

$(\in, \in \vee q)$ -Interval-valued Fuzzy Dot d-ideals of d-algebras

S. R. Barbhuiya¹, K. D. Choudhury²

¹Department of Mathematics, Srikishan Sarda College, Hailakandi-788151, Assam, India

²Department of Mathematics Assam University, Silchar-788011, Assam, India

saidurbarbhuiya@gmail.com

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Abstract. The concept of $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideals in d-algebras is introduced. Relationship among interval-valued fuzzy d-ideal, interval-valued fuzzy dot d-ideal, (\in, \in) -interval-valued fuzzy d-ideal, (\in, \in) -interval-valued fuzzy dot d-ideal, and $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideals are discussed. Conditions for an interval-valued fuzzy d-ideal to be an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideals are given. Some properties of interval-valued fuzzy relations and interval-valued fuzzy ideals under homomorphism are investigated.

1 Introduction

In 1991 Xi [12] applied the concept of fuzzy sets to BCK-algebras which are introduced by Imai and Iseki[7]in 1996. Neggers and Kim [11]introduced the class of d-algebras which is a generalisation of BCK-algebras and investigated relation between d-algebras and BCK- algebras. Akram and Dar[1] introduced the concepts fuzzy d-algebra, they introduced fuzzy subalgebra and fuzzy d-ideals of d-algebras. Kim [8]introduced the notion of a fuzzy dot subalgebra of d-algebra and investigated some related properties. Bhakat and Das [5, 6] used the relation of "belongs to" and quasi-coincident" between fuzzy point and fuzzy set to introduced the concept of $(\in, \in \vee q)$ -fuzzy subgroup and $(\in, \in \vee q)$ -fuzzy subring. Al-Shehrie[2] introduced the notion of fuzzy dot d-ideals of a d-algebra. Interval-valued fuzzy sets were first introduced by Zadeh [15] in 1975. After that many researchers consider the interval-valued fuzzification of ideals and subalgebras in BG/ BCK-algebras. The concept of $(\in, \in \vee q)$ -interval-valued fuzzification of ideals in ring was introduced in [9]. Here in this paper, we introduce the notion of $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of d-algebra and then we investigates some of its interesting properties.

2 Preliminaries

Definition 0.1 [1, 8] A d-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

- (i) $x * x = 0$
- (ii) $0 * x = 0$
- (iii) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ for all $x, y \in X$.

For brevity we also call X a d-algebra.

Definition 0.2 [8] A non-empty subset S of a d-algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Definition 0.3 [2] A nonempty subset I of a d-algebra X is called an ideal of X if

- (i) $0 \in I$
- (ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$
- (iii) $x \in I$ and $y \in X \Rightarrow x * y \in I$

Definition 0.4 [2, 8] A fuzzy subset μ of X is called a fuzzy dot subalgebra of a d -algebra X if $\mu(x * y) \geq \mu(x) \cdot \mu(y)$ for all $x, y \in X$.

Definition 0.5 [2] A fuzzy subset μ of X is called a fuzzy d -ideal of X if it satisfies the following conditions:

- (i) $\mu(0) \geq \mu(x)$
- (ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$
- (iii) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

Definition 0.6 [2] A fuzzy subset μ of X is called a fuzzy dot d -ideal of X if it satisfies the following conditions:

- (i) $\mu(0) \geq \mu(x)$
- (ii) $\mu(x) \geq \mu(x * y) \cdot \mu(y)$
- (iii) $\mu(x * y) \geq \mu(x) \cdot \mu(y)$ for all $x, y \in X$.

Definition 0.7 [5] A fuzzy set μ of the form

$$\mu(y) = \begin{cases} t & \text{if } y = x, \quad t \in (0, 1] \\ 0 & \text{if } y \neq x \end{cases}$$

is called a fuzzy point with support x and value t and it is denoted by x_t .

