

## $(\in, \in \vee q)$ -Interval-valued Fuzzy Dot d-ideals of d-algebras

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**Abstract.** The concept of  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideals in d-algebras is introduced. Relationship among interval-valued fuzzy d-ideal, interval-valued fuzzy dot d-ideal,  $(\in, \in)$ -interval-valued fuzzy d-ideal,  $(\in, \in)$ -interval-valued fuzzy dot d-ideal, and  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideals are discussed. Conditions for an interval-valued fuzzy d-ideal to be an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideals are given. Some properties of interval-valued fuzzy relations and interval-valued fuzzy ideals under homomorphism are investigated.

### 1 Introduction

In 1991 Xi [12] applied the concept of fuzzy sets to BCK-algebras which are introduced by Imai and Iseki[7]in 1996. Neggers and Kim [11]introduced the class of d-algebras which is a generalisation of BCK-algebras and investigated relation between d-algebras and BCK- algebras. Akram and Dar[1] introduced the concepts fuzzy d-algebra, they introduced fuzzy subalgebra and fuzzy d-ideals of d-algebras. Kim [8]introduced the notion of a fuzzy dot subalgebra of d-algebra and investigated some related properties. Bhakat and Das [5, 6] used the relation of "belongs to" and quasi-coincident" between fuzzy point and fuzzy set to introduced the concept of  $(\in, \in \vee q)$ -fuzzy subgroup and  $(\in, \in \vee q)$ -fuzzy subring. Al-Shehrie[2] introduced the notion of fuzzy dot d-ideals of a d-algebra. Interval-valued fuzzy sets were first introduced by Zadeh [15] in 1975. After that many researchers consider the interval-valued fuzzification of ideals and subalgebras in BG/ BCK-algebras. The concept of  $(\in, \in \vee q)$ -interval-valued fuzzification of ideals in ring was introduced in [9]. Here in this paper, we introduce the notion of  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of d-algebra and then we investigates some of its interesting properties.

### 2 Preliminaries

**Definition 0.1** [1, 8] A d-algebra is a non-empty set  $X$  with a constant  $0$  and a binary operation  $*$  satisfying the following axioms:

- (i)  $x * x = 0$
- (ii)  $0 * x = 0$
- (iii)  $x * y = 0$  and  $y * x = 0 \Rightarrow x = y$  for all  $x, y \in X$ .

For brevity we also call  $X$  a d-algebra.

**Definition 0.2** [8] A non-empty subset  $S$  of a d-algebra  $X$  is called a subalgebra of  $X$  if  $x * y \in S$ , for all  $x, y \in S$ .

**Definition 0.3** [2] A nonempty subset  $I$  of a d-algebra  $X$  is called an ideal of  $X$  if

- (i)  $0 \in I$
- (ii)  $x * y \in I$  and  $y \in I \Rightarrow x \in I$
- (iii)  $x \in I$  and  $y \in X \Rightarrow x * y \in I$

**Definition 0.4** [2, 8] A fuzzy subset  $\mu$  of  $X$  is called a fuzzy dot subalgebra of a  $d$ -algebra  $X$  if  $\mu(x * y) \geq \mu(x) \cdot \mu(y)$  for all  $x, y \in X$ .

**Definition 0.5** [2] A fuzzy subset  $\mu$  of  $X$  is called a fuzzy  $d$ -ideal of  $X$  if it satisfies the following conditions:

- (i)  $\mu(0) \geq \mu(x)$
- (ii)  $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$
- (iii)  $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

**Definition 0.6** [2] A fuzzy subset  $\mu$  of  $X$  is called a fuzzy dot  $d$ -ideal of  $X$  if it satisfies the following conditions:

- (i)  $\mu(0) \geq \mu(x)$
- (ii)  $\mu(x) \geq \mu(x * y) \cdot \mu(y)$
- (iii)  $\mu(x * y) \geq \mu(x) \cdot \mu(y)$  for all  $x, y \in X$ .

**Definition 0.7** [5] A fuzzy set  $\mu$  of the form

$$\mu(y) = \begin{cases} t & \text{if } y = x, \quad t \in (0, 1] \\ 0 & \text{if } y \neq x \end{cases}$$

is called a fuzzy point with support  $x$  and value  $t$  and it is denoted by  $x_t$ .

**Definition 0.8** [5] Let  $\mu$  be a fuzzy set in  $X$  and  $x_t$  be a fuzzy point then

- (i) If  $\mu(x) \geq t$  then we say  $x_t$  belongs to  $\mu$  and write  $x_t \in \mu$
- (ii) If  $\mu(x) + t > 1$  then we say  $x_t$  quasi coincident  $\mu$  and write  $x_t q \mu$
- (iii) If  $x_t \in \vee q \mu \Leftrightarrow x_t \in \mu$  or  $x_t q \mu$
- (iv) If  $x_t \in \wedge q \mu \Leftrightarrow x_t \in \mu$  and  $x_t q \mu$

The symbol  $x_t \bar{\alpha} \mu$  means  $x_t \alpha \mu$  does not hold and  $\overline{\in \wedge q}$  means  $\bar{\in} \vee \bar{q}$

For a fuzzy point  $x_t$ . and a fuzzy set  $\mu$  in set  $X$ , Pu and Liu [10] gave meaning to the symbol  $x_t \alpha \mu$  where  $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$

**Definition 0.9** A fuzzy subset  $\mu$  of  $X$  is called a  $(\in, \in \vee q)$ -fuzzy  $d$ -ideal of  $X$  if it satisfies the following conditions:

- (i)  $x_t \in \mu \Rightarrow 0_t \in \vee q \mu$
  - (ii)  $(x * y)_t, y_s \in \mu \Rightarrow x_{m(t,s)} \in \vee q \mu$
  - (iii)  $x_t, y_s \in \mu \Rightarrow (x * y)_{m(t,s)} \in \vee q \mu \quad \forall t, s \in (0, 1], \quad \forall x, y \in X$
- Where  $m(t, s) = \min\{t, s\}$

**Definition 0.10** A fuzzy subset  $\mu$  of  $X$  is called a  $(\in, \in \vee q)$ -fuzzy dot  $d$ -ideal of  $X$  if it satisfies the following conditions:

1.  $x_t \in \mu \Rightarrow 0_t \in \vee q \mu$
2.  $(x * y)_t, y_s \in \mu \Rightarrow x_{t.s} \in \vee q \mu$
3.  $x_t, y_s \in \mu \Rightarrow (x * y)_{t.s} \in \vee q \mu \quad \forall t, s \in (0, 1], \quad \forall x \in X$

**Example 0.11** Consider  $d$ -algebra  $X = \{0, a, b, c\}$  with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Define  $\mu : X \rightarrow [0, 1]$  by  $\mu$  by  $\mu(0) = 0.9, \mu(a) = \mu(b) = 0.8, \mu(c) = 0.7$ , then it is easy to verify that  $\mu$  is  $(\in, \in \vee q)$ -fuzzy dot  $d$ -ideal  $X$ .

**Definition 0.12** [2] Let  $\lambda$  and  $\mu$  be two fuzzy sets in a set  $X$ . The their cartesian product  $\lambda \times \mu : X \times X \rightarrow [0, 1]$  is defined by  $(\lambda \times \mu)(x, y) = \lambda(x) \cdot \mu(y)$ , for all  $x, y \in X$ . Let  $\sigma$  be a fuzzy subset of  $X$ , then the strongest fuzzy  $\sigma$  relation on  $d$  algebra  $X$  is the fuzzy subset  $\mu_\sigma$  of  $X \times X$  given by  $\mu_\sigma(x, y) = \sigma(x) \cdot \sigma(y) \forall x, y \in X$ . A fuzzy relation  $\mu$  on  $d$  algebra  $X$  is called a fuzzy  $\sigma$  product relation if  $\mu(x, y) \geq \sigma(x) \cdot \sigma(y) \forall x, y \in X$ . A fuzzy relation  $\mu$  on  $d$  algebra  $X$  is called a left fuzzy relation on  $\sigma$  if  $\mu(x, y) = \sigma(x) \forall x, y \in X$ . Note that a left fuzzy relation on  $\sigma$  is a fuzzy  $\sigma$  product relation.

**Remark 0.13** If  $X$  and  $Y$  be two  $d$ -algebras, then  $X \times X$  is also a  $d$ -algebra under the binary operation  $'*'$  defined in  $X \times X$  by  $(x, y) * (p, q) = (x * p, y * q)$  for all  $(x, y), (p, q) \in X \times X$ .

