu_A^t \times B^t >$ is defined by
$\mu_A^t \times B^t(x, y) = \min\{\mu_A(x), \mu_B(y)\}$
$\nu_A^t \times B^t(x, y) = \max\{\nu_A(x), \nu_B(y)\}, \forall x, y \in X.$

**t-Intuitionistic Fuzzy Subalgebra BG-algebra**

Now onwards, let $X$ denote a BG-algebra unless otherwise stated.

Definition 0.14 Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set of BG-algebra $X$. Let $t \in [0, 1]$ then $A$ is called t-intuitionistic fuzzy subalgebra (t-IFSA) of $X$ if $A^t$ is IFSA of $X$ i.e. if $A^t$ satisfies following conditions:
\[
\mu_A^t(x \ast y) \geq \min\{\mu_A^t(x), \mu_A^t(y)\}
\]
\[
\nu_A^t(x \ast y) \leq \max\{\nu_A^t(x), \nu_A^t(y)\}
\]

**Theorem 0.15** If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebra BG-algebra $X$, then $A$ is also t-intuitionistic fuzzy subalgebra of $X$.

**Proof.** Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subalgebra BG-algebra $X$, therefore
\[
\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}
\]
\[
\nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}, \forall x, y \in X.
\]

Now, $\mu_A^t(x \ast y) = \min\{\mu_A(x \ast y), t\}$
\[
\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, t\}
\]
\[
= \min\{\min(\mu_A(x), t), \min(\mu_A(y), t)\}
\]
\[
= \min\{\mu_A(x), \mu_A(y)\}
\]

$\Rightarrow \mu_A^t(x \ast y) \geq \min\{\mu_A^t(x), \mu_A^t(y)\}$

Similarly we can show
\[
\nu_A^t(x \ast y) \leq \max\{\nu_A^t(x), \nu_A^t(y)\}
\]

Hence $A$ is also t-intuitionistic fuzzy subalgebra BG-algebra $X$. 
Remark 0.16 The converse of above Theorem is not true.

Example 1. Consider a BG-algebra $X = \{0, 1, 2\}$ with the following Cayley table:

Table 1: Example of intuitionistic fuzzy BG-subalgebra.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

The intuitionistic fuzzy subset $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X \}$ given by $\mu_A(0) = 0.4$, $\mu_A(1) = 0.5, \mu_A(2) = 0.3$ and $\nu_A(0) = 0.5, \nu_A(1) = 0.4, \nu_A(2) = 0.6$. Since $\mu_A(0) = 0.4 \not\leq \min\{\mu_A(1), \mu_A(1)\}$. Therefore $A$ is not an intuitionistic fuzzy BG-subalgebra of $X$.

Take $t = 0.2$. Then $\mu_A(x) > t$ for all $x \in X$ and also $\nu_A(x) < 1 - t$ for all $x \in X$.

Therefore $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}$ for all $x \in X$.

Hence $A$ is $t$-intuitionistic fuzzy subalgebra of $X$.

Theorem 0.17 If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set of BG-algebra $X$ and let $t < \min\{p, 1-q\}$, where $p = \min\{\mu_A(x)|x \in X\}$ and $q = \max\{\nu_A(x)|x \in X\}$ then $A$ is also $t$-intuitionistic fuzzy subalgebra BG-algebra $X$.

Proof. Since $t < \min\{p, 1-q\}$

\[
\begin{align*}
t &< \min\{p, 1-q\} \\
\Rightarrow &p > t \quad \text{and} \quad 1 - q > t \\
\Rightarrow &p > t \quad \text{and} \quad q < 1 - t \\
\Rightarrow &\min\{\mu_A(x)|x \in X\} > t \quad \text{and} \quad \max\{\nu_A(x)|x \in X\} < 1 - t \\
\Rightarrow &\mu_A(x) > t, \forall \ x \in X \quad \text{and} \quad \nu_A(x) < 1 - t, \forall \ x \in X
\end{align*}
\]

Therefore $\mu_A'(x * y) \geq \min\{\mu_A'(x), \mu_A'(y)\}$ and $\nu_A'(x * y) \leq \max\{\nu_A'(x), \nu_A'(y)\}$ for all $x \in X$ hold. Hence $A$ is $t$-intuitionistic fuzzy subalgebra of $X$.