Definition 0.8 [5] Let μ be a fuzzy set in X and x_t be a fuzzy point then

- (i) If $\mu(x) \geq t$ then we say x_t belongs to μ and write $x_t \in \mu$
- (ii) If $\mu(x) + t > 1$ then we say x_t quasi coincident μ and write $x_t q \mu$
- (iii) If $x_t \in \vee q \mu \Leftrightarrow x_t \in \mu$ or $x_t q \mu$
- (iv) If $x_t \in \wedge q \mu \Leftrightarrow x_t \in \mu$ and $x_t q \mu$

The symbol $x_t \bar{\alpha} \mu$ means $x_t \alpha \mu$ does not hold and $\overline{\in \wedge q}$ means $\bar{\in} \vee \bar{q}$

For a fuzzy point x_t . and a fuzzy set μ in set X , Pu and Liu [10] gave meaning to the symbol $x_t \alpha \mu$ where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$

Definition 0.9 A fuzzy subset μ of X is called a $(\in, \in \vee q)$ -fuzzy d -ideal of X if it satisfies the following conditions:

- (i) $x_t \in \mu \Rightarrow 0_t \in \vee q \mu$
 - (ii) $(x * y)_t, y_s \in \mu \Rightarrow x_{m(t,s)} \in \vee q \mu$
 - (iii) $x_t, y_s \in \mu \Rightarrow (x * y)_{m(t,s)} \in \vee q \mu \quad \forall t, s \in (0, 1], \quad \forall x, y \in X$
- Where $m(t, s) = \min\{t, s\}$

Definition 0.10 A fuzzy subset μ of X is called a $(\in, \in \vee q)$ -fuzzy dot d -ideal of X if it satisfies the following conditions:

1. $x_t \in \mu \Rightarrow 0_t \in \vee q \mu$
2. $(x * y)_t, y_s \in \mu \Rightarrow x_{t.s} \in \vee q \mu$
3. $x_t, y_s \in \mu \Rightarrow (x * y)_{t.s} \in \vee q \mu \quad \forall t, s \in (0, 1], \quad \forall x \in X$

Example 0.11 Consider d -algebra $X = \{0, a, b, c\}$ with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Define $\mu : X \rightarrow [0, 1]$ by μ by $\mu(0) = 0.9, \mu(a) = \mu(b) = 0.8, \mu(c) = 0.7$, then it is easy to verify that μ is $(\in, \in \vee q)$ -fuzzy dot d -ideal X .

Definition 0.12 [2] Let λ and μ be two fuzzy sets in a set X . The their cartesian product $\lambda \times \mu : X \times X \rightarrow [0, 1]$ is defined by $(\lambda \times \mu)(x, y) = \lambda(x) \cdot \mu(y)$, for all $x, y \in X$. Let σ be a fuzzy subset of X , then the strongest fuzzy σ relation on d algebra X is the fuzzy subset μ_σ of $X \times X$ given by $\mu_\sigma(x, y) = \sigma(x) \cdot \sigma(y) \forall x, y \in X$. A fuzzy relation μ on d algebra X is called a fuzzy σ product relation if $\mu(x, y) \geq \sigma(x) \cdot \sigma(y) \forall x, y \in X$. A fuzzy relation μ on d algebra X is called a left fuzzy relation on σ if $\mu(x, y) = \sigma(x) \forall x, y \in X$. Note that a left fuzzy relation on σ is a fuzzy σ product relation.

Remark 0.13 If X and Y be two d -algebras, then $X \times X$ is also a d -algebra under the binary operation $'*'$ defined in $X \times X$ by $(x, y) * (p, q) = (x * p, y * q)$ for all $(x, y), (p, q) \in X \times X$.

3 $(\in, \in \vee q)$ -interval-valued fuzzy sets

The notion of interval-valued fuzzy set was introduced by L.A.Zadeh[25]. To consider the notion of interval-valued fuzzy sets, we need the following notations. By an interval number \hat{a} , we mean an interval $[\underline{a}, \overline{a}]$, where $0 \leq \underline{a} \leq \overline{a} \leq 1$. Let $D[0, 1]$ denote the set of all such interval numbers of $[0, 1]$. Define on $D[0, 1]$ the relations $\leq, =, <, \cdot$ by