### 3 $(\in, \in \vee q)$ -interval-valued fuzzy sets

The notion of interval-valued fuzzy set was introduced by L.A.Zadeh[25]. To consider the notion of interval-valued fuzzy sets, we need the following notations. By an interval number  $\hat{a}$ , we mean an interval  $[\underline{a}, \bar{a}]$ , where  $0 \leq \underline{a} \leq \bar{a} \leq 1$ . Let  $D[0, 1]$  denote the set of all such interval numbers of  $[0, 1]$ . Define on  $D[0, 1]$  the relations  $\leq, =, <, \cdot$  by

- $\hat{a}_1 \leq \hat{a}_2 \Leftrightarrow \underline{a}_1 \leq \underline{a}_2$  and  $\bar{a}_1 \leq \bar{a}_2$
- $\hat{a}_1 = \hat{a}_2 \Leftrightarrow \underline{a}_1 = \underline{a}_2$  and  $\bar{a}_1 = \bar{a}_2$
- $\hat{a}_1 < \hat{a}_2 \Leftrightarrow \underline{a}_1 < \underline{a}_2$  and  $\bar{a}_1 < \bar{a}_2$
- $\hat{a}_1 \cdot \hat{a}_2 \Leftrightarrow [\min(\underline{a}_1 \underline{a}_2, \underline{a}_1 \bar{a}_2, \bar{a}_1 \underline{a}_2, \bar{a}_1 \bar{a}_2), \max(\underline{a}_1 \underline{a}_2, \underline{a}_1 \bar{a}_2, \bar{a}_1 \underline{a}_2, \bar{a}_1 \bar{a}_2)] = [\underline{a}_1 \underline{a}_2, \bar{a}_1 \bar{a}_2]$
- $k\hat{a} = [k\underline{a}, k\bar{a}]$  where  $0 \leq k \leq 1$

Now consider two intervals  $\hat{a}_1 = [\underline{a}_1, \bar{a}_1], \hat{a}_2 = [\underline{a}_2, \bar{a}_2] \in D[0, 1]$  then we define refine minimum  $rmin$  as  $rmin(\hat{a}_1, \hat{a}_2) = [\min(\underline{a}_1, \underline{a}_2), \min(\bar{a}_1, \bar{a}_2)]$  and refine maximum as

$rmax$   $rmax(\hat{a}_1, \hat{a}_2) = [\max(\underline{a}_1, \underline{a}_2), \max(\bar{a}_1, \bar{a}_2)]$  generally if  $\hat{a}_i = [\underline{a}_i, \bar{a}_i], \hat{b}_i = [\underline{b}_i, \bar{b}_i] \in D[0, 1]$  for  $i=1,2,3,\dots$  then we define

$rmax(\hat{a}_i, \hat{b}_i) = [\max(\underline{a}_i, \underline{b}_i), \max(\bar{a}_i, \bar{b}_i)]$  and  $rmin(\hat{a}_i, \hat{b}_i) = [\min(\underline{a}_i, \underline{b}_i), \min(\bar{a}_i, \bar{b}_i)]$

and  $rinf_i(\hat{a}_i) = [\wedge_i \underline{a}_i, \wedge_i \bar{a}_i]$  and  $rsup_i(\hat{a}_i) = [\vee_i \underline{a}_i, \vee_i \bar{a}_i]$

$(D[0, 1], \leq)$  is a complete lattice with  $\wedge = rmin, \vee = rmax, \hat{0} = [0 0]$  and  $\hat{1} = [1 1]$  being the least and the greatest element respectively.

**Definition 0.14** An interval-valued fuzzy set defined on a non empty set  $X$  as an objects having the form  $\hat{\mu} = \{x, [\underline{\mu}(x), \bar{\mu}(x)]\}, \forall x \in X$  where  $\underline{\mu}$  and  $\bar{\mu}$  are two fuzzy sets in  $X$  such that  $\underline{\mu}(x) \leq \bar{\mu}(x)$  for all  $x \in X$ . Let  $\hat{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)], \forall x \in X$ . Then  $\hat{\mu}(x) \in D[0 1], \forall x \in X$   
If  $\hat{\mu}$  and  $\hat{\nu}$  be two interval-valued fuzzy sets in  $X$ , then we define

- $\hat{\mu} \subset \hat{\nu} \Leftrightarrow$  for all  $\underline{\mu}(x) \leq \underline{\nu}(x)$  and  $\bar{\mu}(x) \leq \bar{\nu}(x)$ .

- $\hat{\mu} = \hat{\nu} \Leftrightarrow$  for all  $\underline{\mu}(x) = \underline{\nu}(x)$  and  $\overline{\mu}(x) = \overline{\nu}(x)$ .
- $(\hat{\mu} \cup \hat{\nu})(x) = \hat{\mu}(x) \vee \hat{\nu}(x) = [\max\{\underline{\mu}(x), \underline{\nu}(x)\}, \max\{\overline{\mu}(x), \overline{\nu}(x)\}]$ .
- $(\hat{\mu} \cap \hat{\nu})(x) = \hat{\mu}(x) \wedge \hat{\nu}(x) = [\min\{\underline{\mu}(x), \underline{\nu}(x)\}, \min\{\overline{\mu}(x), \overline{\nu}(x)\}]$ .
- $\hat{\mu}^c(x) = [1 - \overline{\mu}(x), 1 - \underline{\mu}(x)]$ .

**Definition 0.15** Let  $\hat{\mu}$  be an interval-valued fuzzy set in  $X$ . Then, for every  $[0, 0] < \hat{t} \leq [1, 1]$ , the crisp set  $\hat{\mu}_{\hat{t}} = \{x \in X \mid \hat{\mu}(x) \geq \hat{t}\}$  is called the level subset of  $\hat{\mu}$ .

**Definition 0.16** A interval-valued fuzzy subset  $\hat{\mu}$  of  $X$  is called an interval-valued fuzzy dot subalgebra of a  $d$ -algebra  $X$  if  $\hat{\mu}(x * y) \geq \hat{\mu}(x) \cdot \hat{\mu}(y)$  for all  $x, y \in X$ .

**Definition 0.17** An interval-valued fuzzy set  $\hat{\mu}$  in  $d$ -algebra  $X$  is called an interval-valued fuzzy ideal of  $X$  if it satisfies

- (i)  $\hat{\mu}(0) \geq \hat{\mu}(x)$
- (ii)  $\hat{\mu}(x) \geq r\min\{\hat{\mu}(x * y), \hat{\mu}(y)\}$  for all  $x, y \in X$
- (ii)  $\hat{\mu}(x * y) \geq r\min\{\hat{\mu}(x), \hat{\mu}(y)\}$  for all  $x, y \in X$

**Definition 0.18** An interval-valued fuzzy set  $\hat{\mu}$  in  $d$ -algebra  $X$  is called an interval-valued fuzzy dot  $d$ -ideal of  $X$  if it satisfies

- (i)  $\hat{\mu}(0) \geq \hat{\mu}(x)$
- (ii)  $\hat{\mu}(x) \geq \hat{\mu}(x * y) \cdot \hat{\mu}(y)$  for all  $x, y \in X$
- (ii)  $\hat{\mu}(x * y) \geq \hat{\mu}(x) \cdot \hat{\mu}(y)$  for all  $x, y \in X$

**Example 0.19** Consider  $d$ -algebra  $X = \{0, a, b, c\}$  with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Define  $\hat{\mu} : X \rightarrow D[0, 1]$  by  $\hat{\mu}(0) = [0.8, 0.9]$ ,  $\hat{\mu}(a) = \hat{\mu}(b) = [0.6, 0.8]$ ,  $\hat{\mu}(c) = [0.35, 0.5]$ , then it is easy to verify that  $\hat{\mu}$  is  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal  $X$ .

**Definition 0.20** Let  $\hat{\mu}(x) = [\underline{\mu}(x), \overline{\mu}(x)]$  and  $\hat{t} = [\underline{t}, \overline{t}] \in D[0, 1]$ , then we define  $\hat{\mu}(x) + \hat{t} = [\underline{\mu}(x) + \underline{t}, \overline{\mu}(x) + \overline{t}] \forall x \in X$ . In particular  $\underline{\mu}(x) + \underline{t} > 1$ , we write  $\hat{\mu}(x) + \hat{t} > [1, 1]$ . Let  $x \in X$ .

An interval-valued fuzzy set  $\hat{\mu}$  of the form

$$\hat{\mu}(y) = \begin{cases} \hat{t} & \text{if } y = x, \hat{t} \in D(0, 1) \\ \hat{0} & \text{if } y \neq x \end{cases}$$

is called an interval-valued fuzzy point with support  $x$  and value  $\hat{t}$  and it is denoted by  $x_{\hat{t}}$ .

