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$(\in, \in \vee q)$-Interval-valued Fuzzy Dot d-ideals of d-algebras

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Abstract. The concept of $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideals in d-algebras is introduced. Relationship among interval-valued fuzzy d-ideal, interval-valued fuzzy dot d-ideal, $(\in, \in)$-interval-valued fuzzy d-ideal,$(\in, \in)$-interval-valued fuzzy dot d-ideal, and $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideals are discussed. Conditions for an interval-valued fuzzy d-ideal to be an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideals are given. Some properties of interval-valued fuzzy relations and interval-valued fuzzy ideals under homomorphism are investigated.

1 Introduction

In 1991 Xi [12] applied the concept of fuzzy sets to BCK-algebras which are introduced by Imai and Iseki[7]in 1996. Neggers and Kim [11]introduced the class of d-algebras which is a generalisation of BCK-algebras and investigated relation between d-algebras and BCK- algebras. Akram and Dar[1] introduced the concepts fuzzy d-algebra, they introduced fuzzy subalgebra and fuzzy d-ideals of d-algebras. Kim [8]introduced the notion of a fuzzy dot subalgebra of d-algebra and investigated some related properties. Bhakat and Das [5, 6] used the relation of ”belongs to” and quasi-coincident” between fuzzy point and fuzzy set to introduced the concept of$(\in, \in \vee q)$-fuzzy subgroup and $(\in, \in \vee q)$-fuzzy subring. Al-Shehrie[2] introduced the notion of fuzzy dot d-ideals of a d-algebra. Interval-valued fuzzy sets were first introduced by Zadeh [15] in 1975. After that many researchers consider the interval-valued fuzzification of ideals and subalgebras in BG/ BCK-algebras. The concept of $(\in, \in \vee q)$-interval-valued fuzzification of ideals in ring was introduced in [9]. Here in this paper, we introduce the notion of $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of d-algebra and then we investigates some of its interesting properties.

2 Preliminaries

Definition 0.1 *[1, 8] A d-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:*
(i) $x * x = 0$
(ii) $0 * x = 0$
(iii) $x * y = 0$ *and* $y * x = 0 \Rightarrow x = y$ *for all* $x, y \in X$.
For brevity we also call X a d-algebra.

Definition 0.2 *[8] A non-empty subset S of a d-algebra X is called a subalgebra of X if* $x * y \in S$, *for all* $x, y \in S$.

Definition 0.3 *[2] A nonempty subset I of a d-algebra X is called an ideal of X if*
(i) $0 \in I$
(ii) $x * y \in I$ *and* $y \in I \Rightarrow x \in I$
(iii) $x \in I$ *and* $y \in X \Rightarrow x * y \in I$

Definition 0.4 *[2, 8] A fuzzy subset μ of X is called a fuzzy dot subalgebra of a d-algebra X if* $\mu(x * y) \geq \mu(x).\mu(y)$ *for all* $x, y \in X$.

Definition 0.5 *[2] A fuzzy subset μ of X is called a fuzzy d-ideal of X if it satisfies the following conditions:*
(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(x) \geq min\{\mu(x * y), \mu(y)\}$
(iii) $\mu(x * y) \geq min\{\mu(x), \mu(y)\}$

Definition 0.6 *[2] A fuzzy subset μ of X is called a fuzzy dot d-ideal of X if it satisfies the following conditions:*
(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(x) \geq \mu(x * y).\mu(y)$
(iii) $\mu(x * y) \geq \mu(x).\mu(y)$ *for all* $x, y \in X$.

Definition 0.7 *[5] A fuzzy set μ of the form*
$$\mu(y) = \begin{cases} t \ if \ \ y = x, \ \ t \ \in \ (0, 1] \\ 0 \ if \ \ y \neq x \end{cases}$$
is called a fuzzy point with support x and value t and it is denoted by x_t.

Definition 0.8 *[5] Let μ be a fuzzy set in X and x_t be a fuzzy point then*
(i) If $\mu(x) \geq t$ then we say x_t belongs to μ and write $x_t \in \mu$
(ii)If $\mu(x) + t > 1$ then we say x_t quasi coincidenc μ and write $x_t q\mu$
(iii)If $x_t \in \vee q\mu \Leftrightarrow x_t \in \mu$ or $x_t q\mu$
(iv) If $x_t \in \wedge q\mu \Leftrightarrow x_t \in \mu$ and $x_t q\mu$
The symbol $x_t\overline{\alpha}\mu$ means $x_t\alpha\mu$ does not hold and $\overline{\in \wedge q}$ means $\overline{\in} \vee \overline{q}$
For a fuzzy point x_t. and a fuzzy set μ in set X ,Pu and Liu [10]gave meaning to the symbol $x_t\alpha\mu$ where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$

Definition 0.9 *A fuzzy subset μ of X is called a $(\in, \in \vee q)$-fuzzy d-ideal of X if it satisfies the following conditions:*
(i)$x_t \in \mu \Rightarrow 0_t \in \vee q\mu$
(ii)$(x * y)_t, y_s \in \mu \Rightarrow x_{m(t,s)} \in \vee q\mu$
(iii)$x_t, y_s \in \mu \Rightarrow (x * y)_{m(t,s)} \in \vee q\mu \quad \forall \quad t, s \in (0, 1], \quad \forall x, y \in X$
Where $m(t, s) = min\{t, s\}$

Definition 0.10 *A fuzzy subset μ of X is called a $(\in, \in \vee q)$-fuzzy dot d-ideal of X if it satisfies the following conditions:*

1. $x_t \in \mu \Rightarrow 0_t \in \vee q\mu$

2. $(x * y)_t, y_s \in \mu \Rightarrow x_{t.s} \in \vee q\mu$

3. $x_t, y_s \in \mu \Rightarrow (x * y)_{t.s} \in \vee q\mu \, \forall \quad t, s \in (0, 1], \quad \forall x \in X$

Example 0.11 *Consider d-algebra $X = \{0, a, b, c\}$ with the following cayley table.*

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Define $\mu : X \to [0,1]$ *by* μ *by* $\mu(0) = 0.9, \mu(a) = \mu(b) = 0.8, \mu(c) = 0.7$*, then it is easy to verify that* μ *is* $(\in, \in \vee q)$*-fuzzy dot d-ideal X.*

Definition 0.12 *[2] Let* λ *and* μ *be two fuzzy sets in a set X.The their cartesian product* $\lambda \times \mu : X \times X \to [0,1]$ *is defined by* $(\lambda \times \mu)(x,y) = \lambda(x).\mu(y)$*, for all* $x, y \in X$*. Let* σ *be a fuzzy subset of X, then the strongest fuzzy* σ *relation on d algebra X is the fuzzy subset* μ_σ *of* $X \times X$ *given by* $\mu_\sigma(x,y) = \sigma(x).\sigma(y) \forall x, y \in X$*. A fuzzy relation* μ *on d algebra X is called a fuzzy* σ *product relation if* $\mu(x,y) \geq \sigma(x).\sigma(y) \forall x, y \in X$*. A fuzzy relation* μ *on d algebra X is called a left fuzzy relation on* σ *if* $\mu(x,y) = \sigma(x) \forall x, y \in X$*. Note that a left fuzzy relation on* σ *is a fuzzy* σ *product relation.*

Remark 0.13 *If X and Y be two d-algebras, then* $X \times X$ *is also a d-algebra under the binary operation* $'*'$ *defined in* $X \times X$ *by* $(x,y) * (p,q) = (x * p, y * q)$ *for all* $(x,y), (p,q) \in X \times X$*.*

3 $(\in, \in \vee q)$-interval-valued fuzzy sets

The notion of interval-valued fuzzy set was introduced by L.A.Zadeh[25]. To consider the notion of interval-valued fuzzy sets, we need the following notations.By an interval number $\hat{a}$, we mean an interval $[\underline{a}, \overline{a}]$, where $0 \leq \underline{a} \leq \overline{a} \leq 1$. Let $D[0,\ 1]$ denote the set of all such interval numbers of $[0,\ 1]$. Define on D[0, 1] the relations $\leq, =, <, .$ by

1. $\hat{a_1} \leq \hat{a_2} \Leftrightarrow \underline{a_1} \leq \underline{a_2}$ and $\overline{a_1} \leq \overline{a_2}$
2. $\hat{a_1} = \hat{a_2} \Leftrightarrow \underline{a_1} = \underline{a_2}$ and $\overline{a_1} = \overline{a_2}$
3. $\hat{a_1} < \hat{a_2} \Leftrightarrow \underline{a_1} < \underline{a_2}$ and $\overline{a_1} < \overline{a_2}$
4. $\hat{a_1}.\hat{a_2} \Leftrightarrow [min(\underline{a_1 a_2}, \underline{a_1}\overline{a_2}, \overline{a_1}\underline{a_2}, \overline{a_1 a_2}), max(\underline{a_1 a_2}, \underline{a_1}\overline{a_2}, \overline{a_1}\underline{a_2}, \overline{a_1 a_2})] = [\underline{a_1 a_2}, \overline{a_1 a_2}]$
5. $k\hat{a} = [k\underline{a}, k\overline{a}]$ where $0 \leq k \leq 1$