- $\hat{a}_1 \leq \hat{a}_2 \Leftrightarrow \underline{a}_1 \leq \underline{a}_2$ and $\overline{a}_1 \leq \overline{a}_2$
- $\hat{a}_1 = \hat{a}_2 \Leftrightarrow \underline{a}_1 = \underline{a}_2$ and $\overline{a}_1 = \overline{a}_2$
- $\hat{a}_1 < \hat{a}_2 \Leftrightarrow \underline{a}_1 < \underline{a}_2$ and $\overline{a}_1 < \overline{a}_2$
- $\hat{a}_1 \cdot \hat{a}_2 \Leftrightarrow [\min(\underline{a}_1 \underline{a}_2, \underline{a}_1 \overline{a}_2, \overline{a}_1 \underline{a}_2, \overline{a}_1 \overline{a}_2), \max(\underline{a}_1 \underline{a}_2, \underline{a}_1 \overline{a}_2, \overline{a}_1 \underline{a}_2, \overline{a}_1 \overline{a}_2)] = [\underline{a}_1 \underline{a}_2, \overline{a}_1 \overline{a}_2]$
- $k\hat{a} = [k\underline{a}, k\overline{a}]$ where $0 \leq k \leq 1$

Now consider two intervals $\hat{a}_1 = [\underline{a}_1, \overline{a}_1], \hat{a}_2 = [\underline{a}_2, \overline{a}_2] \in D[0, 1]$ then we define refine minimum $rmin$ as $rmin(\hat{a}_1, \hat{a}_2) = [\min(\underline{a}_1, \underline{a}_2), \min(\overline{a}_1, \overline{a}_2)]$ and refine maximum as

$rmax$ $rmax(\hat{a}_1, \hat{a}_2) = [\max(\underline{a}_1, \underline{a}_2), \max(\overline{a}_1, \overline{a}_2)]$ generally if $\hat{a}_i = [\underline{a}_i, \overline{a}_i], \hat{b}_i = [\underline{b}_i, \overline{b}_i] \in D[0, 1]$ for $i= 1, 2, 3, \dots$ then we define

$rmax(\hat{a}_i, \hat{b}_i) = [\max(\underline{a}_i, \underline{b}_i), \max(\overline{a}_i, \overline{b}_i)]$ and $rmin(\hat{a}_i, \hat{b}_i) = [\min(\underline{a}_i, \underline{b}_i), \min(\overline{a}_i, \overline{b}_i)]$
 and $rinf_i(\hat{a}_i) = [\wedge_i \underline{a}_i, \wedge_i \overline{a}_i]$ and $rsup_i(\hat{a}_i) = [\vee_i \underline{a}_i, \vee_i \overline{a}_i]$

$(D[0, 1], \leq)$ is a complete lattice with $\wedge = rmin, \vee = rmax, \hat{0} = [0 0]$ and $\hat{1} = [1 1]$ being the least and the greatest element respectively.

Definition 0.14 An interval-valued fuzzy set defined on a non empty set X as an objects having the form $\hat{\mu} = \{x, [\underline{\mu}(x), \overline{\mu}(x)]\}, \forall x \in X$ where $\underline{\mu}$ and $\overline{\mu}$ are two fuzzy sets in X such that $\underline{\mu}(x) \leq \overline{\mu}(x)$ for all $x \in X$. Let $\hat{\mu}(x) = [\underline{\mu}(x), \overline{\mu}(x)], \forall x \in X$. Then $\hat{\mu}(x) \in D[0 1], \forall x \in X$
 If $\hat{\mu}$ and $\hat{\nu}$ be two interval-valued fuzzy sets in X , then we define

- $\hat{\mu} \subset \hat{\nu} \Leftrightarrow$ for all $\underline{\mu}(x) \leq \underline{\nu}(x)$ and $\overline{\mu}(x) \leq \overline{\nu}(x)$.

- $\hat{\mu} = \hat{\nu} \Leftrightarrow$ for all $\underline{\mu}(x) = \underline{\nu}(x)$ and $\overline{\mu}(x) = \overline{\nu}(x)$.
- $(\hat{\mu} \cup \hat{\nu})(x) = \hat{\mu}(x) \vee \hat{\nu}(x) = [\max\{\underline{\mu}(x), \underline{\nu}(x)\}, \max\{\overline{\mu}(x), \overline{\nu}(x)\}]$.
- $(\hat{\mu} \cap \hat{\nu})(x) = \hat{\mu}(x) \wedge \hat{\nu}(x) = [\min\{\underline{\mu}(x), \underline{\nu}(x)\}, \min\{\overline{\mu}(x), \overline{\nu}(x)\}]$.
- $\hat{\mu}^c(x) = [1 - \overline{\mu}(x), 1 - \underline{\mu}(x)]$.