**Definition 0.21** ([5]) Let  $\hat{\mu}$  be an interval-value fuzzy set in  $X$  and  $x_{\hat{t}}$  be an interval-value fuzzy point then

- (i) If  $\hat{\mu}(x) \geq \hat{t}$  then we say  $x_{\hat{t}}$  belongs to  $\hat{\mu}$  and write  $x_{\hat{t}} \in \hat{\mu}$
- (ii) If  $\hat{\mu}(x) + \hat{t} > [1, 1]$  then we say  $x_{\hat{t}}$  quasi coincident  $\hat{\mu}$  and write  $x_{\hat{t}} q \hat{\mu}$

(iii) If  $x_{\hat{t}} \in \vee q \hat{\mu} \Leftrightarrow x_t \in \hat{\mu}$  or  $x_{\hat{t}} q \hat{\mu}$

(iv) If  $x_{\hat{t}} \in \wedge q \hat{\mu} \Leftrightarrow x_{\hat{t}} \in \hat{\mu}$  and  $x_{\hat{t}} q \hat{\mu}$

The symbol  $x_{\hat{t}} \overline{\wedge} \hat{\mu}$  means  $x_{\hat{t}} \wedge \hat{\mu}$  does not hold and  $\overline{\wedge} \hat{\mu}$  means  $\overline{\wedge} \hat{\mu}$

For an interval-valued fuzzy point  $x_{\hat{t}}$  and an interval-valued fuzzy set  $\hat{\mu}$  in set  $X$ , Pu and Liu [10] gave meaning to the symbol  $x_{\hat{t}} \alpha \hat{\mu}$  where  $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$

**Definition 0.22** A interval-valued fuzzy set  $\hat{\mu}$  of a  $d$ -algebra  $X$  is said to be  $(\alpha, \beta)$ -interval-valued fuzzy  $d$ -ideal of  $X$ , Where  $\alpha \neq \in \wedge q$  if

(i)  $x_{\hat{t}} \alpha \hat{\mu} \Rightarrow 0_{\hat{t}} \beta \hat{\mu}$

(ii)  $(x * y)_{\hat{t}}, y_{\hat{s}} \alpha \hat{\mu} \Rightarrow x_{rmin(\hat{t}, \hat{s})} \beta \hat{\mu}$

(iii)  $(x)_{\hat{t}}, y_{\hat{s}} \alpha \hat{\mu} \Rightarrow (x * y)_{rmin(\hat{t}, \hat{s})} \beta \hat{\mu}$

for all  $x, y \in X$  where  $[0, 0] < \hat{t}, \hat{s} \leq [1, 1]$ , where  $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$

**Definition 0.23** A interval-valued fuzzy set  $\hat{\mu}$  of a  $d$ -algebra  $X$  is said to be  $(\alpha, \beta)$ -interval-valued fuzzy dot  $d$ -ideal of  $X$ , Where  $\alpha \neq \in \wedge q$  if

(i)  $x_{\hat{t}} \alpha \hat{\mu} \Rightarrow 0_{\hat{t}} \beta \hat{\mu}$

(ii)  $(x * y)_{\hat{t}}, y_{\hat{s}} \alpha \hat{\mu} \Rightarrow x_{(\hat{t}, \hat{s})} \beta \hat{\mu}$

(iii)  $(x)_{\hat{t}}, y_{\hat{s}} \alpha \hat{\mu} \Rightarrow (x * y)_{(\hat{t}, \hat{s})} \beta \hat{\mu}$

for all  $x, y \in X$  where  $[0, 0] < \hat{t}, \hat{s} \leq [1, 1]$

**Definition 0.24** A interval-valued fuzzy set  $\hat{\mu}$  of a  $d$ -algebra  $X$  is said to be  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal of  $X$ , if

(i)  $x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \vee q \hat{\mu}$

(ii)  $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{(\hat{t}, \hat{s})} \in \vee q \hat{\mu}$

(iii)  $(x)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{(\hat{t}, \hat{s})} \in \vee q \hat{\mu}$

for all  $x, y \in X$  where  $[0, 0] < \hat{t}, \hat{s} \leq [1, 1]$

**Example 0.25** Consider  $d$ -algebra  $X$  as in Example 0.11 and  $\hat{\mu}$  by  $\hat{\mu}(0) = [0.75, 0.9], \hat{\mu}(a) = \hat{\mu}(b) = [0.7, 0.8], \hat{\mu}(c) = [0.65, 0.7]$ , then it is easy to verify that  $\hat{\mu}$  is  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal  $X$ .

**Theorem 0.26** Every  $(\in, \in)$ -interval-valued fuzzy  $d$ -ideal of  $d$  algebra  $X$  is an  $(\in, \in)$ -interval-valued fuzzy dot  $d$ -ideal of  $X$ .

**Proof** Straightforward

**Remark 0.27** The converse of Theorem 0.26 is not true as shown in following Example.

**Example 0.28** Consider  $d$ -algebra  $X = \{0, a, b\}$  with the following cayley table.

*	0	a	b
0	0	0	0
a	b	0	b
b	a	a	0

Define  $\hat{\mu} : X \rightarrow D[0, 1]$  by  $\hat{\mu}(0) = [0.6, 0.7]$ ,  $\hat{\mu}(a) = [0.7, 0.8]$ ,  $\hat{\mu}(b) = [0.8, 0.9]$  then it is easy to verify that  $\hat{\mu}$  is  $(\in, \in)$ -interval-valued fuzzy dot d ideal X. Here  $(a * b)_{[0.77, 0.88]}$ ,  $b_{[0.77, 0.88]} \in \hat{\mu}$  But  $a_{[0.77, 0.88]} \notin \hat{\mu}$ . Therefore  $\hat{\mu}$  is not an  $(\in, \in)$ -interval-valued fuzzy d-ideal X.

**Theorem 0.29** Every interval-valued fuzzy d-ideal of a d algebra X is an interval-valued fuzzy dot d-ideal of X.

**Remark 0.30** The converse of Theorem 0.29 is not true as shown in following Example.

**Example 0.31** Consider d-algebra X and  $\hat{\mu}$  as in Example 0.28 then it is easy to verify that  $\hat{\mu}$  is an interval-valued fuzzy dot d ideal X. But  $rmin\{\hat{\mu}(a * b), \hat{\mu}(b)\} = \hat{\mu}(b) = [0.8, 0.9] \not\subseteq \hat{\mu}(a) = [0.7, 0.8]$  Therefore  $\hat{\mu}$  is not an interval-valued fuzzy d-ideal X.

**Theorem 0.32** An interval-valued fuzzy subset  $\hat{\mu}$  of a d algebra X is a interval-valued fuzzy d-ideal iff  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy d-ideal of X

**Proof** Let  $\hat{\mu}$  be an interval-valued fuzzy d-ideal of X.

To prove  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy d-ideal of X.

Let  $x \in X$  such that  $x_{\hat{t}} \in \hat{\mu}$  where  $\hat{t} \in D(0, 1)$  then  $\hat{\mu}(x) \geq \hat{t} \Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x) \geq \hat{t}$  [ Since  $\hat{\mu}$ -is interval-valued fuzzy d-ideal ]

$\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\mu} \Rightarrow x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$

Again let  $x, y \in X$  such that  $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$  where  $\hat{t}, \hat{s} \in D(0, 1)$

then  $\hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$

Now  $\hat{\mu}(x) \geq rmin\{\hat{\mu}(x * y), \hat{\mu}(y)\} \geq rmin(\hat{t}, \hat{s})$  [ Since  $\hat{\mu}$ -is interval-valued fuzzy d-ideal ]

$\Rightarrow x_{rmin(\hat{t}, \hat{s})} \in \hat{\mu}$

$\Rightarrow (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{rmin(\hat{t}, \hat{s})} \in \hat{\mu}$

Again let  $x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$

$\Rightarrow \hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$

Now  $\hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x), \hat{\mu}(y)\} \geq rmin(\hat{t}, \hat{s})$  [ Since  $\hat{\mu}$ -interval-valued fuzzy d-ideal ]

$\Rightarrow (x * y)_{rmin(\hat{t}, \hat{s})} \in \hat{\mu}$

$x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{rmin(\hat{t}, \hat{s})} \in \hat{\mu}$

Hence  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy d-ideal of X.

Conversely suppose  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy d-ideal of X.

to prove  $\hat{\mu}$  is an interval-valued fuzzy d-ideal of X.

Let  $x \in X$  Now since  $x_{\hat{\mu}(x)} \in \hat{\mu}$  therefore  $0_{\hat{\mu}(x)} \in \hat{\mu}$

$\Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x)$  [ Since  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy d-ideal of X ]

Let  $x, y \in X$  again since  $(x * y)_{\hat{\mu}(x * y)} \in \hat{\mu}$  and  $y_{\hat{\mu}(y)} \in \hat{\mu}$

$\Rightarrow x_{rmin\{\hat{\mu}(x * y), \hat{\mu}(y)\}} \in \hat{\mu}$  [ Since  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy d-ideal of X ]

$\Rightarrow \hat{\mu}(x) \geq rmin\{\hat{\mu}(x * y), \hat{\mu}(y)\}$

Let  $x, y \in X$  again since  $x_{\hat{\mu}(x)} \in \hat{\mu}$  and  $y_{\hat{\mu}(y)} \in \mu$

$\Rightarrow (x * y)_{rmin\{\hat{\mu}(x), \hat{\mu}(y)\}} \in \hat{\mu}$  [ Since  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy d-ideal of X ]

$\Rightarrow \hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x), \hat{\mu}(y)\}$

Hence  $\hat{\mu}$  is an interval-valued fuzzy d-ideal of X.