Now consider two intervals $\hat{a_1} = [\underline{a_1}, \overline{a_1}], \hat{a_2} = [\underline{a_2}, \overline{a_2}] \in D[0,\ 1]$ then we define refine minimum rmin as $rmin(\hat{a_1}, \hat{a_2}) = [min(\underline{a_1}, \underline{a_2}), min(\overline{a_1}, \overline{a_2})]$ and refine maximum as
rmax $rmax(\hat{a_1}, \hat{a_2}) = [max(\underline{a_1}, \underline{a_2}), max(\overline{a_1}, \overline{a_2})]$ generally if $\hat{a_i} = [\underline{a_1}, \overline{a_i}], \hat{b_i} = [\underline{b_1}, \overline{b_i}] \in D[0,\ 1]$ for i= 1,2,3,...then we define
$rmax(\hat{a_i}, \hat{b_i}) = [max(\underline{a_i}, \underline{b_i}), max(\overline{a_i}, \overline{b_i})]$ and $rmin(\hat{a_i}, \hat{b_i}) = [min(\underline{a_i}, \underline{b_i}), min(\overline{a_i}, \overline{b_i})]$
and $rinf_i(\hat{a_i}) = [\wedge_i \underline{a_i}, \wedge_i \overline{a_i}]$ and $rsup_i(\hat{a_i}) = [\vee_i \underline{a_i}, \vee_i \overline{a_i}]$
$(D[0,\ 1], \leq)$ is a complete lattice with $\wedge = rmin,\ \vee = rmax, \hat{0} = [0\,0]$ and $\hat{1} = [1\,1]$ being the least and the greatest element respectively.

Definition 0.14 *An interval-valued fuzzy set defined on a non empty set X as an objects having the form* $\hat{\mu} = \{x, [\underline{\mu}(x), \overline{\mu}(x)]\}, \forall x \in X$ *where* $\underline{\mu}$ *and* $\overline{\mu}$ *are two fuzzy sets in X such that* $\underline{\mu}(x) \leq \overline{\mu}(x)$ *for all* $x \in X$*. Let* $\hat{\mu}(x) = [\underline{\mu}(x), \overline{\mu}(x)], \forall x \in X$*. Then* $\hat{\mu}(x) \in D[0\ 1], \forall x \in X$
If $\hat{\mu}$ *and* $\hat{\nu}$ *be two interval-valued fuzzy sets in X, then we define*

- $\hat{\mu} \subset \hat{\nu} \Leftrightarrow$ *for all* $\underline{\mu}(x) \leq \underline{\nu}(x)$ *and* $\overline{\mu}(x) \leq \overline{\nu}(x)$*.*

- $\hat{\mu} = \hat{\nu} \Leftrightarrow$ *for all* $\underline{\mu}(x) = \underline{\nu}(x)$ *and* $\overline{\mu}(x) = \overline{\nu}(x)$.
- $(\hat{\mu} \cup \hat{\nu})(x) = \hat{\mu}(x) \vee \hat{\nu}(x) = [max\{\underline{\mu}(x), \underline{\nu}(x)\}, max\{\overline{\mu}(x), \overline{\nu}(x)\}]$.
- $(\hat{\mu} \cap \hat{\nu})(x) = \hat{\mu}(x) \wedge \hat{\nu}(x) = [min\{\underline{\mu}(x), \underline{\nu}(x)\}, min\{\overline{\mu}(x), \overline{\nu}(x)\}]$.
- $\hat{\mu}^c(x) = [1 - \overline{\mu}(x), 1 - \underline{\mu}(x)]$.

Definition 0.15 *Let $\hat{\mu}$ be an interval-valued fuzzy set in X.Then, for every $[0,\ 0] < \hat{t} \leq [1,\ 1]$, the crisp set $\hat{\mu}_t = \{x \in X | \hat{\mu}(x) \geq \hat{t}\}$ is called the level subset of $\hat{\mu}$.*

Definition 0.16 *A interval-valued fuzzy subset $\hat{\mu}$ of X is called an interval-valued fuzzy dot subalgebra of a d-algebra X if $\hat{\mu}(x * y)) \geq \hat{\mu}(x).\hat{\mu}(y)$ for all $x, y \in X$.*

Definition 0.17 *An interval-valued fuzzy set $\hat{\mu}$ in d-algebra X is called an interval-valued fuzzy ideal of X if it satisfies*
(i) $\hat{\mu}(0) \geq \hat{\mu}(x)$
(ii) $\hat{\mu}(x) \geq rmin\{\hat{\mu}(x * y), \hat{\mu}(y)\}$ for all $x, y \in X$
(ii) $\hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x), \hat{\mu}(y)\}$ for all $x, y \in X$

Definition 0.18 *An interval-valued fuzzy set $\hat{\mu}$ in d-algebra X is called an interval-valued fuzzy dot d-ideal of X if it satisfies*
(i) $\hat{\mu}(0) \geq \hat{\mu}(x)$
(ii) $\hat{\mu}(x) \geq \hat{\mu}(x * y).\hat{\mu}(y)$ for all $x, y \in X$
(ii) $\hat{\mu}(x * y) \geq \hat{\mu}(x).\hat{\mu}(y)$ for all $x, y \in X$

Example 0.19 *Consider d-algebra $X = \{0, a, b, c\}$ with the following cayley table.*

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Define $\hat{\mu} : X \to D[0, 1]$ by $\hat{\mu}(0) = [0.8,\ 0.9], \hat{\mu}(a) = \hat{\mu}(b) = [0.6,\ 0.8], \hat{\mu}(c) = [0.35,\ 0.5]$, then it is easy to verify that $\hat{\mu}$ is $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal X.

Definition 0.20 *Let $\hat{\mu}(x) = [\underline{\mu}(x), \overline{\mu}(x)]$ and $\hat{t} = [\underline{t}, \overline{t}] \in D[0,\ 1]$, then we define $\hat{\mu}(x) + \hat{t} = [\underline{\mu}(x) + \underline{t}, \overline{\mu}(x) + \overline{t}]\ \forall x \in X$. In particular $\underline{\mu}(x) + \underline{t} > 1$, we write $\hat{\mu}(x) + \hat{t} > [1,\ 1]$. Let $x \in X$. An interval-valued fuzzy set $\hat{\mu}$ of the form*

$$\hat{\mu}(y) = \begin{cases} \hat{t}\ if\ \ y = x,\ \ \hat{t}\ \in\ D(0, 1] \\ \hat{0}\ if\ \ y \neq x \end{cases}$$

is called an interval-valued fuzzy point with support x and value $\hat{t}$ and it is denoted by $x_{\hat{t}}$.

Definition 0.21 *([5]) Let $\hat{\mu}$ be an interval-value fuzzy set in X and $x_{\hat{t}}$ be an interval-value fuzzy point then*
(i) If $\hat{\mu}(x) \geq \hat{t}$ then we say $x_{\hat{t}}$ belongs to $\hat{\mu}$ and write $x_{\hat{t}} \in \hat{\mu}$
(ii)If $\hat{\mu}(x) + \hat{t} > [1,\ 1]$ then we say $x_{\hat{t}}$ quasi coincidenc $\hat{\mu}$ and write $x_{\hat{t}} q \hat{\mu}$

(iii)If $x_{\hat{t}} \in \vee q\hat{\mu} \Leftrightarrow x_t \in \hat{\mu}$ *or* $x_{\hat{t}} q\hat{\mu}$
(iv) If $x_{\hat{t}} \in \wedge q\hat{\mu} \Leftrightarrow x_{\hat{t}} \in \hat{\mu}$ *and* $x_{\hat{t}} q\hat{\mu}$
The symbol $x_{\hat{t}}\overline{\alpha}\hat{\mu}$ *means* $x_{\hat{t}}\alpha\hat{\mu}$ *does not hold and* $\overline{\in \wedge q}$ *means* $\overline{\in} \vee \overline{q}$
For an interval-valued fuzzy point $x_{\hat{t}}$ *and an interval-valued fuzzy set* $\hat{\mu}$ *in set X, Pu and Liu [10]gave meaning to the symbol* $x_{\hat{t}}\alpha\hat{\mu}$ *where* $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$