Definition 0.15 Let $\hat{\mu}$ be an interval-valued fuzzy set in X . Then, for every $[0, 0] < \hat{t} \leq [1, 1]$, the crisp set $\hat{\mu}_{\hat{t}} = \{x \in X \mid \hat{\mu}(x) \geq \hat{t}\}$ is called the level subset of $\hat{\mu}$.

Definition 0.16 A interval-valued fuzzy subset $\hat{\mu}$ of X is called an interval-valued fuzzy dot subalgebra of a d -algebra X if $\hat{\mu}(x * y) \geq \hat{\mu}(x) \cdot \hat{\mu}(y)$ for all $x, y \in X$.

Definition 0.17 An interval-valued fuzzy set $\hat{\mu}$ in d -algebra X is called an interval-valued fuzzy ideal of X if it satisfies

- (i) $\hat{\mu}(0) \geq \hat{\mu}(x)$
- (ii) $\hat{\mu}(x) \geq r\min\{\hat{\mu}(x * y), \hat{\mu}(y)\}$ for all $x, y \in X$
- (ii) $\hat{\mu}(x * y) \geq r\min\{\hat{\mu}(x), \hat{\mu}(y)\}$ for all $x, y \in X$

Definition 0.18 An interval-valued fuzzy set $\hat{\mu}$ in d -algebra X is called an interval-valued fuzzy dot d -ideal of X if it satisfies

- (i) $\hat{\mu}(0) \geq \hat{\mu}(x)$
- (ii) $\hat{\mu}(x) \geq \hat{\mu}(x * y) \cdot \hat{\mu}(y)$ for all $x, y \in X$
- (ii) $\hat{\mu}(x * y) \geq \hat{\mu}(x) \cdot \hat{\mu}(y)$ for all $x, y \in X$

Example 0.19 Consider d -algebra $X = \{0, a, b, c\}$ with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Define $\hat{\mu} : X \rightarrow D[0, 1]$ by $\hat{\mu}(0) = [0.8, 0.9]$, $\hat{\mu}(a) = \hat{\mu}(b) = [0.6, 0.8]$, $\hat{\mu}(c) = [0.35, 0.5]$, then it is easy to verify that $\hat{\mu}$ is $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal X .

Definition 0.20 Let $\hat{\mu}(x) = [\underline{\mu}(x), \overline{\mu}(x)]$ and $\hat{t} = [\underline{t}, \overline{t}] \in D[0, 1]$, then we define $\hat{\mu}(x) + \hat{t} = [\underline{\mu}(x) + \underline{t}, \overline{\mu}(x) + \overline{t}] \forall x \in X$. In particular $\underline{\mu}(x) + \underline{t} > 1$, we write $\hat{\mu}(x) + \hat{t} > [1, 1]$. Let $x \in X$.

An interval-valued fuzzy set $\hat{\mu}$ of the form

$$\hat{\mu}(y) = \begin{cases} \hat{t} & \text{if } y = x, \hat{t} \in D(0, 1) \\ \hat{0} & \text{if } y \neq x \end{cases}$$

is called an interval-valued fuzzy point with support x and value \hat{t} and it is denoted by $x_{\hat{t}}$.

Definition 0.21 ([5]) Let $\hat{\mu}$ be an interval-value fuzzy set in X and $x_{\hat{t}}$ be an interval-value fuzzy point then

- (i) If $\hat{\mu}(x) \geq \hat{t}$ then we say $x_{\hat{t}}$ belongs to $\hat{\mu}$ and write $x_{\hat{t}} \in \hat{\mu}$
- (ii) If $\hat{\mu}(x) + \hat{t} > [1, 1]$ then we say $x_{\hat{t}}$ quasi coincident $\hat{\mu}$ and write $x_{\hat{t}} q \hat{\mu}$

(iii) If $x_{\hat{t}} \in \vee q \hat{\mu} \Leftrightarrow x_t \in \hat{\mu}$ or $x_{\hat{t}} q \hat{\mu}$

(iv) If $x_{\hat{t}} \in \wedge q \hat{\mu} \Leftrightarrow x_{\hat{t}} \in \hat{\mu}$ and $x_{\hat{t}} q \hat{\mu}$

The symbol $x_{\hat{t}} \overline{\wedge} \hat{\mu}$ means $x_{\hat{t}} \wedge \hat{\mu}$ does not hold and $\overline{\wedge} \hat{\mu}$ means $\overline{\wedge} \hat{\mu}$