**Theorem 0.33** An interval-valued fuzzy subset  $\hat{\mu}$  of a d algebra X is a interval-valued fuzzy dot d-ideal iff  $\hat{\mu}$  is a  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X

**Proof**  $\hat{\mu}$  be an interval-valued fuzzy dot d-ideal of X. To prove  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X.

Let  $x \in X$  such that  $x_{\hat{t}} \in \hat{\mu}$  where  $\hat{t} \in D(0, 1)$  then  $\hat{\mu}(x) \geq \hat{t}$   
 $\Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x) \geq \hat{t}$  [ Since  $\hat{\mu}$  is an interval-valued fuzzy dot d-ideal ]  
 $\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$   
 $\Rightarrow x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$

Let  $x, y \in X$  such that  $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$  where  $\hat{t}, \hat{s} \in D(0, 1)$   
 then  $\hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$   
 Now  $\hat{\mu}(x) \geq \hat{\mu}(x * y) \cdot \hat{\mu}(y) \geq \hat{t} \cdot \hat{s}$  [ Since  $\hat{\mu}$  is an interval-valued fuzzy dot d-ideal ]  
 $\Rightarrow x_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$   
 $\Rightarrow (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$

Again let  $x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$   
 $\Rightarrow \hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$   
 Now  $\hat{\mu}(x * y) \geq \hat{\mu}(x) \cdot \hat{\mu}(y) \geq \hat{t} \cdot \hat{s}$  [ Since  $\hat{\mu}$  is an interval-valued fuzzy dot d-ideal ]  
 $\Rightarrow (x * y)_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$   
 $\Rightarrow x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$

Hence  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X.

Conversely suppose  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X. to prove  $\hat{\mu}$  is an interval-valued fuzzy dot d-ideal of X.

Let  $x \in X$  and  $\hat{t} = \hat{\mu}(x)$   
 then  $\hat{\mu}(x) \geq \hat{t} \Rightarrow x_{\hat{t}} \in \hat{\mu}$   
 $\Rightarrow 0_{\hat{t}} \in \hat{\mu}$  [ Since  $\hat{\mu}$  is  $(\in, \in)$ -is an interval-valued fuzzy dot d-ideal of X.]  
 $\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x)$

Let  $x, y \in X$  and  $\hat{t} = \hat{\mu}(x * y), \hat{s} = \hat{\mu}(y)$   
 then  $\hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$   
 $\Rightarrow (x * y)_{\hat{t}} \in \hat{\mu}, y_{\hat{s}} \in \hat{\mu}$   
 $\Rightarrow x_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$  [ Since  $\hat{\mu}$  is  $(\in, \in)$ -is an interval-valued fuzzy dot d-ideal of X ]  
 $\Rightarrow \hat{\mu}(x) \geq \hat{t} \cdot \hat{s} = \hat{\mu}(x * y) \cdot \hat{\mu}(y)$

Again let  $\hat{t} = \hat{\mu}(x), \hat{s} = \hat{\mu}(y)$   
 then  $\hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$   
 $\Rightarrow x_{\hat{t}} \in \hat{\mu}, y_{\hat{s}} \in \hat{\mu}$   
 $\Rightarrow (x * y)_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$  [ Since  $\hat{\mu}$  is  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X ]  
 $\Rightarrow \hat{\mu}(x * y) \geq \hat{t} \cdot \hat{s} = \hat{\mu}(x) \cdot \hat{\mu}(y)$

Hence  $\hat{\mu}$  is an interval-valued fuzzy dot d-ideal of X.

**Theorem 0.34** Every interval-valued fuzzy d-ideal of a d algebra X is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X.

**Proof** It follows from Theorem 0.29 and Theorem 0.33.

**Remark 0.35** The converse of Theorem 0.34 is not true as shown in following Example.

**Example 0.36** Consider d-algebra X and  $\hat{\mu}$  as in Example 0.28 then it is easy to verify that  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d ideal X. But  $\min\{\hat{\mu}(a * b), \hat{\mu}(b)\} = \hat{\mu}(b) = [0.8, 0.9] \not\leq \hat{\mu}(a) = [0.7, 0.8]$  Therefore  $\hat{\mu}$  is not an interval-valued fuzzy d-ideal X.

**Theorem 0.37** Every  $(\in, \in)$ -interval-valued fuzzy d-ideal of a d algebra X is an interval-valued fuzzy dot d-ideal of X.

**Proof** It follows from Theorem 0.32 and Theorem 0.29.

**Remark 0.38** The converse of Theorem 0.37 is not true as shown in following Example.

**Example 0.39** Consider  $d$ -algebra  $X$  and  $\hat{\mu}$  as in Example 0.28 then it is easy to verify that  $\hat{\mu}$  is an interval-valued fuzzy dot  $d$  ideal  $X$ . But  $(a * b)_{[0.77, 0.88]}, b_{[0.77, 0.88]} \in \hat{\mu}$  But  $a_{[0.77, 0.88]} \notin \hat{\mu}$ . Therefore  $\hat{\mu}$  is not an  $(\in, \in)$ -interval-valued fuzzy  $d$ -ideal  $X$ .

**Definition 0.40** A fuzzy subset  $\hat{\mu}$  of  $X$  is called a  $(\in, \in)$ -interval-valued fuzzy dot subalgebra if  $x_{\hat{\mu}}, y_{\hat{\mu}} \in \hat{\mu} \Rightarrow (x * y)_{\hat{\mu}} \in \hat{\mu}$  for all  $x, y \in X$ .

**Proposition 0.41** Every  $(\in, \in)$ -interval-valued fuzzy dot  $d$ -ideal of a  $d$ -algebra  $X$  is an  $(\in, \in)$ -interval-valued fuzzy dot sub algebra of  $X$ .

**Remark 0.42** The converse of Proposition 0.41 is not true as shown in following example.

**Example 0.43** Consider  $d$  algebra  $X = \{0, a, b, c\}$  with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Define  $\hat{\mu} : X \rightarrow [0, 1]$  by  $\hat{\mu}(0) = \hat{\mu}(b) = [0.8, 0.9]$ ,  $\hat{\mu}(a) = \hat{\mu}(c) = [0.7, 0.8]$  then it is easy to verify that  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot subalgebra of  $X$ . But  $\hat{\mu}$  is not an  $(\in, \in)$ -interval-valued fuzzy dot  $d$ -ideal of  $X$  because  $\hat{\mu}(a * b) = \hat{\mu}(0) = [0.8, 0.9]$  and  $\hat{\mu}(b) = [0.8, 0.9]$  therefore  $\hat{\mu}(a * b) \geq [0.8, 0.9]$  and  $\hat{\mu}(b) \geq [0.8, 0.9] \not\Rightarrow \hat{\mu}(a) \geq [0.8, 0.9]. [0.9, 0.9] = [0.72, 0.81]$

**Theorem 0.44** Every  $(\in, \in)$ -interval-valued fuzzy dot  $d$ -ideal of a  $d$ -algebra  $X$  is a  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal  $X$ . But the converse is not true as shown in the following Example.

**Example 0.45** Consider  $d$ -algebra  $X = \{0, a, b, c\}$  with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Define  $\hat{\mu} : X \rightarrow [0, 1]$  by  $\hat{\mu}(0) = [0.85, 0.95]$ ,  $\hat{\mu}(a) = \hat{\mu}(b) = [0.8, 0.9]$ ,  $\hat{\mu}(c) = [0.6, 0.7]$ , then it is easy to verify that  $\hat{\mu}$  is  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal  $X$ . Since  $a_{[0.8, 0.9]} = (c * b)_{[0.8, 0.9]}, b_{[0.8, 0.9]} \in \hat{\mu}$  but  $\hat{\mu}(c) \not\geq [0.8, 0.9]. [0.8, 0.9]$  i.e  $c_{[0.64, 0.81]} \notin \hat{\mu}$

**Theorem 0.46** If  $\hat{\mu}, \hat{\nu}$  are  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal of a  $d$ -algebra  $X$ , then so is  $\hat{\mu} \cap \hat{\nu}$ .