Theorem 0.18 Any IF set of BG-algebra $X$ can be realised as $t$-intuitionistic fuzzy subalgebra $X$.

Proof. It follows from Theorem 0.17 and Theorem 0.15.

Theorem 0.19 The intersection of two $t$-intuitionistic fuzzy subalgebra BG-algebra $X$ is also a $t$-intuitionistic fuzzy subalgebra of $X$. 

Proof. Let \( x, y \in X \). Then
\[
\mu_{(A \cap B)^t}(x * y) = \min\{\mu_{(A \cap B)}(x * y), t\}
\]
\[
\geq \min\{\min\{\mu_A(x * y), \mu_A(x * y)\}, t\}
\]
\[
= \min\{\min\{\mu_A(x * y), t\}, \min\{\nu_B(x * y), t\}\}
\]
\[
= \min\{\mu_A(x * y), \mu_B^t(x * y)\}
\]
\[
\geq \min\{\min\{\mu_A(x), \mu_A^t(y)\}, \min\{\mu_B(x), \mu_B^t(y)\}\}
\]
\[
= \min\{\mu_A(x), \mu_B^t(x)\}
\]
\[
= \min\{\mu_{(A \cap B)^t}(x), \mu_{(A \cap B)^t}(y)\}
\]

\[
\Rightarrow \mu_{(A \cap B)^t}(x * y) \geq \min\{\mu_{(A \cap B)^t}(x), \mu_{(A \cap B)^t}(y)\}
\]

Similarly we can show that
\[
\nu_{(A \cap B)^t}(x * y) \leq \max\{\nu_{(A \cap B)^t}(x), \nu_{(A \cap B)^t}(y)\}
\]

**Theorem 0.20** The intersection of any number of t-intuitionistic fuzzy subalgebra BG-algebra \( X \) is also a t-intuitionistic fuzzy subalgebra of \( X \).

**Theorem 0.21** For every t-intuitionistic fuzzy subalgebra \( A^t \) of \( X \), the following properties hold

(i) \( \mu_{A^t}(0) \geq \mu_{A^t}(x) \)
(ii) \( \nu_{A^t}(0) \leq \nu_{A^t}(x) , \forall x \in X \).

\[
\mu_{A^t}(0) = \mu_{A^t}(x * x) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(x)\} = \mu_{A^t}(x)
\]

\[
\nu_{A^t}(0) = \nu_{A^t}(x * x) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(x)\} = \nu_{A^t}(x)
\]

**Theorem 0.22** If \( A \) be IF subalgebra of BG-algebra \( X \), then \( \Box A, \Diamond A \) and \( F_{\alpha, \beta}(A) \) are also t-intuitionistic fuzzy subalgebra of \( X \).

Proof. Here \( A \) be IF subalgebra of BG-algebra \( X \), By Theorem 0.15 \( A \) is also t-intuitionistic fuzzy subalgebra of \( X \).

\[
\mu_{A^t}(x * y) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\}
\]

\[
\nu_{A^t}(x * y) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\}
\]

Now \( \Box A^t = \{x, \mu_{A^t}(x), 1 - \mu_{A^t}(x) | x \in X \} = \{x, \mu_{A^t}(x), \mu_{A^t}(x) | x \in X \} \)

\( \Diamond A^t = \{x, 1 - \nu_{A^t}(x), \mu_{A^t}(x) | x \in X \} = \{x, \nu_{A^t}(x), \mu_{A^t}(x) | x \in X \} \)

Now
\[
\mu_{A^t}(x * y) = 1 - \mu_{A^t}(x * y)
\]
\[
\leq 1 - \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \quad \text{By (1)}
\]
\[
= \max\{1 - \mu_{A^t}(x), 1 - \mu_{A^t}(y)\}
\]
\[
= \max\{\mu_{A^t}(x), \mu_{A^t}(y)\}
\]

\[
\Rightarrow \mu_{A^t}(x * y) \leq \max\{\mu_{A^t}(x), \mu_{A^t}(y)\}
\]

Hence by Eq (1) and (3) \( \Box A^t = \{x, \mu_{A^t}(x), \mu_{A^t}(x) | x \in X \} \) is t-intuitionistic fuzzy subalgebra of \( X \).