For an interval-valued fuzzy point $x_{\hat{t}}$ and an interval-valued fuzzy set $\hat{\mu}$ in set X , Pu and Liu [10] gave meaning to the symbol $x_{\hat{t}} \alpha \hat{\mu}$ where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$

Definition 0.22 A interval-valued fuzzy set $\hat{\mu}$ of a d -algebra X is said to be (α, β) -interval-valued fuzzy d -ideal of X , Where $\alpha \neq \in \wedge q$ if

(i) $x_{\hat{t}} \alpha \hat{\mu} \Rightarrow 0_{\hat{t}} \beta \hat{\mu}$

(ii) $(x * y)_{\hat{t}}, y_{\hat{s}} \alpha \hat{\mu} \Rightarrow x_{rmin(\hat{t}, \hat{s})} \beta \hat{\mu}$

(iii) $(x)_{\hat{t}}, y_{\hat{s}} \alpha \hat{\mu} \Rightarrow (x * y)_{rmin(\hat{t}, \hat{s})} \beta \hat{\mu}$

for all $x, y \in X$ where $[0, 0] < \hat{t}, \hat{s} \leq [1, 1]$, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$

Definition 0.23 A interval-valued fuzzy set $\hat{\mu}$ of a d -algebra X is said to be (α, β) -interval-valued fuzzy dot d -ideal of X , Where $\alpha \neq \in \wedge q$ if

(i) $x_{\hat{t}} \alpha \hat{\mu} \Rightarrow 0_{\hat{t}} \beta \hat{\mu}$

(ii) $(x * y)_{\hat{t}}, y_{\hat{s}} \alpha \hat{\mu} \Rightarrow x_{(\hat{t}, \hat{s})} \beta \hat{\mu}$

(iii) $(x)_{\hat{t}}, y_{\hat{s}} \alpha \hat{\mu} \Rightarrow (x * y)_{(\hat{t}, \hat{s})} \beta \hat{\mu}$

for all $x, y \in X$ where $[0, 0] < \hat{t}, \hat{s} \leq [1, 1]$

Definition 0.24 A interval-valued fuzzy set $\hat{\mu}$ of a d -algebra X is said to be $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal of X , if

(i) $x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \vee q \hat{\mu}$

(ii) $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{(\hat{t}, \hat{s})} \in \vee q \hat{\mu}$

(iii) $(x)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{(\hat{t}, \hat{s})} \in \vee q \hat{\mu}$

for all $x, y \in X$ where $[0, 0] < \hat{t}, \hat{s} \leq [1, 1]$

Example 0.25 Consider d -algebra X as in Example 0.11 and $\hat{\mu}$ by $\hat{\mu}(0) = [0.75, 0.9], \hat{\mu}(a) = \hat{\mu}(b) = [0.7, 0.8], \hat{\mu}(c) = [0.65, 0.7]$, then it is easy to verify that $\hat{\mu}$ is $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal X .

Theorem 0.26 Every (\in, \in) -interval-valued fuzzy d -ideal of d algebra X is an (\in, \in) -interval-valued fuzzy dot d -ideal of X .

Proof Straightforward

Remark 0.27 The converse of Theorem 0.26 is not true as shown in following Example.

Example 0.28 Consider d -algebra $X = \{0, a, b\}$ with the following cayley table.

*	0	a	b
0	0	0	0
a	b	0	b
b	a	a	0

Define $\hat{\mu} : X \rightarrow D[0, 1]$ by $\hat{\mu}(0) = [0.6, 0.7]$, $\hat{\mu}(a) = [0.7, 0.8]$, $\hat{\mu}(b) = [0.8, 0.9]$ then it is easy to verify that $\hat{\mu}$ is (\in, \in) -interval-valued fuzzy dot d ideal X. Here $(a * b)_{[0.77, 0.88]}$, $b_{[0.77, 0.88]} \in \hat{\mu}$ But $a_{[0.77, 0.88]} \notin \hat{\mu}$. Therefore $\hat{\mu}$ is not an (\in, \in) -interval-valued fuzzy d-ideal X.

Theorem 0.29 Every interval-valued fuzzy d-ideal of a d algebra X is an interval-valued fuzzy dot d-ideal of X.

Remark 0.30 The converse of Theorem 0.29 is not true as shown in following Example.