**Proof** Let  $x, y \in X$  such that  $x_{\hat{t}} \in (\hat{\mu} \cap \hat{\nu}) \Rightarrow (\hat{\mu} \cap \hat{\nu})(x) \geq \hat{t} \Rightarrow rmin\{\hat{\mu}(x), \hat{\nu}(x)\} \geq \hat{t} \Rightarrow \hat{\mu}(x) \geq \hat{t}, \hat{\nu}(x) \geq \hat{t} \Rightarrow x_{\hat{t}} \in \hat{\mu}$  and  $x_{\hat{t}} \in \hat{\nu} \Rightarrow 0_{\hat{t}} \in \vee q\hat{\mu}$  and  $0_{\hat{t}} \in \vee q\hat{\nu}$  [ Since  $\hat{\mu}, \hat{\nu}$  both are  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideals of X]

$$\Rightarrow 0_{\hat{t}} \in \vee q(\hat{\mu} \cap \hat{\nu})$$

Again let  $(x * y)_{\hat{t}}, y_{\hat{s}} \in (\hat{\mu} \cap \hat{\nu})$

$$\Rightarrow (\hat{\mu} \cap \hat{\nu})(x * y) \geq \hat{t}, (\hat{\mu} \cap \hat{\nu})(y) \geq \hat{s}$$

$$\Rightarrow rmin\{\hat{\mu}(x * y), \hat{\nu}(x * y)\} \geq \hat{t} \text{ and } rmin\{\hat{\mu}(y), \hat{\nu}(y)\} \geq \hat{s}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}, \hat{\nu}(x * y) \geq \hat{t} \text{ and } \hat{\mu}(y) \geq \hat{s}, \hat{\nu}(y) \geq \hat{s}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s} \text{ and } \hat{\nu}(x * y) \geq \hat{t}, \hat{\nu}(y) \geq \hat{s}$$

$$\Rightarrow (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \text{ and } (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\nu}$$

$$\Rightarrow x_{\hat{t}, \hat{s}} \in \vee q\hat{\mu} \text{ and } x_{\hat{t}, \hat{s}} \in \vee q\hat{\nu} \text{ [ Since } \hat{\mu}, \hat{\nu} \text{ both are } (\in, \in \vee q)\text{-interval-valued fuzzy dot d-ideals of X]}$$

$$\Rightarrow x_{\hat{t}, \hat{s}} \in \vee q(\hat{\mu} \cap \hat{\nu})$$

Similarly we can prove that

$$x_{\hat{t}}, y_{\hat{s}} \in (\hat{\mu} \cap \hat{\nu}) \Rightarrow (x * y)_{\hat{t}, \hat{s}} \in \vee q(\hat{\mu} \cap \hat{\nu})$$

Hence the proof.

**Theorem 0.47** If  $\hat{\mu}$  is a  $(q, q)$ -interval-valued fuzzy dot d-ideal of a d algebra X then it is also  $an(\in, \in)$ -interval-valued fuzzy dot d-ideal of X.

**Proof** Let  $\hat{\mu}$  be an  $(q, q)$ -interval-valued fuzzy dot d-ideal of a d-algebra X, to prove  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X. let  $x \in X$  such that  $x_{\hat{t}} \in \hat{\mu} \Rightarrow \hat{\mu}(x) \geq \hat{t}$

$$\Rightarrow \hat{\mu}(x) + \hat{\delta} > \hat{t} \text{ where } \hat{\delta} \text{ is a arbitrary small positive interval in } D[0 \ 1]$$

$$\Rightarrow \hat{\mu}(x) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1} \Rightarrow (x)_{\hat{1} + \hat{\delta} - \hat{t}} q\hat{\mu} \Rightarrow (0)_{\hat{1} + \hat{\delta} - \hat{t}} q\hat{\mu} \Rightarrow \hat{\mu}(0) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1} \Rightarrow \hat{\mu}(0) + \hat{\delta} > \hat{t}$$

$$\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$$

$$\text{Therefore } x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$$

Again let  $x, y \in X$  such that

$$(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \text{ therefore } \hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$$

$$\Rightarrow \hat{\mu}(x * y) + \hat{\delta} > \hat{t}, \hat{\mu}(y) + \hat{\delta} > \hat{s} \text{ where } \hat{\delta} \text{ is a arbitrary small positive interval in } D[0 \ 1]$$

$$\Rightarrow \hat{\mu}(x * y) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1}, \hat{\mu}(y) + \hat{\delta} - \hat{s} + \hat{1} > \hat{1}$$

$$\Rightarrow (x * y)_{\hat{1} + \hat{\delta} - \hat{t}} q\hat{\mu} \text{ and } (y)_{\hat{1} + \hat{\delta} - \hat{s}} q\hat{\mu}$$

$$\Rightarrow (x)_{(\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s})} q\hat{\mu} \text{ [since } \hat{\mu} \text{ is } (q, q)\text{-interval-valued fuzzy dot d-ideal of X]}$$

$$\Rightarrow \hat{\mu}(x) + (\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s}) > \hat{1}$$

$$\Rightarrow \hat{\mu}(x) + (\hat{1} - \hat{t}).(\hat{1} - \hat{s}) \geq \hat{1} \text{ taking } \hat{\delta} = \hat{0}$$

$$\Rightarrow \hat{\mu}(x) + \hat{1} - \hat{t} - \hat{s} + \hat{t}.\hat{s} \geq \hat{1}$$

$$\Rightarrow \hat{\mu}(x) \geq \hat{s} + \hat{t} - \hat{t}.\hat{s}$$

$$\Rightarrow \hat{\mu}(x) \geq \hat{t}.\hat{s} + \hat{t}.\hat{s} - \hat{t}.\hat{s} \text{ [ Since } \hat{t} \geq \hat{t}.\hat{s} \text{ and } \hat{s} \geq \hat{t}.\hat{s}]}$$

$$\Rightarrow \hat{\mu}(x) \geq \hat{t}.\hat{s}$$

$$\Rightarrow x_{\hat{t}, \hat{s}} \in \hat{\mu}$$

Again let

$$x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \text{ therefore } \hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$$

$$\Rightarrow \hat{\mu}(x) + \hat{\delta} > \hat{t}, \hat{\mu}(y) + \hat{\delta} > \hat{s} \text{ where } \hat{\delta} \text{ is a arbitrary small positive interval in } D[0 \ 1]$$

$$\Rightarrow \hat{\mu}(x) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1}, \hat{\mu}(y) + \hat{\delta} - \hat{s} + \hat{1} > \hat{1}$$

$$\Rightarrow (x)_{\hat{1} + \hat{\delta} - \hat{t}} q\hat{\mu} \text{ and } (y)_{\hat{1} + \hat{\delta} - \hat{s}} q\hat{\mu}$$

$$\Rightarrow (x * y)_{(\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s})} q\hat{\mu} \text{ [since } \hat{\mu} \text{ is } (q, q)\text{-interval-valued fuzzy dot d-ideal of X]}$$

$$\Rightarrow \hat{\mu}(x * y) + (\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s}) > \hat{1}$$

$$\Rightarrow \hat{\mu}(x * y) + (\hat{1} - \hat{t}).(\hat{1} - \hat{s}) \geq \hat{1} \text{ taking } \hat{\delta} = \hat{0}$$

$$\Rightarrow \hat{\mu}(x * y) + \hat{1} - \hat{t} - \hat{s} + \hat{t}.\hat{s} \geq \hat{1}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{s} + \hat{t} - \hat{t}.\hat{s}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}.\hat{s} + \hat{t}.\hat{s} - \hat{t}.\hat{s} \text{ [ Since } \hat{t} \geq \hat{t}.\hat{s} \text{ and } \hat{s} \geq \hat{t}.\hat{s}]}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t} \cdot \hat{s}$$

$$\Rightarrow (x * y)_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$$

Hence  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X.

**Remark 0.48** *The converse of Theorem 0.47 is not true as shown in following Example.*

**Example 0.49** *Consider d-algebra X and  $\hat{\mu}$  as in Example 0.11, then  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X. But  $\hat{\mu}$  is not an  $(q, q)$ -interval-valued fuzzy dot d-ideal of X because if  $x = c, y = a, x * y = c * a = c, t = 0.7, s = 0.4$  then  $\mu(x * y) + t = 0.35 + 0.7 > 1, \mu(y) + s = 0.7 + 0.4 > 1$  but  $\mu(x) + t \cdot s = 0.35 + 0.7 \times 0.4 = 0.35 + 0.28 < 1$  i.e  $x_{\hat{t} \cdot \hat{s}} \bar{q} \hat{\mu}$ .*

**Theorem 0.50** *A fuzzy subset  $\hat{\mu}$  of a d-algebra X is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X iff*

$$(i) \hat{\mu}(0) \geq rmin\{\hat{\mu}(x), [0.5, 0.5]\}$$

$$(ii) \hat{\mu}(x) \geq rmin\{\hat{\mu}(x * y) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x * y) \cdot \hat{\mu}(y)\}$$

$$(iii) \hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x) \cdot \hat{\mu}(y)\}$$

**Proof** Let  $\hat{\mu}$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X

$$(i) \text{ Assume } \hat{\mu}(0) < rmin\{\hat{\mu}(x), [0.5, 0.5]\}$$

SubCase I: If  $\hat{\mu}(x) < [0.5, 0.5]$ , then  $rmin\{\hat{\mu}(x), [0.5, 0.5]\} = \hat{\mu}(x)$

$$\Rightarrow \hat{\mu}(0) < \hat{\mu}(x) \Rightarrow \hat{\mu}(0) < \hat{t} < \hat{\mu}(x) \text{ for some } (0, 0) < \hat{t} < (0.5, 0.5)$$

$$\Rightarrow x_{\hat{t}} \in \hat{\mu} \text{ and } 0_{\hat{t}} \bar{\in} \hat{\mu} \text{ also } \hat{\mu}(0) + \hat{t} < \hat{1} \text{ i.e } 0_{\hat{t}} \bar{q} \hat{\mu} \Rightarrow 0_{\hat{t}} \bar{\in} \vee q \hat{\mu} \text{ Which is a contradiction}$$