Similarly we can show that
$\Diamond A^t = \{ x, \overline{\mu^{A^t}(x)}, \mu^{A^t}(x) | x \in X \}$ is t-intuitionistic fuzzy subalgebra of X.

Again, we have $F_{\alpha,\beta}(A) = \langle \mu_{F_{\alpha,\beta}(A)}, \nu_{F_{\alpha,\beta}(A)} \rangle$ let $x, y \in X$, then $F_{\alpha,\beta}(x \ast y) = (\mu_{F_{\alpha,\beta}(A)}(x \ast y), \nu_{F_{\alpha,\beta}(A)}(x \ast y))$ where $\mu_{F_{\alpha,\beta}(A)}(x \ast y) = \mu_{\alpha}(x \ast y) + \alpha \pi_{\alpha}(x \ast y)$ and $\nu_{F_{\alpha,\beta}(A)}(x \ast y) = \nu_{\alpha}(x \ast y) + \beta \pi_{\alpha}(x \ast y)$

\[ \mu_{F_{\alpha,\beta}A^t}(x \ast y) = \mu_{A^t}(x \ast y) + \alpha \pi_{A^t}(x \ast y) \]
\[ = \mu_{A^t}(x \ast y) + \alpha (1 - \mu_{A^t}(x \ast y) - \nu_{A^t}(x \ast y)) \]
\[ \geq \alpha + (1 - \alpha) \mu_{A^t}(x \ast y) - \alpha \nu_{A^t}(x \ast y) \]
\[ \geq \alpha (1 - \max(\nu_{A^t}(x), \nu_{A^t}(y))) + (1 - \alpha) \min(\mu_{A^t}(x), \mu_{A^t}(y)) \]
\[ \geq \min\{\alpha(1-\nu_{A^t}(x)) + (1-\alpha)\mu_{A^t}(x), \alpha(1-\nu_{A^t}(y)) + (1-\alpha)\mu_{A^t}(y)\} \]
\[ \geq \min\{\mu_{A^t}(x) + \alpha(1-\mu_{A^t}(x) - \nu_{A^t}(x)), \mu_{A^t}(y) + \alpha(1-\mu_{A^t}(y) - \nu_{A^t}(y))\} \]
\[ \geq \min\{\mu_{F_{\alpha,\beta}A^t}(x), \mu_{F_{\alpha,\beta}A^t}(y)\} \]

\[ \therefore \nu_{F_{\alpha,\beta}A^t}(x \ast y) \geq \min\{\mu_{F_{\alpha,\beta}A^t}(x), \mu_{F_{\alpha,\beta}A^t}(y)\} \]

Similarly we can prove that $\nu_{F_{\alpha,\beta}A^t}(x \ast y) \leq \max\{\nu_{F_{\alpha,\beta}A^t}(x), \nu_{F_{\alpha,\beta}A^t}(y)\}$

Hence $F_{\alpha,\beta}(A)$ is t-intuitionistic fuzzy subalgebra of X.

**Theorem 0.23** Cartesian product of two t-intuitionistic fuzzy subalgebra of X is again a t-intuitionistic fuzzy subalgebra of $X \times X$.

**Proof.** Let $A^t = \langle \mu_{A^t}, \nu_{A^t} \rangle$ and $B^t = \langle \mu_{B^t}, \nu_{B^t} \rangle$ be two t-intuitionistic fuzzy subalgebra of BG-algebra X

Then their cartesian product $A^t \times B^t = \langle \mu_{A^t \times B^t}, \nu_{A^t \times B^t} \rangle$, where
\[ \mu_{A^t \times B^t}(x, y) = \min\{\mu_{A^t}(x), \mu_{B^t}(y)\} \]
\[ \nu_{A^t \times B^t}(x, y) = \max\{\nu_{A^t}(x), \nu_{B^t}(y)\} \quad \forall x, y \in X. \]