Definition 0.22 *A interval-valued fuzzy set* $\hat{\mu}$ *of a d-algebra X is said to be* (α, β)*-interval-valued fuzzy d-ideal of X, Where* $\alpha \neq \in \wedge q$ *if*
(i) $x_{\hat{t}}\alpha\hat{\mu} \Rightarrow 0_{\hat{t}}\beta\hat{\mu}$
(ii) $(x * y)_{\hat{t}}, y_{\hat{s}}\alpha\hat{\mu} \Rightarrow x_{rmin(\hat{t},\hat{s})}\beta\hat{\mu}$
(iii) $(x)_{\hat{t}}, y_{\hat{s}}\alpha\hat{\mu} \Rightarrow (x * y)_{rmin(\hat{t},\hat{s})}\beta\hat{\mu}$
for all $x, y \in X$ *where* $[0,\ 0] < \hat{t}, \hat{s} \leq [1,\ 1]$, *where* $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$

Definition 0.23 *A interval-valued fuzzy set* $\hat{\mu}$ *of a d-algebra X is said to be* (α, β)*-interval-valued fuzzy dot d-ideal of X, Where* $\alpha \neq \in \wedge q$ *if*
(i) $x_{\hat{t}}\alpha\hat{\mu} \Rightarrow 0_{\hat{t}}\beta\hat{\mu}$
(ii) $(x * y)_{\hat{t}}, y_{\hat{s}}\alpha\hat{\mu} \Rightarrow x_{(\hat{t}.\hat{s})}\beta\hat{\mu}$
(iii) $(x)_{\hat{t}}, y_{\hat{s}}\alpha\hat{\mu} \Rightarrow (x * y)_{(\hat{t}.\hat{s})}\beta\hat{\mu}$
for all $x, y \in X$ *where* $[0,\ 0] < \hat{t}, \hat{s} \leq [1,\ 1]$

Definition 0.24 *A interval-valued fuzzy set* $\hat{\mu}$ *of a d-algebra X is said to be (*$\in, \in \vee q$*)-interval-valued fuzzy dot d-ideal of X, if*
(i) $x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \vee q\hat{\mu}$
(ii) $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{(\hat{t}.\hat{s})} \in \vee q\hat{\mu}$
(iii) $(x)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{(\hat{t}.\hat{s})} \in \vee q\hat{\mu}$
for all $x, y \in X$ *where* $[0,\ 0] < \hat{t}, \hat{s} \leq [1,\ 1]$

Example 0.25 *Consider d-algebra X as in Example 0.11 and* $\hat{\mu}$ *by* $\hat{\mu}(0) = [0.75,\ 0.9], \hat{\mu}(a) = \hat{\mu}(b) = [0.7,\ 0.8], \hat{\mu}(c) = [0.65,\ 0.7]$*, then it is easy to verify that* $\hat{\mu}$ *is (*$\in, \in \vee q$*)-interval-valued fuzzy dot d-ideal X.*

Theorem 0.26 *Every (*$\in, \in$*)-interval-valued fuzzy d-ideal of d algebra X is an (*$\in, \in$*)-interval-valued fuzzy dot d-ideal of X.*

Proof Straightforward

Remark 0.27 *The converse of Theorem 0.26 is not true as shown in following Example.*

Example 0.28 *Consider d-algebra* $X = \{0, a, b\}$ *with the following cayley table.*

*	0	a	b
0	0	0	0
a	b	0	b
b	a	a	0

Define $\hat{\mu} : X \to D[0,1]$ *by* $\hat{\mu}(0) = [0.6, 0.7], \hat{\mu}(a) = [0.7, 0.8], \hat{\mu}(b) = [0.8, 0.9]$ *then it is easy to verify that* $\hat{\mu}$ *is* $(\in, \in)$*-interval-valued fuzzy dot d ideal X. Here* $(a * b)_{[0.77,0.88]}, b_{[0.77,0.88]} \in \hat{\mu}$ *But* $a_{[0.77,0.88]} \notin \hat{\mu}$*. Therefore* $\hat{\mu}$ *is not an* $(\in, \in)$*-interval-valued fuzzy d-ideal X.*

Theorem 0.29 *Every interval-valued fuzzy d-ideal of a d algebra X is an interval-valued fuzzy dot d-ideal of X.*

Remark 0.30 *The converse of Theorem 0.29 is not true as shown in following Example.*

Example 0.31 *Consider d-algebra X and* $\hat{\mu}$ *as in Example 0.28 then it is easy to verify that* $\hat{\mu}$ *is an interval-valued fuzzy dot d ideal X. But* $rmin\{\hat{\mu}(a * b), \hat{\mu}(b)\} = \hat{\mu}(b) = [0.8, 0.9] \not\leq \hat{\mu}(a) = [0.7, 0.8]$ *Therefore* $\hat{\mu}$ *is not an interval-valued fuzzy d-ideal X.*

Theorem 0.32 *An interval-valued fuzzy subset* $\hat{\mu}$ *of a d algebra X is a interval-valued fuzzy d-ideal iff* $\hat{\mu}$ *is an* $(\in, \in)$*-interval-valued fuzzy d-ideal of X*

Proof Let $\hat{\mu}$ be an interval-valued fuzzy d-ideal of X.
To prove $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy d-ideal of X.
Let $x \in X$ such that $x_{\hat{t}} \in \hat{\mu}$ where $\hat{t} \in D(0,1)$ then $\hat{\mu}(x) \geq \hat{t} \Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x) \geq \hat{t}$ [Since $\hat{\mu}$-is interval-valued fuzzy d-ideal]
$\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\mu} \Rightarrow x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$
Again let $x, y \in X$ such that $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$ where $\hat{t}, \hat{s} \in D(0,1)$
then $\hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
Now $\hat{\mu}(x) \geq rmin\{\hat{\mu}(x * y), \hat{\mu}(y)\} \geq rmin(\hat{t}, \hat{s})$[Since $\hat{\mu}$-is interval-valued fuzzy d-ideal]
$\Rightarrow x_{rmin(\hat{t},\hat{s})} \in \hat{\mu}$
$\Rightarrow (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{rmin(\hat{t},\hat{s})} \in \hat{\mu}$
Again let $x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$
$\Rightarrow \hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
Now $\hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x), \hat{\mu}(y)\} \geq rmin(\hat{t}, \hat{s})$[Since $\hat{\mu}$-interval-valued fuzzy d-ideal]
$\Rightarrow (x * y)_{rmin(\hat{t},\hat{s})} \in \hat{\mu}$
$x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{rmin(\hat{t},\hat{s})} \in \hat{\mu}$
Hence $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy d-ideal of X.
Conversely suppose $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy d-ideal of X.
to prove $\hat{\mu}$ is an interval-valued fuzzy d-ideal of X.
Let $x \in X$ Now since $x_{\hat{\mu}(x)} \in \hat{\mu}$ therefore $0_{\hat{\mu}(x)} \in \hat{\mu}$
$\Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x)$ [Since $\hat{\mu}$ is an$(\in, \in)$-interval-valued fuzzy d-ideal of X]
Let $x, y \in X$ again since $(x * y)_{\hat{\mu}(x*y)} \in \hat{\mu}$ and $y_{\hat{\mu}(y)} \in \hat{\mu}$
$\Rightarrow x_{rmin\{\hat{\mu}(x*y),\hat{\mu}(y)\}} \in \mu$[Since $\hat{\mu}$ is an$(\in, \in)$-interval-valued fuzzy d-ideal of X]
$\Rightarrow \hat{\mu}(x) \geq rmin\{\hat{\mu}(x * y), \hat{\mu}(y)\}$
Let $x, y \in X$ again since $(x)_{\hat{\mu}(x)} \in \hat{\mu}$ and $y_{\hat{\mu}(y)} \in \mu$
$\Rightarrow (x * y)_{rmin\{\hat{\mu}(x),\hat{\mu}(y)\}} \in \hat{\mu}$[Since $\hat{\mu}$ is an$(\in, \in)$-interval-valued fuzzy d-ideal of X]
$\Rightarrow \hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x), \hat{\mu}(y)\}$
Hence $\hat{\mu}$ is an interval-valued fuzzy d-ideal of X.