SubCase II: If  $\hat{\mu}(x) \geq [0.5, 0.5]$  i.e  $x_{[0.5, 0.5]} \in \hat{\mu}$ , then  $rmin\{\hat{\mu}(x), [0.5, 0.5]\} = [0.5, 0.5]$

$$\hat{\mu}(0) < rmin\{\hat{\mu}(x), [0.5, 0.5]\} = [0.5, 0.5]$$

$$\Rightarrow 0_{[0.5, 0.5]} \bar{\in} \hat{\mu} \text{ and } \hat{\mu}(0) + [0.5, 0.5] < [0.5, 0.5] + [0.5, 0.5] = \hat{1} \text{ i.e } 0_{[0.5, 0.5]} \bar{q} \hat{\mu}$$

$$\Rightarrow 0_{[0.5, 0.5]} \bar{\in} \vee q \hat{\mu} \text{ Which is again a contradiction, therefore}$$

Therefore we must have  $\hat{\mu}(0) \geq rmin\{\hat{\mu}(x), [0.5, 0.5]\}$

$$(ii) \text{ Assume } \hat{\mu}(x) < rmin\{\hat{\mu}(x * y) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x * y) \cdot \hat{\mu}(y)\}$$

choose  $[0, 0] \leq \hat{t}, \hat{s} \leq [0, 1]$  such that  $\hat{\mu}(x) < \hat{t} \cdot \hat{s} < rmin\{\hat{\mu}(x * y) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x * y) \cdot \hat{\mu}(y)\}$

where  $\hat{\mu}(x * y) \geq \hat{t}$  and  $\hat{\mu}(y) \geq \hat{s}$

SubCase I: If  $\hat{\mu}(x * y) \cdot \hat{\mu}(y) < [0.5, 0.5]$ , then  $rmin\{\hat{\mu}(x * y) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x * y) \cdot \hat{\mu}(y)\} = \hat{\mu}(x * y) \cdot \hat{\mu}(y)$

$$\Rightarrow \hat{\mu}(x) < \hat{t} \cdot \hat{s} < \hat{\mu}(x * y) \cdot \hat{\mu}(y) < [0.5, 0.5]$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t} \text{ and } \hat{\mu}(y) \geq \hat{s}$$

$\Rightarrow (x * y)_{\hat{t}} \in \hat{\mu}$  and  $y_{\hat{s}} \in \hat{\mu}$  but  $x_{\hat{t} \cdot \hat{s}} \bar{\in} \hat{\mu}$  and  $\hat{\mu}(x) + \hat{t} \cdot \hat{s} < \hat{1}$  and  $x_{\hat{t} \cdot \hat{s}} \bar{q} \hat{\mu}$  Which contradict the fact that  $\hat{\mu}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X

SubCase II: If  $\hat{\mu}(x * y) \cdot \hat{\mu}(y) \geq [0.5, 0.5]$ , then  $\hat{\mu}(x) < \hat{t} \cdot \hat{s} < [0.5, 0.5] < \hat{\mu}(x * y) \cdot \hat{\mu}(y)$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t} \text{ and } \hat{\mu}(y) \geq \hat{s}$$

$\Rightarrow (x * y)_{\hat{t}} \in \hat{\mu}$  and  $y_{\hat{s}} \in \hat{\mu}$  but  $\hat{\mu}(x) \leq \hat{t} \cdot \hat{s}$  and  $\hat{\mu}(x) + \hat{t} \cdot \hat{s} < \hat{1}$  that is  $x_{\hat{t} \cdot \hat{s}} \bar{\in} \hat{\mu}$  and  $x_{\hat{t} \cdot \hat{s}} \bar{q} \hat{\mu}$  that is  $x_{\hat{t} \cdot \hat{s}} \bar{\in} \vee q \hat{\mu}$

Which is again a contradiction, therefore in both subcases

Therefore we must have  $\hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x) \cdot \hat{\mu}(y)\}$

(iii) Proof is similar to case (ii) above

**Theorem 0.51** *A fuzzy subset  $\hat{\mu}$  of d- algebra X is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X and  $\hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \forall x \in X$  then  $\hat{\mu}$  is also an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of X.*

**Proof** Let  $\hat{\mu}$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X and  $\hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \forall x \in X$   
Let  $x_{\hat{t}} \in \hat{\mu} \Rightarrow \hat{\mu}(x) \geq \hat{t}$

$$\Rightarrow \hat{t} \leq \hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \text{ and also } \hat{\mu}(0) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$$

$$\hat{\mu}(0) + \hat{t} < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] + [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] = [\sqrt{5} - 1, \sqrt{5} - 1] \Rightarrow \hat{\mu}(0) + \hat{t} \not\geq \hat{1}$$

$$\Rightarrow 0_{\hat{t}}\bar{q}\hat{\mu} \text{ therefore } x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}}\bar{q}\hat{\mu}$$

Since  $\hat{\mu}$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of  $X$ , therefore we must have  $x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$

Again let  $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$

$$\Rightarrow \hat{t} \leq \hat{\mu}(x * y) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \text{ and } \hat{t} \leq \hat{\mu}(y) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$$

$$\Rightarrow \hat{t}.\hat{s} \leq \hat{\mu}(x * y).\hat{\mu}(y) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}].[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \text{ and}$$

$$\Rightarrow \hat{t}.\hat{s} \leq [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}].[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \text{ and } \hat{\mu} \text{ is also an } (\in, \in)\text{-interval-valued fuzzy dot d-ideal of } X$$

$$\hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$$

$$\Rightarrow \mu(x) + \hat{t}.\hat{s} < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}].[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] + [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] = \hat{1}$$

$$\Rightarrow \hat{\mu}(x) + \hat{t}.\hat{s} < \hat{1}$$

$$\Rightarrow x_{\hat{t}.\hat{s}}\bar{q}\hat{\mu}$$

therefore  $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{\hat{t}.\hat{s}}\bar{q}\hat{\mu}$

since  $\hat{\mu}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of  $X$ , therefore we must have  $x_{\hat{t}.\hat{s}} \in \hat{\mu}$ .

Therefore  $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{\hat{t}.\hat{s}} \in \hat{\mu}$

Similarly we can prove that  $x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \hat{\mu}$

Hence  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of  $X$ .

**Theorem 0.52** *Let  $\hat{\mu}$  be a left fuzzy relation on a fuzzy subset  $\hat{\sigma}$  of a d-algebra  $X$ . If  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of  $X \times X$  then  $\hat{\sigma}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of d-algebra  $X$ .*

**Proof** Suppose that a left fuzzy relation  $\hat{\mu}$  on  $\hat{\sigma}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of  $X \times X$ .

To prove that  $\hat{\sigma}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of  $X$

Let  $x, y \in X$  since  $\hat{\sigma}(x) = \hat{\mu}(x, y)$  let  $x_{\hat{t}} \in \hat{\sigma} \Rightarrow \hat{\sigma}(x) \geq \hat{t} \Rightarrow \hat{\sigma}(x) = \hat{\mu}(x, y) \geq \hat{t}$

$(x, y)_{\hat{t}} \in \hat{\mu} \Rightarrow (0, 0)_{\hat{t}} \in \hat{\mu}$  [ Since  $\hat{\mu}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of  $X \times X$  ]

$\Rightarrow \hat{\mu}(0, 0) \geq \hat{t} \Rightarrow \hat{\sigma}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\sigma}$

therefore  $x_{\hat{t}} \in \hat{\sigma} \Rightarrow 0_{\hat{t}} \in \hat{\sigma}$

Let  $x, x', y, y' \in X$

$\hat{\sigma}(x) = \hat{\mu}(x, y)$

now Let  $(x * x')_{\hat{t}}, (x')_{\hat{s}} \in \hat{\sigma}$

$\Rightarrow \hat{\sigma}(x * x') \geq \hat{t}$  and  $\hat{\sigma}(x') \geq \hat{s}$

Now  $\hat{\sigma}(x) = \hat{\mu}(x, y) \geq \hat{\mu}((x, y) * (x', y')).\hat{\mu}(x', y')$  [ Since  $\hat{\mu}$  is also an interval-valued fuzzy dot d-ideal of  $X \times X$ , see Theorem 0.33 ]

$= \hat{\mu}(x * x', y * y').\hat{\mu}(x', y') = \hat{\sigma}(x * x').\hat{\sigma}(x') \geq \hat{t}.\hat{s}$

$\Rightarrow (x)_{\hat{t}.\hat{s}} \in \hat{\sigma}$

$(x * x')_{\hat{t}}, (x')_{\hat{s}} \in \hat{\sigma} \Rightarrow (x)_{\hat{t}.\hat{s}} \in \hat{\sigma}$

Again let  $x_{\hat{t}}, x'_{\hat{s}} \in \hat{\sigma} \Rightarrow \hat{\sigma}(x) \geq \hat{t}$  and  $\hat{\sigma}(x') \geq \hat{s}$

$\hat{\sigma}(x * x') = \hat{\mu}(x * x', y * y')$

$= \hat{\mu}((x, y) * (x', y')) \geq \hat{\mu}(x, y).\hat{\mu}(x', y') \geq \hat{\sigma}(x).\hat{\sigma}(x') \geq \hat{t}.\hat{s}$

$(x * x')_{\hat{t}.\hat{s}} \in \hat{\sigma}$

$x_{\hat{t}}, x'_{\hat{s}} \in \hat{\sigma} \Rightarrow (x * x')_{\hat{t}.\hat{s}} \in \hat{\sigma}$

Hence from above  $\hat{\sigma}$  is an  $(\in, \in)$ -interval-valued fuzzy dot d-ideal of  $X$

#### 4 Homomorphism of $d$ -algebras and $(\in, \in \vee q)$ -interval-valued Fuzzy Dot $d$ -ideals

**Definition 0.53** Let  $X$  and  $X'$  be two  $d$ -algebras, then a mapping  $f : X \rightarrow X'$  is said to be homomorphism if  $f(x * y) = f(x) * f(y) \forall x, y \in X$ .