Example 0.31 Consider d-algebra X and $\hat{\mu}$ as in Example 0.28 then it is easy to verify that $\hat{\mu}$ is an interval-valued fuzzy dot d ideal X. But $rmin\{\hat{\mu}(a * b), \hat{\mu}(b)\} = \hat{\mu}(b) = [0.8, 0.9] \not\leq \hat{\mu}(a) = [0.7, 0.8]$ Therefore $\hat{\mu}$ is not an interval-valued fuzzy d-ideal X.

Theorem 0.32 An interval-valued fuzzy subset $\hat{\mu}$ of a d algebra X is a interval-valued fuzzy d-ideal iff $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy d-ideal of X

Proof Let $\hat{\mu}$ be an interval-valued fuzzy d-ideal of X.

To prove $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy d-ideal of X.

Let $x \in X$ such that $x_{\hat{t}} \in \hat{\mu}$ where $\hat{t} \in D(0, 1)$ then $\hat{\mu}(x) \geq \hat{t} \Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x) \geq \hat{t}$ [Since $\hat{\mu}$ -is interval-valued fuzzy d-ideal]

$\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\mu} \Rightarrow x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$

Again let $x, y \in X$ such that $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$ where $\hat{t}, \hat{s} \in D(0, 1)$

then $\hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$

Now $\hat{\mu}(x) \geq rmin\{\hat{\mu}(x * y), \hat{\mu}(y)\} \geq rmin(\hat{t}, \hat{s})$ [Since $\hat{\mu}$ -is interval-valued fuzzy d-ideal]

$\Rightarrow x_{rmin(\hat{t}, \hat{s})} \in \hat{\mu}$

$\Rightarrow (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{rmin(\hat{t}, \hat{s})} \in \hat{\mu}$

Again let $x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$

$\Rightarrow \hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$

Now $\hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x), \hat{\mu}(y)\} \geq rmin(\hat{t}, \hat{s})$ [Since $\hat{\mu}$ -interval-valued fuzzy d-ideal]

$\Rightarrow (x * y)_{rmin(\hat{t}, \hat{s})} \in \hat{\mu}$

$x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{rmin(\hat{t}, \hat{s})} \in \hat{\mu}$

Hence $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy d-ideal of X.

Conversely suppose $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy d-ideal of X.

to prove $\hat{\mu}$ is an interval-valued fuzzy d-ideal of X.

Let $x \in X$ Now since $x_{\hat{\mu}(x)} \in \hat{\mu}$ therefore $0_{\hat{\mu}(x)} \in \hat{\mu}$

$\Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x)$ [Since $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy d-ideal of X]

Let $x, y \in X$ again since $(x * y)_{\hat{\mu}(x * y)} \in \hat{\mu}$ and $y_{\hat{\mu}(y)} \in \hat{\mu}$

$\Rightarrow x_{rmin\{\hat{\mu}(x * y), \hat{\mu}(y)\}} \in \hat{\mu}$ [Since $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy d-ideal of X]

$\Rightarrow \hat{\mu}(x) \geq rmin\{\hat{\mu}(x * y), \hat{\mu}(y)\}$

Let $x, y \in X$ again since $x_{\hat{\mu}(x)} \in \hat{\mu}$ and $y_{\hat{\mu}(y)} \in \mu$

$\Rightarrow (x * y)_{rmin\{\hat{\mu}(x), \hat{\mu}(y)\}} \in \hat{\mu}$ [Since $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy d-ideal of X]

$\Rightarrow \hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x), \hat{\mu}(y)\}$

Hence $\hat{\mu}$ is an interval-valued fuzzy d-ideal of X.

Theorem 0.33 An interval-valued fuzzy subset $\hat{\mu}$ of a d algebra X is a interval-valued fuzzy dot d-ideal iff $\hat{\mu}$ is a (\in, \in) -interval-valued fuzzy dot d-ideal of X

Proof $\hat{\mu}$ be an interval-valued fuzzy dot d-ideal of X. To prove $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of X.