Theorem 0.33 *An interval-valued fuzzy subset* $\hat{\mu}$ *of a d algebra X is a interval-valued fuzzy dot d-ideal iff* $\hat{\mu}$ *is a* $(\in, \in)$*-interval-valued fuzzy dot d-ideal of X*

Proof $\hat{\mu}$ be an interval-valued fuzzy dot d-ideal of X.To prove $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X.
Let $x \in X$ such that $x_{\hat{t}} \in \hat{\mu}$ where $\hat{t} \in D(0,1)$ then $\hat{\mu}(x) \geq \hat{t}$
$\Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x) \geq \hat{t}$[Since $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal]
$\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$
$\Rightarrow x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$
Let $x, y \in X$ such that $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$ where $\hat{t}, \hat{s} \in D(0,1)$
then $\hat{\mu}(x * y) \geq \hat{t}, \mu(y) \geq \hat{s}$
Now $\hat{\mu}(x) \geq \hat{\mu}(x * y).\hat{\mu}(y) \geq \hat{t}.\hat{s}$ [Since $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal]
$\Rightarrow x_{\hat{t}.\hat{s}} \in \hat{\mu}$
$\Rightarrow (x * y)_{\hat{t}}, y_{\hat{t}} \in \hat{\mu} \Rightarrow x_{\hat{t}.\hat{s}} \in \hat{\mu}$
Again let $x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$
$\Rightarrow \hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
Now $\hat{\mu}(x * y) \geq \hat{\mu}(x).\hat{\mu}(y) \geq \hat{t}.\hat{s}$ [Since $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal]
$\Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \hat{\mu}$
$\Rightarrow x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \hat{\mu}$
Hence $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X.
Conversely suppose $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X.
to prove $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal of X.
Let $x \in X$ and $\hat{t} = \hat{\mu}(x)$
then $\hat{\mu}(x) \geq \hat{t} \Rightarrow x_{\hat{t}} \in \hat{\mu}$
$\Rightarrow 0_{\hat{t}} \in \hat{\mu}$[Since $\hat{\mu}$ is $(\in, \in)$-is an interval-valued fuzzy dot d-ideal of X.]
$\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x)$
Let $x, y \in X$ and $\hat{t} = \hat{\mu}(x * y), \hat{s} = \hat{\mu}(y)$
then $\hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
$\Rightarrow (x * y)_{\hat{t}} \in \hat{\mu}, y_{\hat{s}} \in \hat{\mu}$
$\Rightarrow x_{\hat{t}.\hat{s}} \in \hat{\mu}$ [Since $\hat{\mu}$ is $(\in, \in)$-is an interval-valued fuzzy dot d-ideal of X]
$\Rightarrow \hat{\mu}(x) \geq \hat{t}.\hat{s} = \hat{\mu}(x * y).\hat{\mu}(y)$
Again let $\hat{t} = \hat{\mu}(x), \hat{s} = \hat{\mu}(y)$
then $\hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
$\Rightarrow x_{\hat{t}} \in \hat{\mu}, y_{\hat{s}} \in \hat{\mu}$
$\Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \hat{\mu}$ [Since $\hat{\mu}$ is $(\in, \in)$-interval-valued fuzzy dot d-ideal of X]
$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}.\hat{s} = \hat{\mu}(x).\hat{\mu}(y)$
Hence $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal of X.

Theorem 0.34 *Every interval-valued fuzzy d-ideal of a d algebra X is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X.*

Proof It follows from Theorem 0.29 and Theorem 0.33.

Remark 0.35 *The converse of Theorem 0.34 is not true as shown in following Example.*

Example 0.36 *Consider d-algebra X and $\hat{\mu}$ as in Example 0.28 then it is easy to verify that $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d ideal X. But $min\{\hat{\mu}(a * b), \hat{\mu}(b)\} = \hat{\mu}(b) = [0.8, 0.9] \not\leq \hat{\mu}(a) = [0.7, 0.8]$ Therefore $\hat{\mu}$ is not an interval-valued fuzzy d-ideal X.*

Theorem 0.37 *Every $(\in, \in)$-interval-valued fuzzy d-ideal of a d algebra X is an interval-valued fuzzy dot d-ideal of X.*

Proof It follows from Theorem 0.32 and Theorem 0.29.

Remark 0.38 *The converse of Theorem0.37 is not true as shown in following Example.*

Example 0.39 *Consider d-algebra X and $\hat{\mu}$ as in Example 0.28 then it is easy to verify that $\hat{\mu}$ is an interval-valued fuzzy dot d ideal X. But $(a * b)_{[0.77,0.88]}, b_{[0.77,0.88]} \in \hat{\mu}$ But $a_{[0.77,0.88]} \notin \hat{\mu}$. Therefore $\hat{\mu}$ is not an ($\in, \in$)-interval-valued fuzzy d-ideal X.*

Definition 0.40 *A fuzzy subset $\hat{\mu}$ of X is called a ($\in, \in$)-interval-valued fuzzy dot subalgebra if $x_{\hat{t}}, y_{\hat{t}} \in \hat{\mu} \Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \hat{\mu}$ for all $x, y \in X$.*

Proposition 0.41 *Every ($\in, \in$)-interval-valued fuzzy dot d-ideal of a d-algebra X is an ($\in, \in$)-interval-valued fuzzy dot sub algebra of X.*

Remark 0.42 *The converse of Proposition 0.41 is not true as shown in following example.*

Example 0.43 *Consider d algebra $X = \{0, a, b, c\}$ with the following cayley table.*

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Define $\hat{\mu} : X \to [0, 1]$ by $\hat{\mu}(0) = \hat{\mu}(b) = [0.8, 0.9], \hat{\mu}(a) = \hat{\mu}(c) = [0.7, 0.8]$ then it is easy to verify that $\hat{\mu}$ is an ($\in, \in$)-interval-valued fuzzy dot subalgebra of X. But $\hat{\mu}$ is not an ($\in, \in$)-interval-valued fuzzy dot d-ideal of X because $\hat{\mu}(a * b) = \hat{\mu}(0) = [0.8, 0.9]$ and $\hat{\mu}(b) = [0.8, 0.9]$
therefore $\hat{\mu}(a * b) \geq [0.8, 0.9]$ and $\hat{\mu}(b) \geq [0.8, 0.9] \not\Rightarrow \hat{\mu}(a) \geq [0.8, 0.9].[0.9, 0.9] = [0.72, 0.81]$

Theorem 0.44 *Every($\in, \in$)-interval-valued fuzzy dot d-ideal of a d-algebra X is a ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal X.But the converse is not true as shown in the following Example.*

Example 0.45 *Consider d-algebra $X = \{0, a, b, c\}$ with the following cayley table.*

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Define $\hat{\mu} : X \to [0, 1]$ by $\hat{\mu}(0) = [0.85, 0.95], \hat{\mu}(a) = \hat{\mu}(b) = [0.8, 0.9], \hat{\mu}(c) = [0.6, 0.7]$, then it is easy to verify that $\hat{\mu}$ is ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal X.Since $a_{[0.8,0.9]} = (c * b)_{[0.8,0.9]}, b_{[0.8,0.9]} \in \hat{\mu}$ but $\hat{\mu}(c) \not\geq [0.8, 0.9].[0.8, 0.9]$ i.e $c_{[0.64,0.81]} \notin \hat{\mu}$

Theorem 0.46 *If $\hat{\mu}, \hat{\nu}$ are ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal of a d-algebra X, then so is $\hat{\mu} \cap \hat{\nu}$.*

Proof Let $x, y \in X$ such that $x_{\hat{t}} \in (\hat{\mu} \cap \hat{\nu}) \Rightarrow (\hat{\mu} \cap \hat{\nu})(x) \geq \hat{t} \Rightarrow rmin\{\hat{\mu}(x), \hat{\nu}(x)\} \geq \hat{t} \Rightarrow \hat{\mu}(x) \geq \hat{t}, \hat{\nu}(x) \geq \hat{t} \Rightarrow x_{\hat{t}} \in \hat{\mu}$ and $x_{\hat{t}} \in \hat{\nu} \Rightarrow 0_{\hat{t}} \in \vee q\hat{\mu}$ and $0_{\hat{t}} \in \vee q\hat{\nu}$ [Since $\hat{\mu}, \hat{\nu}$ both are$(\in, \in \vee q)$-interval-valued fuzzy dot d-ideals of X]
$\Rightarrow 0_{\hat{t}} \in \vee q(\hat{\mu} \cap \hat{\nu})$
Again let $(x * y)_{\hat{t}}, y_{\hat{s}} \in (\hat{\mu} \cap \hat{\nu})$
$\Rightarrow (\hat{\mu} \cap \hat{\nu})(x * y) \geq \hat{t}, (\hat{\mu} \cap \hat{\nu})(y) \geq \hat{s}$
$\Rightarrow rmin\{\hat{\mu}(x * y), \hat{\nu}(x * y)\} \geq \hat{t}$ and $rmin\{\hat{\mu}(y), \hat{\nu}(y)\} \geq \hat{s}$
$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}, \hat{\nu}(x * y) \geq \hat{t}$ and $\hat{\mu}(y) \geq s, \hat{\nu}(y) \geq \hat{s}$
$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq s$ and $\hat{\nu}(x * y) \geq \hat{t}, \hat{\nu}(y) \geq \hat{s}$
$\Rightarrow (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$ and $(x * y)_{\hat{t}}, y_s \in \hat{\nu}$
$\Rightarrow x_{\hat{t}.\hat{s}} \in \vee q\hat{\mu}$ and $x_{\hat{t}.\hat{s}} \in \vee q\hat{\nu}$ [Since $\hat{\mu}, \hat{\nu}$ both are$(\in, \in \vee q)$-interval-valued fuzzy dot d-ideals of X]
$\Rightarrow x_{\hat{t}.\hat{s}} \in \vee q(\hat{\mu} \cap \hat{\nu})$
Similarly we can prove that
$x_{\hat{t}}, y_{\hat{s}} \in (\hat{\mu} \cap \hat{\nu}) \Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \vee q(\hat{\mu} \cap \hat{\nu})$
Hence the proof.