**Theorem 0.54** Let  $X$  and  $X'$  be two  $d$ -algebras and  $f : X \rightarrow X'$  be a homomorphism. Then  $f(0) = 0'$

**Proof** Let  $x \in X$  therefore  $f(x) \in X'$ . Now  $f(0) = f(x * x) = f(x) * f(x) = 0 * 0 = 0'$ .

**Theorem 0.55** Let  $f : X \rightarrow X'$  be an onto homomorphism of  $d$ -algebras,  $\hat{\nu}$  be a  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal of  $X'$ , then the pre-image  $f^{-1}(\hat{\nu})$  of  $\hat{\nu}$  under  $f$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal of  $X$ .

**Proof**  $f^{-1}(\hat{\nu})$  is defined as  $f^{-1}(\hat{\nu})(x) = \hat{\nu}(f(x)) \forall x \in X$

Let  $\hat{\nu}$  be an  $(\in, \in)$ -interval-valued fuzzy dot  $d$ -ideal of  $X'$

Let  $x \in X$  such that  $x_{\hat{t}} \in f^{-1}(\hat{\nu}) \Rightarrow f^{-1}(\hat{\nu})(x) \geq \hat{t} \Rightarrow \hat{\nu}f(x) \geq \hat{t} \Rightarrow (f(x))_{\hat{t}} \in \hat{\nu} \Rightarrow x'_{\hat{t}} \in \hat{\nu}$  where  $f(x) = x'$

$\Rightarrow 0'_{\hat{t}} \in \vee q \hat{\nu}$  [Since  $\hat{\nu}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal of  $X'$ ]

$\Rightarrow [f(0)]_{\hat{t}} \in \vee q \hat{\nu} \Rightarrow \hat{\nu}f(0) \geq \hat{t}$  or  $\hat{\nu}f(0) + \hat{t} \geq \hat{1} \Rightarrow f^{-1}(\hat{\nu})(0) \geq \hat{t}$  or  $f^{-1}(\hat{\nu})(0) + \hat{t} \geq \hat{1}$

$\Rightarrow 0_{\hat{t}} \in f^{-1}(\hat{\nu})$  or  $0_{\hat{t}} q f^{-1}(\hat{\nu})$

$\Rightarrow 0_{\hat{t}} \in \vee q f^{-1}(\hat{\nu})$

Therefore  $x_{\hat{t}} \in f^{-1}(\hat{\nu}) \Rightarrow 0_{\hat{t}} \in \vee q f^{-1}(\hat{\nu})$

Let  $x, y \in X$  such that  $(x * y)_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu})$

$\Rightarrow f^{-1}(\hat{\nu})(x * y) \geq \hat{t}$  and  $f^{-1}(\hat{\nu})(y) \geq \hat{s}$

$\Rightarrow \hat{\nu}f(x * y) \geq \hat{t}$  and  $\hat{\nu}f(y) \geq \hat{s}$

$\Rightarrow (f(x * y))_{\hat{t}} \in \hat{\nu}$  and  $(f(y))_{\hat{s}} \in \hat{\nu}$

$\Rightarrow (f(x) * f(y))_{\hat{t}} \in \hat{\nu}$  and  $(f(y))_{\hat{s}} \in \hat{\nu}$  [ Since  $f$  is homomorphism ]

$\Rightarrow (f(x))_{\hat{t}, \hat{s}} \in \vee q \hat{\nu}$  [Since  $\hat{\nu}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal of  $X'$ ]

$\Rightarrow \hat{\nu}f(x) \geq \hat{t}, \hat{s}$  or  $\hat{\nu}f(x) + \hat{t}, \hat{s} \geq \hat{1}$

$\Rightarrow f^{-1}(\hat{\nu})(x) \geq \hat{t}, \hat{s}$  or  $f^{-1}(\hat{\nu})(x) + \hat{t}, \hat{s} \geq \hat{1}$

$\Rightarrow x_{\hat{t}, \hat{s}} \in f^{-1}(\hat{\nu})$  or  $x_{\hat{t}, \hat{s}} q f^{-1}(\hat{\nu})$

$\Rightarrow x_{\hat{t}, \hat{s}} \in \vee q f^{-1}(\hat{\nu})$  Therefore  $(x * y)_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu}) \Rightarrow x_{\hat{t}, \hat{s}} \in \vee q f^{-1}(\hat{\nu})$

Again let  $x, y \in X$  such that  $x_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu})$

$\Rightarrow f^{-1}(\hat{\nu})(x) \geq \hat{t}$  and  $f^{-1}(\hat{\nu})(y) \geq \hat{s}$

$\Rightarrow \hat{\nu}f(x) \geq \hat{t}$  and  $\hat{\nu}f(y) \geq \hat{s}$

$\Rightarrow (f(x))_{\hat{t}} \in \hat{\nu}$  and  $(f(y))_{\hat{s}} \in \hat{\nu}$

$\Rightarrow (f(x) * f(y))_{\hat{t}, \hat{s}} \in \vee q \hat{\nu}$  [Since  $\hat{\nu}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal of  $X'$ ]

$\Rightarrow \hat{\nu}(f(x) * f(y)) \geq \hat{t}, \hat{s}$  or  $\hat{\nu}(f(x) * f(y)) + \hat{t}, \hat{s} \geq \hat{1}$

$\Rightarrow \hat{\nu}(f(x * y)) \geq \hat{t}, \hat{s}$  or  $\hat{\nu}(f(x * y)) + \hat{t} \geq \hat{1}$  [ Since  $f$  is homomorphism ]

$\Rightarrow f^{-1}(\hat{\nu})(x * y) \geq \hat{t}, \hat{s}$  or  $f^{-1}(\hat{\nu})(x * y) + \hat{t}, \hat{s} \geq \hat{1}$

$\Rightarrow (x * y)_{\hat{t}, \hat{s}} \in f^{-1}(\hat{\nu})$  or  $(x * y)_{\hat{t}, \hat{s}} q f^{-1}(\hat{\nu})$

$\Rightarrow (x * y)_{\hat{t}, \hat{s}} \in \vee q f^{-1}(\hat{\nu})$

Therefore  $x_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu}) \Rightarrow (x * y)_{\hat{t}, \hat{s}} \in \vee q f^{-1}(\hat{\nu})$

Hence  $f^{-1}(\hat{\nu})$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal of  $X$ .

**Theorem 0.56** An onto homomorphic image of an  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal with the sup property is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot  $d$ -ideal.

**Proof** Let  $f : X \rightarrow X'$  be an onto homomorphism of d-algebras,  $\hat{\mu}$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of  $X$ , and the image of  $\hat{\mu}$  under  $f$  be  $f(\hat{\mu})$ . To prove  $f(\hat{\mu})$  is an  $(\in, \in \vee q)$ -fuzzy dot d-ideal of  $X'$

Since  $0 \in f^{-1}(0')$  therefore, for any  $x', y' \in X'$ , let  $x_0, y_0 \in X$  such that

$$\hat{\mu}(x_0) = \sup_{z \in f^{-1}(x')} \hat{\mu}(z) \quad \hat{\mu}(y_0) = \sup_{z \in f^{-1}(y')} \hat{\mu}(z)$$

and

$$\hat{\mu}(x_0 * y_0) = \sup_{z \in f^{-1}(x' * y')} \hat{\mu}(z)$$

then, let

$$\begin{aligned} x'_t \in f(\hat{\mu}) &\Rightarrow f(\hat{\mu})(x') \geq \hat{t} \Rightarrow \sup_{z \in f^{-1}(x')} \hat{\mu}(z) = \hat{t} \\ &\Rightarrow \hat{\mu}(x_0) \geq \hat{t} \Rightarrow (x_0)_t \in \hat{\mu} \Rightarrow 0_t \in \vee q \hat{\mu} \end{aligned}$$