Let $x \in X$ such that $x_{\hat{t}} \in \hat{\mu}$ where $\hat{t} \in D(0, 1)$ then $\hat{\mu}(x) \geq \hat{t}$
 $\Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x) \geq \hat{t}$ [Since $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal]
 $\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$
 $\Rightarrow x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$

Let $x, y \in X$ such that $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$ where $\hat{t}, \hat{s} \in D(0, 1)$
 then $\hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
 Now $\hat{\mu}(x) \geq \hat{\mu}(x * y) \cdot \hat{\mu}(y) \geq \hat{t} \cdot \hat{s}$ [Since $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal]
 $\Rightarrow x_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$
 $\Rightarrow (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$

Again let $x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$
 $\Rightarrow \hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
 Now $\hat{\mu}(x * y) \geq \hat{\mu}(x) \cdot \hat{\mu}(y) \geq \hat{t} \cdot \hat{s}$ [Since $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal]
 $\Rightarrow (x * y)_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$
 $\Rightarrow x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$

Hence $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of X.

Conversely suppose $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of X. to prove $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal of X.

Let $x \in X$ and $\hat{t} = \hat{\mu}(x)$
 then $\hat{\mu}(x) \geq \hat{t} \Rightarrow x_{\hat{t}} \in \hat{\mu}$
 $\Rightarrow 0_{\hat{t}} \in \hat{\mu}$ [Since $\hat{\mu}$ is (\in, \in) -is an interval-valued fuzzy dot d-ideal of X.]
 $\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow \hat{\mu}(0) \geq \hat{\mu}(x)$

Let $x, y \in X$ and $\hat{t} = \hat{\mu}(x * y), \hat{s} = \hat{\mu}(y)$
 then $\hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
 $\Rightarrow (x * y)_{\hat{t}} \in \hat{\mu}, y_{\hat{s}} \in \hat{\mu}$
 $\Rightarrow x_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$ [Since $\hat{\mu}$ is (\in, \in) -is an interval-valued fuzzy dot d-ideal of X]
 $\Rightarrow \hat{\mu}(x) \geq \hat{t} \cdot \hat{s} = \hat{\mu}(x * y) \cdot \hat{\mu}(y)$

Again let $\hat{t} = \hat{\mu}(x), \hat{s} = \hat{\mu}(y)$
 then $\hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$
 $\Rightarrow x_{\hat{t}} \in \hat{\mu}, y_{\hat{s}} \in \hat{\mu}$
 $\Rightarrow (x * y)_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$ [Since $\hat{\mu}$ is (\in, \in) -interval-valued fuzzy dot d-ideal of X]
 $\Rightarrow \hat{\mu}(x * y) \geq \hat{t} \cdot \hat{s} = \hat{\mu}(x) \cdot \hat{\mu}(y)$

Hence $\hat{\mu}$ is an interval-valued fuzzy dot d-ideal of X.

Theorem 0.34 Every interval-valued fuzzy d-ideal of a d algebra X is an (\in, \in) -interval-valued fuzzy dot d-ideal of X.

Proof It follows from Theorem 0.29 and Theorem 0.33.

Remark 0.35 The converse of Theorem 0.34 is not true as shown in following Example.

Example 0.36 Consider d-algebra X and $\hat{\mu}$ as in Example 0.28 then it is easy to verify that $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot d ideal X. But $\min\{\hat{\mu}(a * b), \hat{\mu}(b)\} = \hat{\mu}(b) = [0.8, 0.9] \not\subseteq \hat{\mu}(a) = [0.7, 0.8]$ Therefore $\hat{\mu}$ is not an interval-valued fuzzy d-ideal X.

Theorem 0.37 Every (\in, \in) -interval-valued fuzzy d-ideal of a d algebra X is an interval-valued fuzzy dot d-ideal of X.

Proof It follows from Theorem 0.32 and Theorem 0.29.

Remark 0.38 The converse of Theorem 0.37 is not true as shown in following Example.

Example 0.39 Consider d -algebra X and $\hat{\mu}$ as in Example 0.28 then it is easy to verify that $\hat{\mu}$ is an interval-valued fuzzy dot d ideal X . But $(a * b)_{[0.77, 0.88]}, b_{[0.77, 0.88]} \in \hat{\mu}$ But $a_{[0.77, 0.88]} \notin \hat{\mu}$. Therefore $\hat{\mu}$ is not an (\in, \in) -interval-valued fuzzy d -ideal X .

Definition 0.40 A fuzzy subset $\hat{\mu}$ of X is called a (\in, \in) -interval-valued fuzzy dot subalgebra if $x_{\hat{\mu}}, y_{\hat{\mu}} \in \hat{\mu} \Rightarrow (x * y)_{\hat{\mu}} \in \hat{\mu}$ for all $x, y \in X$.