Theorem 0.47 *If $\hat{\mu}$ is a (q,q)-interval-valued fuzzy dot d-ideal of a d algebra X then it is also an$(\in, \in)$-interval-valued fuzzy dot d-ideal of X.*

Proof Let $\hat{\mu}$ be an (q, q)-interval-valued fuzzy dot d-ideal of a d-algebra X, to prove $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X. let $x \in X$ such that$x_{\hat{t}} \in \hat{\mu} \Rightarrow \hat{\mu}(x) \geq \hat{t}$
$\Rightarrow \hat{\mu}(x) + \hat{\delta} > \hat{t}$ where $\hat{\delta}$ is a arbitrary small positive interval in $D[0\ 1]$
$\Rightarrow \hat{\mu}(x) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1} \Rightarrow (x)_{\hat{1}+\hat{\delta}-\hat{t}} q\hat{\mu} \Rightarrow (0)_{\hat{1}+\hat{\delta}-\hat{t}} q\hat{\mu} \Rightarrow \hat{\mu}(0) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1} \Rightarrow \hat{\mu}(0) + \hat{\delta} > \hat{t}$
$\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$
Therefore$x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$
Again let $x, y \in X$ such that
$(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$ therefore $\hat{\mu}(x * y) \geq t, \hat{\mu}(y) \geq \hat{s}$
$\Rightarrow \hat{\mu}(x * y) + \hat{\delta} > \hat{t}, \hat{\mu}(y) + \hat{\delta} > \hat{s}$ where $\hat{\delta}$ is a arbitrary small positive interval in $D[0\ 1]$
$\Rightarrow \hat{\mu}(x * y) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1}, \hat{\mu}(y) + \hat{\delta} - \hat{s} + \hat{1} > \hat{1}$
$\Rightarrow (x * y)_{\hat{1}+\hat{\delta}-\hat{t}} q\hat{\mu}$ and $(y)_{\hat{1}+\hat{\delta}-\hat{s}} q\hat{\mu}$
$\Rightarrow (x)_{(\hat{1}+\hat{\delta}-\hat{t}).(\hat{1}+\hat{\delta}-\hat{s})} q\mu$ [since μ is (q,q)-interval-valued fuzzy dot d-ideal of X]
$\Rightarrow \hat{\mu}(x) + (\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s}) > \hat{1}$
$\Rightarrow \hat{\mu}(x) + (\hat{1} - \hat{t}).(\hat{1} - \hat{s}) \geq \hat{1}$ taking $\hat{\delta} = \hat{0}$
$\Rightarrow \hat{\mu}(x) + \hat{1} - \hat{t} - \hat{s} + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow \hat{\mu}(x) \geq \hat{s} + \hat{t} - \hat{t}.\hat{s}$
$\Rightarrow \hat{\mu}(x) \geq \hat{t}.\hat{s} + \hat{t}.\hat{s} - \hat{t}.\hat{s}$ [Since $\hat{t} \geq \hat{t}.\hat{s}$ and $\hat{s} \geq \hat{t}.\hat{s}$]
$\Rightarrow \hat{\mu}(x) \geq \hat{t}.\hat{s}$
$\Rightarrow x_{\hat{t}.\hat{s}} \in \hat{\mu}$
Again let
$x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$ therefore $\hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
$\Rightarrow \hat{\mu}(x) + \hat{\delta} > \hat{t}, \hat{\mu}(y) + \hat{\delta} > \hat{s}$ where $\hat{\delta}$ is a arbitrary small positive interval in $D[0\ 1]$
$\Rightarrow \hat{\mu}(x) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1}, \hat{\mu}(y) + \hat{\delta} - \hat{s} + \hat{1} > \hat{1}$
$\Rightarrow (x)_{\hat{1}+\hat{\delta}-\hat{t}} q\hat{\mu}$ and $(y)_{\hat{1}+\hat{\delta}-\hat{s}} q\hat{\mu}$
$\Rightarrow (x * y)_{(\hat{1}+\hat{\delta}-\hat{t}).(\hat{1}+\delta-s)} q\hat{\mu}$ [since $\hat{\mu}$ is (q,q)-interval-valued fuzzy dot d-ideal of X]
$\Rightarrow \hat{\mu}(x * y) + (\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s}) > \hat{1}$
$\Rightarrow \hat{\mu}(x * y) + (\hat{1} - \hat{t}).(\hat{1} - \hat{s}) \geq \hat{1}$ taking $\hat{\delta} = \hat{0}$
$\Rightarrow \hat{\mu}(x * y) + \hat{1} - \hat{t} - \hat{s} + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow \hat{\mu}(x * y) \geq \hat{s} + \hat{t} - \hat{t}.\hat{s}$
$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}.\hat{s} + \hat{t}.\hat{s} - \hat{t}.\hat{s}$ [Since $\hat{t} \geq \hat{t}.\hat{s}$ and $\hat{s} \geq \hat{t}.\hat{s}$]

$\Rightarrow \hat{\mu}(x*y) \geq \hat{t}.\hat{s}$
$\Rightarrow (x*y)_{\hat{t}.\hat{s}} \in \hat{\mu}$
Hence $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X.

Remark 0.48 *The converse of Theorem 0.47 is not true as shown in following Example.*

Example 0.49 *Consider d-algebra X and $\hat{\mu}$ as in Example 0.11,then $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X.But $\hat{\mu}$ is not an (q, q)-interval-valued fuzzy dot d-ideal of X because if $x = c, y = a, x*y = c*a = c, t = 0.7, s = 0.4$ then $\mu(x*y)+t = 0.35+0.7 > 1, \mu(y)+s = 0.7+0.4 > 1$ but $\mu(x)+t.s = 0.35+0.7\times 0.4 = 0.35+0.28 < 1$ i.e $x_{\hat{t}.\hat{s}}\overline{q}\mu$.*

Theorem 0.50 *A fuzzy subset $\hat{\mu}$ of a d-algebra X is an $(\in, \in\vee q)$-interval-valued fuzzy dot d-ideal of X iff*
(i) $\hat{\mu}(0) \geq rmin\{\hat{\mu}(x), [0.5\ 0.5]\}$
(ii) $\hat{\mu}(x) \geq rmin\{\hat{\mu}(x*y).\hat{\mu}(y), \hat{1}-\hat{\mu}(x*y).\hat{\mu}(y)\}$
(iii $\hat{\mu}(x*y) \geq rmin\{\hat{\mu}(x).\hat{\mu}(y), \hat{1}-\hat{\mu}(x).\hat{\mu}(y)\}$