Also
\[ \mu_{A^t}(x \ast y) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \quad \forall x, y \in X. \tag{4} \]
\[ \nu_{A^t}(x \ast y) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\} \quad \forall x, y \in X. \tag{5} \]

\[ \mu_{A^t \times B^t}((x_1, y_1) \ast (x_2, y_2)) = \mu_{A^t \times B^t}(x_1 \ast x_2, y_1 \ast y_2) \]
\[ = \min\{\mu_{A^t}(x_1 \ast x_2), \mu_{B^t}(y_1 \ast y_2)\} \]
\[ \geq \min\{\min\{\mu_{A^t}(x_1), \mu_{A^t}(x_2)\}, \min\{\mu_{B^t}(y_1), \mu_{B^t}(y_2)\}\} \]
\[ = \min\{\min\{\mu_{A^t}(x_1), \mu_{B^t}(y_1)\}, \min\{\mu_{A^t}(x_2), \mu_{B^t}(y_2)\}\} \]
\[ = \min\{\mu_{A^t \times B^t}((x_1, y_1), 0), \mu_{A^t \times B^t}((x_2, y_2))\} \]
\[ \Rightarrow \mu_{A^t \times B^t}((x_1, y_1) \ast (x_2, y_2)) \geq \min\{\mu_{A^t \times B^t}((x_1, y_1), \mu_{A^t \times B^t}((x_2, y_2))\} \]

Similarly we can show
\[ \nu_{A^t \times B^t}((x_1, y_1) \ast (x_2, y_2)) \leq \max\{\nu_{A^t \times B^t}((x_1, y_1), \nu_{A^t \times B^t}((x_2, y_2))\} \]
Corollary 0.24 If $A^t = \langle \mu_A, \nu_A \rangle$ and $B^t = \langle \mu_B, \nu_B \rangle$ be two $t$-intuitionistic fuzzy subalgebra of $BG$-algebra $X$. Then $\square(A^t \times B^t)$ $\Diamond(A^t \times B^t)$, $F_{\alpha, \beta}(A^t \times B^t)$ are also $t$-intuitionistic fuzzy subalgebra of $X \times X$.

Theorem 0.25 If $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy normal subalgebra $BG$-algebra $X$, then $A$ is also $t$-intuitionistic fuzzy normal subalgebra of $X$.

Proof. Since $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy normal subalgebra $BG$-algebra $X$, therefore

\[(i) \mu_A((x * a) \ast (y * b)) \geq \min\{\mu_A(x \ast y), \mu_A(a \ast b)\}\]
\[(ii) \nu_A((x * a) \ast (y * b)) \leq \max\{\nu_A(x \ast y), \nu_A(a \ast b)\}, \forall x, y \in X.\]

Now, $\mu_{A^t}((x * a) \ast (y * b)) = \min\{\mu_A((x * a) \ast (y * b)), t\}$
$\geq \min\{\min\{\mu_A(x \ast y), \mu_A(a \ast b)\}, t\}$
$= \min\{\min\{\mu_A(x \ast y), t\}, \min\{\mu_A(a \ast b), t\}\}$
$= \min\{\mu_{A^t}(x \ast y), \mu_{A^t}(a \ast b)\}$
$\Rightarrow \mu_A((x * a) \ast (y * b)) \geq \min\{\mu_A(x \ast y), \mu_A(a \ast b)\}$

Similarily we can show that $\nu_A((x * a) \ast (y * b)) \leq \max\{\nu_A(x \ast y), \nu_A(a \ast b)\}$

Hence $A$ is also $t$-intuitionistic fuzzy normal subalgebra $BG$-algebra $X$.

Remark 0.26 The converse of above Theorem is not true.

Theorem 0.27 If $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy set of $BG$-algebra $X$ and let $t < \min\{p, 1-q\}$, where $p = \min\{\mu_A(x) | x \in X\}$ and $q = \max\{\nu_A(x) | x \in X\}$ then $A$ is also $t$-intuitionistic fuzzy normal subalgebra $BG$-algebra $X$.

Proof. Same as Theorem 0.17.

Theorem 0.28 The intersection of two $t$-intuitionistic fuzzy normal subalgebra $BG$-algebra $X$ is also a $t$-intuitionistic fuzzy normal subalgebra of $X$.

Proof. Same as Theorem 0.19.

Theorem 0.29 If $A$ be IF normal subalgebra of $BG$-algebra $X$,then $\square A, \Diamond A$ and $F_{\alpha, \beta}(A)$are also $t$-intuitionistic fuzzy normal subalgebra of $X$.

Proof. Same as Theorem 0.22.

Theorem 0.30 Cartesian product of two $t$-intuitionistic fuzzy normal subalgebra of $X$ is again a $t$-intuitionistic fuzzy normal subalgebra of $X \times X$.