[ Since  $\hat{\mu}$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of  $X$  ]

$$\begin{aligned} &\Rightarrow \hat{\mu}(0) \geq \hat{t} \quad \text{or} \quad \hat{\mu}(0) + \hat{t} \geq \hat{1} \\ &\Rightarrow \sup_{z \in f^{-1}(0')} \hat{\mu}(z) \geq \hat{t} \quad \text{or} \quad \sup_{z \in f^{-1}(0')} \hat{\mu}(z) + \hat{t} \geq \hat{1} \\ &\Rightarrow f(\hat{\mu})(0') \geq \hat{t} \quad \text{or} \quad f(\hat{\mu})(0') + \hat{t} \geq \hat{1} \\ &\Rightarrow 0'_t \in f(\hat{\mu}) \quad \text{or} \quad 0'_t q f(\hat{\mu}) \\ &\Rightarrow 0'_t \in \vee q f(\hat{\mu}) \end{aligned}$$

Therefore  $\Rightarrow x'_t \in f(\hat{\mu}) \Rightarrow 0'_t \in \vee q f(\hat{\mu})$  Again, let  $(x' * y')_{t, \hat{s}} \in f(\hat{\mu}) \Rightarrow f(\hat{\mu})(x' * y') \geq \hat{t}, f(\hat{\mu})(y') \geq \hat{s}$

$$\begin{aligned} &\Rightarrow \sup_{z \in f^{-1}(x' * y')} \hat{\mu}(z) \geq \hat{t}, \sup_{z \in f^{-1}(y')} \hat{\mu}(z) \geq \hat{s} \\ &\Rightarrow \hat{\mu}(x_0 * y_0) \geq \hat{t}, \hat{\mu}(y_0) \geq \hat{s} \\ &\Rightarrow (x_0 * y_0)_{t, \hat{s}} \in \hat{\mu}, (y_0)_{\hat{s}} \in \hat{\mu} \Rightarrow (x_0)_{t, \hat{s}} \in \vee q \hat{\mu} \end{aligned}$$

[ Since  $\hat{\mu}$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of  $X$ , ]

$$\begin{aligned} &\Rightarrow \hat{\mu}(x_0) \geq \hat{t} \cdot \hat{s} \quad \text{or} \quad \hat{\mu}(x_0) + \hat{t} \cdot \hat{s} \geq \hat{1} \\ &\Rightarrow \sup_{z \in f^{-1}(x')} \hat{\mu}(z) \geq \hat{t} \cdot \hat{s} \quad \text{or} \quad \sup_{z \in f^{-1}(x')} \hat{\mu}(z) + \hat{t} \cdot \hat{s} \geq \hat{1} \end{aligned}$$

$$\begin{aligned} &\Rightarrow f(\hat{\mu})(x') \geq \hat{t} \cdot \hat{s} \quad \text{or} \quad f(\hat{\mu})(x') + \hat{t} \cdot \hat{s} \geq \hat{1} \\ &\Rightarrow (x')_{t, \hat{s}} \in f(\hat{\mu}) \quad \text{or} \quad (x')_{t, \hat{s}} q f(\hat{\mu}) \end{aligned}$$

$$\Rightarrow (x')_{t, \hat{s}} \in \vee q f(\hat{\mu})$$

$$\Rightarrow (x' * y')_{t, \hat{s}} \in f(\hat{\mu}) \Rightarrow (x')_{t, \hat{s}} \in \vee q f(\hat{\mu}) \text{ Similarly we can prove}$$

$$x'_t, y'_{\hat{s}} \in f(\hat{\mu}) \Rightarrow (x' * y')_{t, \hat{s}} \in \vee q f(\hat{\mu})$$

Hence from above  $f(\hat{\mu})$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of  $X'$ .

**Theorem 0.57** Let  $f : X \rightarrow X'$  be an onto homomorphism of d-algebras,  $\hat{\mu}$  be a fuzzy subset of  $X'$  such that  $f^{-1}(\hat{\mu})$  is an  $(\in, \in \vee q)$ -fuzzy dot d-ideal of  $X$ , then  $\hat{\mu}$  is also an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of  $X'$ .

**Proof** Let  $f^{-1}(\hat{\mu})$  be an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X

Let  $x', y' \in X'$  since f is onto so there exists  $x, y \in X$  such that  $f(x) = x', f(y) = y'$  and also f is homomorphism therefore  $f(x * y) = f(x) * f(y) = x' * y'$

Let  $x'_t \in \hat{\mu}$  where  $t \in D[0 1]$

Therefore  $\hat{\mu}(x') \geq t \Rightarrow \hat{\mu}(f(x)) \geq t$

$\Rightarrow f^{-1}(\hat{\mu})(x) \geq t$

$\Rightarrow x_t \in f^{-1}(\hat{\mu})$

$\Rightarrow 0_t \in \vee q f^{-1}(\hat{\mu})$  [Since  $f^{-1}(\hat{\mu})$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X.]

$\Rightarrow f^{-1}(\hat{\mu})(0) \geq t$  or  $f^{-1}(\hat{\mu})(0) + t \geq \hat{1}$

$\Rightarrow \hat{\mu}(f(0)) \geq t$  or  $\hat{\mu}(f(0)) + t \geq \hat{1}$

$\Rightarrow \hat{\mu}(0') \geq t$  or  $\hat{\mu}(0') + t \geq \hat{1}$

$\Rightarrow 0'_t \in \hat{\mu}$  or  $0'_t q \hat{\mu}$

$\Rightarrow 0'_t \in \vee q \hat{\mu}$

Therefore  $x'_t \in \hat{\mu} \Rightarrow 0'_t \in \vee q \hat{\mu}$

Again let  $(x' * y')_{t, s} \in \hat{\mu}$  where  $t, s \in D[0 1]$

Therefore  $\hat{\mu}(x' * y') \geq t$  and  $\hat{\mu}(y') \geq s$

$\Rightarrow \hat{\mu}(f(x * y)) \geq t$  and  $\hat{\mu}(f(y)) \geq s$

$\Rightarrow f^{-1}(\hat{\mu})(x * y) \geq t$  and  $f^{-1}(\hat{\mu})(y) \geq s$

$\Rightarrow (x * y)_t \in f^{-1}(\hat{\mu})$  and  $y_s \in f^{-1}(\hat{\mu})$

$\Rightarrow x_{t, s} \in \vee q f^{-1}(\hat{\mu})$  [Since  $f^{-1}(\hat{\mu})$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X.]

$\Rightarrow f^{-1}(\hat{\mu})(x) \geq t.s$  or  $f^{-1}(\hat{\mu})(x) + t.s \geq \hat{1}$

$\Rightarrow \hat{\mu}(f(x)) \geq t.s$  or  $\hat{\mu}(f(x)) + t.s \geq \hat{1}$

$\Rightarrow \hat{\mu}(x') \geq t.s$  or  $\hat{\mu}(x') + t.s \geq \hat{1}$

$\Rightarrow x'_{t, s} \in \hat{\mu}$  or  $x'_{t, s} q \hat{\mu}$

$\Rightarrow x'_{t, s} \in \vee q \hat{\mu}$

$\Rightarrow (x' * y')_{t, s} \in \hat{\mu} \Rightarrow x'_{t, s} \in \vee q \hat{\mu}$

Again let  $x', y' \in X'$  such that  $(x')_t, y'_s \in \hat{\mu}$

therefore  $\hat{\mu}(x') \geq t$  and  $\hat{\mu}(y') \geq s$

$\Rightarrow \hat{\mu}(f(x)) \geq t$  and  $\hat{\mu}(f(y)) \geq s$

$\Rightarrow f^{-1}(\hat{\mu})(x) \geq t$  and  $f^{-1}(\hat{\mu})(y) \geq s$

$\Rightarrow x_t \in f^{-1}(\hat{\mu})$  and  $y_s \in f^{-1}(\hat{\mu})$

$\Rightarrow (x * y)_{t, s} \in \vee q f^{-1}(\hat{\mu})$  [Since  $f^{-1}(\hat{\mu})$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X.]

$\Rightarrow f^{-1}(\hat{\mu})(x * y) \geq t.s$  or  $f^{-1}(\hat{\mu})(x * y) + t.s \geq \hat{1}$

$\Rightarrow \hat{\mu}(f(x * y)) \geq t.s$  or  $\hat{\mu}(f(x * y)) + t.s \geq \hat{1}$

$\Rightarrow \hat{\mu}(f(x) * f(y)) \geq t.s$  or  $\hat{\mu}(f(x) * f(y)) + t.s \geq \hat{1}$

$\Rightarrow (x' * y')_{t, s} \in \hat{\mu}$  or  $(x' * y')_{t, s} q \hat{\mu}$

$\Rightarrow (x' * y')_{t, s} \in \vee q \hat{\mu}$

$\Rightarrow x'_t, y'_s \in \hat{\mu} \Rightarrow (x' * y')_{t, s} \in \vee q \hat{\mu}$

Hence  $\hat{\mu}$  is an  $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of  $X'$ .

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