Proof Let $\hat{\mu}$ be an $(\in, \in\vee q)$-interval-valued fuzzy dot d-ideal of X
(i) Assume $\hat{\mu}(0) < rmin\{\hat{\mu}(x), [0.5, 0.5]\}$
SubCase I: If $\hat{\mu}(x) < [0.5, 0.5]$, them $rmin\{\hat{\mu}(x), [0.5, 0.5]\} = \hat{\mu}(x)$
$\Rightarrow \hat{\mu}(0) < \hat{\mu}(x) \Rightarrow \hat{\mu}(0) < \hat{t} < \hat{\mu}(x)$ for some $(0\ 0) < \hat{t} < (0.5, 0.5)$
$\Rightarrow x_{\hat{t}} \in \hat{\mu}$ and $0_{\hat{t}}\overline{\in}\hat{\mu}$ also $\hat{\mu}(0)+\hat{t} < \hat{1}$ i.e $0_{\hat{t}}\overline{q}\mu \Rightarrow 0_{\hat{t}}\overline{\in\vee q}\hat{\mu}$ Which is a contradiction
SubCase II: If $\hat{\mu}(x) \geq [0.5,\ 0.5]$ i.e $x_{[0.5,\ 0.5]} \in \hat{\mu}$, them $rmin\{\hat{\mu}(x), [0.5,\ 0.5]\} = [0.5,\ 0.5]$
$\hat{\mu}(0) < rmin\{\hat{\mu}(x), [0.5,\ 0.5]\} = [0.5,\ 0.5]$
$\Rightarrow 0_{[0.5,\ 0.5]}\overline{\in}\hat{\mu}$ and $\hat{\mu}(0)+[0.5,\ 0.5] < [0.5,\ 0.5]+[0.5,\ 0.5] = \hat{1} i.e 0_{[0.5,\ 0.5]}\overline{q}\hat{\mu}$
$\Rightarrow 0_{[0.5,\ 0.5]}\overline{\in\vee q}\hat{\mu}$ Which is again a contradiction, therefore
Therefore we must have $\hat{\mu}(0) \geq rmin\{\hat{\mu}(x), [0.5,\ 0.5]\}$
(ii) Assume $\hat{\mu}(x) < rmin\{\hat{\mu}(x*y).\hat{\mu}(y), \hat{1}-\hat{\mu}(x*y).\mu(y)\}$
choose $[0,\ 0] \leq \hat{t}, \hat{s} \leq [0,\ 1]$ such that $\hat{\mu}(x) < \hat{t}.\hat{s} < rmin\{\hat{\mu}(x*y).\hat{\mu}(y), \hat{1}-\hat{\mu}(x*y).\hat{\mu}(y)\}$
where $\hat{\mu}(x*y) \geq \hat{t}$ and $\hat{\mu}(y) \geq \hat{s}$
SubCase I: If $\hat{\mu}(x*y).\hat{\mu}(y) < [0.5,\ 0.5]$, them $m\{\hat{\mu}(x*y).\hat{\mu}(y), \hat{1}-\hat{\mu}(x*y).\hat{\mu}(y)\} = \hat{\mu}(x*y).\hat{\mu}(y)$
$\Rightarrow \hat{\mu}(x) < \hat{t}.\hat{s} < \hat{\mu}(x*y).\hat{\mu}(y) < [0.5,\ 0.5]$
$\Rightarrow \hat{\mu}(x*y) \geq \hat{t}$ and $\hat{\mu}(y) \geq \hat{s}$
$\Rightarrow (x*y)_{\hat{t}} \in \hat{\mu}$ and $y_{\hat{s}} \in \hat{\mu}$ but$x_{\hat{t}.\hat{s}}\overline{\in}\hat{\mu}$ and $\hat{\mu}(x)+\hat{t}.\hat{s} < \hat{1}$ and $x_{\hat{t}.\hat{s}}\overline{q}\hat{\mu}$ Which contradict the fact that $\hat{\mu}$ is an $(\in, \in\vee q)$-interval-valued fuzzy dot d-ideal of X
SubCase II: If $\hat{\mu}(x*y).\hat{\mu}(y) \geq [0.5,\ 0.5]$, them $\hat{\mu}(x) < \hat{t}.\hat{s} < [0.5,\ 0.5] < \hat{\mu}(x*y).\hat{\mu}(y)$
$\Rightarrow \hat{\mu}(x*y) \geq \hat{t}$ and $\hat{\mu}(y) \geq \hat{s}$
$\Rightarrow (x*y)_{\hat{t}} \in \hat{\mu}$ and $y_{\hat{s}} \in \hat{\mu}$ but$\hat{\mu}(x) \leq \hat{t}.\hat{s}$ and $\hat{\mu}(x)+\hat{t}.\hat{s} < \hat{1}$ that is $x_{\hat{t}.\hat{s}}\overline{\in}\hat{\mu}$ and $x_{\hat{t}.\hat{s}}\overline{q}\hat{\mu}$ that is $x_{\hat{t}.\hat{s}}\overline{\in\vee q}\hat{\mu}$
Which is again a contradiction, therefore in both subcases
Therefore we must have$\hat{\mu}(x*y) \geq rmin\{\hat{\mu}(x).\hat{\mu}(y), \hat{1}-\hat{\mu}(x).\hat{\mu}(y)\}$
(iii) Proof is similar to case (ii) above

Theorem 0.51 *A fuzzy subset $\hat{\mu}$ of d- algebra X is an $(\in, \in\vee q)$-interval-valued fuzzy dot d-ideal of X and $\hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]\ \forall x \in X$ then $\hat{\mu}$ is also an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X.*

Proof Let $\hat{\mu}$ be an $(\in, \in\vee q)$-interval-valued fuzzy dot d-ideal of X and $\hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]\ \forall x \in X$
Let $x_{\hat{t}} \in \hat{\mu} \Rightarrow \hat{\mu}(x) \geq \hat{t}$

$\Rightarrow \hat{t} \leq \hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$ and also $\hat{\mu}(0) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$
$\hat{\mu}(0) + \hat{t} < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] + [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] = [\sqrt{5}-1, \sqrt{5}-1] \Rightarrow \hat{\mu}(0) + \hat{t} \not> \hat{1}$
$\Rightarrow 0_{\hat{t}}\overline{q}\hat{\mu}$ therefore $x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}}\overline{q}\hat{\mu}$
Since $\hat{\mu}$ be an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X, therefore we must have $x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$
Again let $(x*y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$
$\Rightarrow \hat{t} \leq \hat{\mu}(x*y) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$ and $\hat{t} \leq \hat{\mu}(y) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$
$\Rightarrow \hat{t}.\hat{s} \leq \hat{\mu}(x*y).\hat{\mu}(y) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}].[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$ and
$\Rightarrow \hat{t}.\hat{s} \leq [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}].[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$ and $\hat{\mu}$ is also an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X $\hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$
$\Rightarrow \mu(x) + \hat{t}.\hat{s} < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}].[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] + [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] = \hat{1}$
$\Rightarrow \hat{\mu}(x) + \hat{t}.\hat{s} < \hat{1}$
$\Rightarrow x_{\hat{t}.\hat{s}}\overline{q}\hat{\mu}$
therefore $(x*y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{\hat{t}.\hat{s}}\overline{q}\hat{\mu}$
since $\hat{\mu}$ is an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X,therefore we must have $x_{\hat{t}.\hat{s}} \in \hat{\mu}$.
Therefore $(x*y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{\hat{t}.\hat{s}} \in \hat{\mu}$
Similarly we can prove that $x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x*y)_{\hat{t}.\hat{s}} \in \hat{\mu}$
Hence $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X.

Theorem 0.52 *Let $\hat{\mu}$ be a left fuzzy relation on a fuzzy subset $\hat{\sigma}$ of a d-algebra X.If $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of $X \times X$ then $\hat{\sigma}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of d-algebra X.*

Proof Suppose that a left fuzzy relation $\hat{\mu}$ on $\hat{\sigma}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of $X \times X$.
To prove that $\hat{\sigma}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X
Let $x, y \in X$ since $\hat{\sigma}(x) = \hat{\mu}(x, y)$ let $x_{\hat{t}} \in \hat{\sigma} \Rightarrow \hat{\sigma}(x) \geq \hat{t} \Rightarrow \hat{\sigma}(x) = \hat{\mu}(x, y) \geq \hat{t}$
$(x, y)_{\hat{t}} \in \hat{\mu} \Rightarrow (0, 0)_{\hat{t}} \in \hat{\mu}$ [Since $\hat{\mu}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of $X \times X$]
$\Rightarrow \hat{\mu}(0, 0) \geq \hat{t} \Rightarrow \hat{\sigma}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\sigma}$
therefore $x_{\hat{t}} \in \hat{\sigma} \Rightarrow 0_{\hat{t}} \in \hat{\sigma}$
Let $x, x', y, y' \in X$
$\hat{\sigma}(x) = \hat{\mu}(x, y)$
now Let $(x*x')_{\hat{t}}, (x')_{\hat{s}} \in \hat{\sigma}$
$\Rightarrow \hat{\sigma}(x*x') \geq \hat{t}$ and $\hat{\sigma}(x') \geq \hat{s}$
Now $\hat{\sigma}(x) = \hat{\mu}(x, y) \geq \hat{\mu}((x, y)*(x', y')).\hat{\mu}(x', y')$[Since $\hat{\mu}$ is also an interval-valued fuzzy dot d-ideal of $X \times X$, see Theorem 0.33]
$= \hat{\mu}(x*x', y*y').\hat{\mu}(x', y') = \hat{\sigma}(x*x').\hat{\sigma}(x') \geq \hat{t}.\hat{s}$
$\Rightarrow (x)_{\hat{t}.\hat{s}} \in \hat{\sigma}$
$(x*x')_{\hat{t}}, (x')_{\hat{s}} \in \hat{\sigma} \Rightarrow (x)_{\hat{t}.\hat{s}} \in \hat{\sigma}$
Again let $x_{\hat{t}}, x'_{\hat{s}} \in \hat{\sigma} \Rightarrow \hat{\sigma}(x) \geq \hat{t}$ and $\hat{\sigma}(x') \geq \hat{s}$
$\hat{\sigma}(x*x') = \hat{\mu}(x*x', y*y')$
$= \hat{\hat{\mu}}((x, y)*(x', y')) \geq \hat{\mu}(x, y).\hat{\mu}(x', y') \geq \hat{\sigma}(x).\hat{\sigma}(x') \geq \hat{t}.\hat{s}$
$(x*x')_{\hat{t}.\hat{s}} \in \hat{\sigma}$
$x_{\hat{t}}, x'_{\hat{s}} \in \hat{\sigma} \Rightarrow (x*x')_{\hat{t}.\hat{s}} \in \hat{\sigma}$
Hence from above $\hat{\sigma}$ is an $(\in, \in)$-interval-valued fuzzy dot d-ideal of X

4 Homomorphism of d-algebras and ($\in, \in \vee q$)-interval-valued Fuzzy Dot d-ideals

Definition 0.53 *Let X and X' be two d-algebras, then a mapping $f : X \to X'$is said to be homomorphism if $f(x * y) = f(x) * f(y) \forall x, y \in X$.*

Theorem 0.54 *Let X and X' be two d-algebras and $f : X \to X'$ be a homomorphism. Then $f(0) = 0'$*

Proof Let $x \in X$ therefore $f(x) \in X'$. Now $f(0) = f(x * x) = f(x) * f(x) = 0 * 0 = 0'$.