Proof. Same as Theorem 0.23.

Corollary 0.31 If $A^t = \langle \mu_A, \nu_A \rangle$ and $B^t = \langle \mu_B, \nu_B \rangle$ be two $t$-intuitionistic fuzzy normal subalgebra of $BG$-algebra $X$. Then $\square(A^t \times B^t)$, $\Diamond(A^t \times B^t)$, $F_{\alpha, \beta}(A^t \times B^t)$ are also $t$-intuitionistic fuzzy normal subalgebra of $X \times X$.

Proof. Same as Corollary 0.24.
Homomorphism of t-intuitionistic fuzzy subalgebra $BG$-algebra

**Definition 0.32** Let $X$ and $Y$ be two $BG$-algebras, then a mapping $f : X \rightarrow Y$ is said to be homomorphism if $f(x * y) = f(x) * f(y)$, $\forall x, y \in X$.

**Theorem 0.33** Let $f : X \rightarrow Y$ be a homomorphism of $BG$-algebras. If $A$ be a t-intuitionistic fuzzy subalgebra of $Y$, then $f^{-1}(A)$ is t-intuitionistic fuzzy subalgebra $X$.

**Proof.** Let $A$ be a t-intuitionistic fuzzy subalgebra of $Y$. Let $x, y \in X$ be any elements, then $f^{-1}(A^t)(x * y) = (\mu_{f^{-1}(A^t)}(x * y), \nu_{f^{-1}(A^t)}(x * y))$

Now, $\mu_{f^{-1}(A)}(x * y)$
$= \mu_{A^t}f(x * y)$
$= \mu_{A^t}[f(x) * f(y)]$
$\geq \min\{\mu_{A^t}(f(x)), \mu_{A^t}(f(y))\}$ [Since $A$ is t-IF subalgebra of $Y$]
$= \min\{\mu_{f^{-1}A^t}(x), \mu_{f^{-1}A^t}(y)\}$

Therefore
$\mu_{f^{-1}A^t}(x * y) \geq \min\{\mu_{f^{-1}A^t}(x), \mu_{f^{-1}A^t}(y)\}$

Similarly we can show that
$\nu_{f^{-1}A^t}(x * y) \leq \max\{\nu_{f^{-1}A^t}(x), \nu_{f^{-1}A^t}(y)\}$

Hence, $f^{-1}(A^t) = (f^{-1}(A))^t$ is t-intuitionistic fuzzy subalgebra $X$.

**Theorem 0.34** Let $f : X \rightarrow Y$ be a homomorphism of $BG$-algebras, If $A$ be a t-intuitionistic fuzzy normal subalgebra of $Y$, then $f^{-1}(A)$ is t-intuitionistic fuzzy normal subalgebra $X$.

**Theorem 0.35** Let $f : X \rightarrow Y$ be a onto homomorphism of $BG$-algebras, If $A$ be t-intuitionistic fuzzy subalgebra $X$. Then $f(A)$ is t-intuitionistic fuzzy subalgebra of $Y$.

**Proof.** Let $y_1, y_2 \in Y$. Since $f$ is onto, therefore there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$.

Now
$f(A)(y_1 * y_2) = (\mu_f(A)(y_1 * y_2), \nu_f(A)(y_1 * y_2))$,

$\mu_f(A)(y_1 * y_2) = \mu_A(t)$ where $f(t) = y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2)$

$\geq \min\{\mu_A(x_1), \mu_A(x_2)\}$, for all $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_1) = y_1$

$= \min\{\nu_f(A)(y_1), \nu_f(A)(y_2)\}$

Therefore
$\mu_f(A^t)(y_1 * y_2) \geq \min\{\mu_f(A^t)(y_1), \mu_f(A^t)(y_2)\}$

Similarly we can show that
$\nu_f(A^t)(y_1 * y_2) \leq \max\{\nu_f(A^t)(y_1), \nu_f(A^t)(y_2)\}$

Hence $f(A)$ is t-intuitionistic fuzzy subalgebra of $Y$. 
Theorem 0.36 Let $f : X \rightarrow Y$ be a onto homomorphism of BG-algebras, If $A$ be t-intuitionistic fuzzy normal subalgebra $X$, then $f(A)$ is t-intuitionistic fuzzy normal subalgebra of $Y$.

References