Theorem 0.55 *Let $f : X \to X'$ be an onto homomorphism of d-algebras, $\hat{\nu}$ be a ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal of X', then the pre-image $f^{-1}(\hat{\nu})$ of $\hat{\nu}$ under f is an ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal of X.*

Proof $f^{-1}(\hat{\nu})$ is defined as $f^{-1}(\hat{\nu})(x) = \hat{\nu}(f(x)) \forall x \in X$
Let $\hat{\nu}$ be an ($\in, \in$)-interval-valued fuzzy dot d-ideal of X'
Let $x \in X$ such that $x_{\hat{t}} \in f^{-1}(\hat{\nu}) \Rightarrow f^{-1}(\hat{\nu})(x) \geq \hat{t} \Rightarrow \hat{\nu}f(x) \geq \hat{t} \Rightarrow (f(x))_{\hat{t}} \in \hat{\nu} \Rightarrow x'_{\hat{t}} \in \hat{\nu}$ where $f(x) = x'$
$\Rightarrow 0'_{\hat{t}} \in \vee q\hat{\nu}$ [Since $\hat{\nu}$ is an ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal of X']
$\Rightarrow [f(0)]_{\hat{t}} \in \vee q\hat{\nu} \Rightarrow \hat{\nu}f(0) \geq \hat{t}$ or $\hat{\nu}f(0) + \hat{t} \geq \hat{1} \Rightarrow f^{-1}(\hat{\nu})(0) \geq \hat{t}$ or $f^{-1}(\hat{\nu})(0) + \hat{t} \geq \hat{1}$
$\Rightarrow 0_{\hat{t}} \in f^{-1}(\hat{\nu})$ or $0_{\hat{t}}qf^{-1}(\hat{\nu})$
$\Rightarrow 0_{\hat{t}} \in \vee qf^{-1}(\hat{\nu})$
Therefore$x_{\hat{t}} \in f^{-1}(\hat{\nu}) \Rightarrow 0_{\hat{t}} \in \vee qf^{-1}(\hat{\nu})$
Let $x, y \in X$ such that $(x * y)_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu})$
$\Rightarrow f^{-1}(\hat{\nu})(x * y) \geq \hat{t}$ and $f^{-1}(\hat{\nu})(y) \geq \hat{s}$
$\Rightarrow \hat{\nu}f(x * y) \geq \hat{t}$ and $\hat{\nu}f(y) \geq \hat{t}$
$\Rightarrow (f(x * y))_{\hat{t}} \in \hat{\nu}$ and $(f(y))_{\hat{s}} \in \hat{\nu}$
$\Rightarrow (f(x) * f(y))_{\hat{t}} \in \hat{\nu}$ and $(f(y))_{\hat{s}} \in \hat{\nu}$ [Since f is homomorphism]
$\Rightarrow (f(x))_{\hat{t}.\hat{s}} \in \vee q\hat{\nu}$ [Since $\hat{\nu}$ is an ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal of X']
$\Rightarrow \hat{\nu}f(x) \geq \hat{t}.\hat{s}$ or $\hat{\nu}f(x) + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow f^{-1}(\hat{\nu})(x) \geq \hat{t}.\hat{s}$ or $f^{-1}(\nu)(x) + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow x_{\hat{t}.\hat{s}} \in f^{-1}(\hat{\nu})$ or $x_{\hat{t}.\hat{s}}qf^{-1}(\hat{\nu})$
$\Rightarrow x_{\hat{t}.\hat{s}} \in \vee qf^{-1}(\hat{\nu})$ Therefore $(x * y)_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu}) \Rightarrow x_{\hat{t}.\hat{s}} \in \vee qf^{-1}(\hat{\nu})$
Again let $x, y \in X$ such that $x_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu})$
$\Rightarrow f^{-1}(\hat{\nu})(x) \geq \hat{t}$ and $f^{-1}(\hat{\nu})(y) \geq \hat{s}$
$\Rightarrow \hat{\nu}f(x) \geq \hat{t}$ and $\hat{\nu}f(y) \geq \hat{s}$
$\Rightarrow (f(x)_{\hat{t}} \in \hat{\nu}$ and $(f(y))_{\hat{s}} \in \hat{\nu}$
$\Rightarrow (f(x) * f(y))_{\hat{t}.\hat{s}} \in \vee q\hat{\nu}$ [Since $\hat{\nu}$ is an ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal of X']
$\Rightarrow \hat{\nu}(f(x) * f(y)) \geq \hat{t}.\hat{s}$ or $\hat{\nu}(f(x) * f(y)) + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow \hat{\nu}(f(x * y)) \geq \hat{t}.\hat{s}$ or $\hat{\nu}(f(x * y)) + \hat{t} \geq \hat{1}$ [Since f is homomorphism]
$\Rightarrow f^{-1}(\hat{\nu})(x * y) \geq \hat{t}.\hat{s}$ or $f^{-1}(\hat{\nu})(x * y) + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow (x * y)_{\hat{t}.\hat{s}} \in f^{-1}(\hat{\nu})$ or $(x * y)_{\hat{t}.\hat{s}}qf^{-1}(\hat{\nu})$
$\Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \vee qf^{-1}(\hat{\nu})$
Therefore $x_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu}) \Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \vee qf^{-1}(\nu)$
Hence $f^{-1}(\hat{\nu})$ is an ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal of X.

Theorem 0.56 *An onto homomorphic image of an ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal with the sup property is an ($\in, \in \vee q$)-interval-valued fuzzy dot d-ideal.*

Proof Let $f : X \to X'$ be an onto homomorphism of d-algebras, $\hat{\mu}$ be an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X, and the image of $\hat{\mu}$ under f be $f(\hat{\mu})$. To prove $f(\hat{\mu})$ is an $(\in, \in \vee q)$-fuzzy dot d-ideal of X'
Since $0 \in f^{-1}(0')$ therefore, for any $x', y' \in X'$, let $x_0, y_0 \in X$ such that

$$\hat{\mu}(x_0) = \sup_{z \in f^{-1}(x')} \hat{\mu}(z) \qquad \hat{\mu}(y) = \sup_{z \in f^{-1}(y')} \hat{\mu}(z)$$

and

$$\hat{\mu}(x_0 * y_0) = \sup_{z \in f^{-1}(x' * y')} \hat{\mu}(z)$$

then, let

$$x'_{\hat{t}} \in f(\hat{\mu}) \Rightarrow f(\hat{\mu})(x') \geq \hat{t} \Rightarrow \sup_{z \in f^{-1}(x')} \hat{\mu}(z) = \hat{t}$$

$$\Rightarrow \hat{\mu}(x_0) \geq \hat{t} \Rightarrow (x_0)_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \vee q \hat{\mu}$$

[Since $\hat{\mu}$ be an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X]

$$\Rightarrow \hat{\mu}(0) \geq \hat{t} \qquad \text{or} \qquad \hat{\mu}(0) + \hat{t} \geq \hat{1}$$

$$\Rightarrow \sup_{z \in f^{-1}(0')} \hat{\mu}(z) \geq \hat{t} \text{ or } \sup_{z \in f^{-1}(0')} \hat{\mu}(z) + \hat{t} \geq \hat{1}$$

$$\Rightarrow f(\hat{\mu})(0') \geq \hat{t} \qquad \text{or} \qquad f(\hat{\mu})(0') + \hat{t} \geq \hat{1}$$

$$\Rightarrow 0'_{\hat{t}} \in f(\hat{\mu}) \qquad \text{or} \qquad 0'_{\hat{t}} q f(\hat{\mu})$$

$$\Rightarrow 0'_{\hat{t}} \in \vee q f(\hat{\mu})$$

Therefore $\Rightarrow x'_{\hat{t}} \in f(\hat{\mu}) \Rightarrow 0'_{\hat{t}} \in \vee q f(\hat{\mu})$ Again, let$(x' * y')_{\hat{t}}, y'_{\hat{s}} \in f(\hat{\mu}) \Rightarrow f(\hat{\mu})(x' * y') \geq \hat{t}, f(\hat{\mu})(y') \geq \hat{s}$

$$\Rightarrow \sup_{z \in f^{-1}(x' * y')} \hat{\mu}(z) \geq \hat{t}, \sup_{z \in f^{-1}(y')} \hat{\mu}(z) \geq \hat{s}$$

$$\Rightarrow \hat{\mu}(x_0 * y_0) \geq \hat{t}, \hat{\mu}(y_0) \geq \hat{s}$$

$$\Rightarrow (x_0 * y_0)_{\hat{t}} \in \hat{\mu}, (y_0)_{\hat{s}} \in \hat{\mu} \Rightarrow (x_0)_{\hat{t}.\hat{s}} \in \vee q \hat{\mu}$$

[Since $\hat{\mu}$ be an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X,]

$$\Rightarrow \hat{\mu}(x_0) \geq \hat{t}.\hat{s} \qquad \text{or} \qquad \hat{\mu}(x_0) + \hat{t}.\hat{s} \geq \hat{1}$$

$$\Rightarrow \sup_{z \in f^{-1}(x')} \hat{\mu}(z) \geq \hat{t}.\hat{s} \text{ or } \sup_{z \in f^{-1}(x')} \hat{\mu}(z) + \hat{t}.\hat{s} \geq \hat{1}$$

$\Rightarrow f(\hat{\mu})(x') \geq \hat{t}.\hat{s}$ or $f(\hat{\mu})(x') + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow (x')_{\hat{t}.\hat{s}} \in f(\hat{\mu})$ or $(x')_{\hat{t}.\hat{s}} q f(\mu)$
$\Rightarrow (x')_{\hat{t}.\hat{s}} \in \vee q f(\hat{\mu})$
$\Rightarrow (x' * y')_{\hat{t}}, y'_{\hat{s}} \in f(\hat{\mu}) \Rightarrow (x')_{\hat{t}.\hat{s}} \in \vee q f(\hat{\mu})$ Similarly we can prove
$x'_{\hat{t}}, y'_{\hat{s}} \in f(\hat{\mu}) \Rightarrow (x' * y')_{\hat{t}.\hat{s}} \in \vee q f(\hat{\mu})$
Hence from above $f(\hat{\mu})$ is an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X.

Theorem 0.57 *Let $f : X \to X'$ be an onto homomorphism of d-algebras, $\hat{\mu}$ be a fuzzy subset of X' such that $f^{-1}(\hat{\mu})$ is an $(\in, \in \vee q)$- fuzzy dot d-ideal of X, then $\hat{\mu}$ is also an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X'.*

Proof Let $f^{-1}(\hat{\mu})$ be an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X
Let $x', y' \in X'$ since f is onto so there exists $x, y \in X$ such that $f(x) = x', f(y) = y'$ and also f is homomorphism therefore $f(x * y) = f(x) * f(y) = x' * y'$
Let $x'_{\hat{t}} \in \hat{\mu}$ where $\hat{t} \in D[0\ 1]$
Therefore $\hat{\mu}(x') \geq \hat{t} \Rightarrow \hat{\mu}(f(x)) \geq \hat{t}$
$\Rightarrow f^{-1}(\hat{\mu})(x) \geq \hat{t}$
$\Rightarrow x_{\hat{t}} \in f^{-1}(\hat{\mu})$
$\Rightarrow 0_t \in \vee q f^{-1}(\hat{\mu})$ [Since $f^{-1}(\hat{\mu})$ is an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X.)
$\Rightarrow f^{-1}(\hat{\mu})(0) \geq \hat{t}$ or $f^{-1}(\hat{\mu})(0) + \hat{t} \geq \hat{1}$
$\Rightarrow \hat{\mu}(f(0)) \geq \hat{t}$ or $\hat{\mu}(f(0)) + \hat{t} \geq \hat{1}$
$\Rightarrow \hat{\mu}(0') \geq \hat{t}$ or $\hat{\mu}(0') + \hat{t} \geq \hat{1}$
$\Rightarrow 0'_{\hat{t}} \in \hat{\mu}$ or $0'_{\hat{t}} q \hat{\mu}$
$\Rightarrow 0'_{\hat{t}} \in \vee q \hat{\mu}$
Therefore $x'_{\hat{t}} \in \hat{\mu} \Rightarrow 0'_{\hat{t}} \in \vee q \hat{\mu}$
Again let $(x' * y')_{\hat{t}}, y'_{\hat{s}} \in \hat{\mu}$ where $\hat{t}, \hat{s} \in D[0\ 1]$
Therefore $\hat{\mu}(x' * y') \geq \hat{t}$ and $\hat{\mu}(y') \geq \hat{s}$
$\Rightarrow \hat{\mu}(f(x * y)) \geq \hat{t}$ and $\hat{\mu}(f(y)) \geq \hat{s}$
$\Rightarrow f^{-1}(\hat{\mu})(x * y) \geq \hat{t}$ and $f^{-1}(\hat{\mu})(y) \geq \hat{s}$
$\Rightarrow (x * y)_{\hat{t}} \in f^{-1}(\hat{\mu})$ and $y_{\hat{t}} \in f^{-1}(\hat{\mu})$
$\Rightarrow x_{\hat{t}.\hat{s}} \in \vee q f^{-1}(\hat{\mu})$ [Since $f^{-1}(\hat{\mu})$ is an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X.)
$\Rightarrow f^{-1}(\hat{\mu})(x) \geq \hat{t}.\hat{s}$ or $f^{-1}(\hat{\mu})(x) + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow \hat{\mu}(f(x)) \geq \hat{t}.\hat{s}$ or $\hat{\mu}(f(x)) + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow \hat{\mu}(x') \geq \hat{t}.\hat{s}$ or $\hat{\mu}(x') + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow x'_{\hat{t}.\hat{s}} \in \hat{\mu}$ or $x'_{\hat{t}.\hat{s}} q \hat{\mu}$
$\Rightarrow x'_{\hat{t}.\hat{s}} \in \vee q \hat{\mu}$
$\Rightarrow (x' * y')_{\hat{t}}, y'_{\hat{s}} \in \hat{\mu} \Rightarrow x'_{\hat{t}.\hat{s}} \in \vee q \hat{\mu}$
Again let $x', y' \in X'$ such that $(x')_{\hat{t}}, y'_{\hat{t}} \in \hat{\mu}$
therefore $\hat{\mu}(x') \geq \hat{t}$ and $\hat{\mu}(y') \geq \hat{s}$
$\Rightarrow \hat{\mu}(f(x)) \geq \hat{t}$ and $\hat{\mu}(f(y)) \geq \hat{s}$
$\Rightarrow f^{-1}(\hat{\mu})(x) \geq \hat{t}$ and $f^{-1}(\hat{\mu})(y) \geq \hat{s}$
$\Rightarrow x_{\hat{t}} \in f^{-1}(\hat{\mu})$ and $y_{\hat{s}} \in f^{-1}(\hat{\mu})$
$\Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \vee q f^{-1}(\hat{\mu})$ [Since $f^{-1}(\hat{\mu})$ is an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X.)
$\Rightarrow f^{-1}(\hat{\mu})(x * y) \geq \hat{t}.\hat{s}$ or $f^{-1}(\hat{\mu})(x * y) + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow \hat{\mu}(f(x * y)) \geq \hat{t}.\hat{s}$ or $\hat{\mu}(f(x * y)) + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow \hat{\mu}(f(x) * f(y)) \geq \hat{t}.\hat{s}$ or $\hat{\mu}(f(x) * f(y)) + \hat{t}.\hat{s} \geq \hat{1}$
$\Rightarrow (x' * y')_{\hat{t}.\hat{s}} \in \hat{\mu}$ or $(x' * y')_{\hat{t}.\hat{s}} q \hat{\mu}$
$\Rightarrow (x' * y')_{\hat{t}.\hat{s}} \in \vee q \hat{\mu}$
$\Rightarrow x'_{\hat{t}}, y'_{\hat{s}} \in \hat{\mu} \Rightarrow (x' * y')_{\hat{t}.\hat{s}} \in \vee q \hat{\mu}$
Hence $\hat{\mu}$ is an $(\in, \in \vee q)$-interval-valued fuzzy dot d-ideal of X'.

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