Proposition 0.41 Every (\in, \in) -interval-valued fuzzy dot d -ideal of a d -algebra X is an (\in, \in) -interval-valued fuzzy dot sub algebra of X .

Remark 0.42 The converse of Proposition 0.41 is not true as shown in following example.

Example 0.43 Consider d algebra $X = \{0, a, b, c\}$ with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Define $\hat{\mu} : X \rightarrow [0, 1]$ by $\hat{\mu}(0) = \hat{\mu}(b) = [0.8, 0.9]$, $\hat{\mu}(a) = \hat{\mu}(c) = [0.7, 0.8]$ then it is easy to verify that $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot subalgebra of X . But $\hat{\mu}$ is not an (\in, \in) -interval-valued fuzzy dot d -ideal of X because $\hat{\mu}(a * b) = \hat{\mu}(0) = [0.8, 0.9]$ and $\hat{\mu}(b) = [0.8, 0.9]$ therefore $\hat{\mu}(a * b) \geq [0.8, 0.9]$ and $\hat{\mu}(b) \geq [0.8, 0.9] \not\Rightarrow \hat{\mu}(a) \geq [0.8, 0.9]. [0.9, 0.9] = [0.72, 0.81]$

Theorem 0.44 Every (\in, \in) -interval-valued fuzzy dot d -ideal of a d -algebra X is a $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal X . But the converse is not true as shown in the following Example.

Example 0.45 Consider d -algebra $X = \{0, a, b, c\}$ with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Define $\hat{\mu} : X \rightarrow [0, 1]$ by $\hat{\mu}(0) = [0.85, 0.95]$, $\hat{\mu}(a) = \hat{\mu}(b) = [0.8, 0.9]$, $\hat{\mu}(c) = [0.6, 0.7]$, then it is easy to verify that $\hat{\mu}$ is $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal X . Since $a_{[0.8, 0.9]} = (c * b)_{[0.8, 0.9]}, b_{[0.8, 0.9]} \in \hat{\mu}$ but $\hat{\mu}(c) \not\geq [0.8, 0.9]. [0.8, 0.9]$ i.e $c_{[0.64, 0.81]} \notin \hat{\mu}$

Theorem 0.46 If $\hat{\mu}, \hat{\nu}$ are $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal of a d -algebra X , then so is $\hat{\mu} \cap \hat{\nu}$.

Proof Let $x, y \in X$ such that $x_{\hat{t}} \in (\hat{\mu} \cap \hat{\nu}) \Rightarrow (\hat{\mu} \cap \hat{\nu})(x) \geq \hat{t} \Rightarrow rmin\{\hat{\mu}(x), \hat{\nu}(x)\} \geq \hat{t} \Rightarrow \hat{\mu}(x) \geq \hat{t}, \hat{\nu}(x) \geq \hat{t} \Rightarrow x_{\hat{t}} \in \hat{\mu}$ and $x_{\hat{t}} \in \hat{\nu} \Rightarrow 0_{\hat{t}} \in \vee q\hat{\mu}$ and $0_{\hat{t}} \in \vee q\hat{\nu}$ [Since $\hat{\mu}, \hat{\nu}$ both are $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideals of X]

$$\Rightarrow 0_{\hat{t}} \in \vee q(\hat{\mu} \cap \hat{\nu})$$

Again let $(x * y)_{\hat{t}}, y_{\hat{s}} \in (\hat{\mu} \cap \hat{\nu})$

$$\Rightarrow (\hat{\mu} \cap \hat{\nu})(x * y) \geq \hat{t}, (\hat{\mu} \cap \hat{\nu})(y) \geq \hat{s}$$

$$\Rightarrow rmin\{\hat{\mu}(x * y), \hat{\nu}(x * y)\} \geq \hat{t} \text{ and } rmin\{\hat{\mu}(y), \hat{\nu}(y)\} \geq \hat{s}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}, \hat{\nu}(x * y) \geq \hat{t} \text{ and } \hat{\mu}(y) \geq \hat{s}, \hat{\nu}(y) \geq \hat{s}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s} \text{ and } \hat{\nu}(x * y) \geq \hat{t}, \hat{\nu}(y) \geq \hat{s}$$

$$\Rightarrow (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \text{ and } (x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\nu}$$

$$\Rightarrow x_{\hat{t}, \hat{s}} \in \vee q\hat{\mu} \text{ and } x_{\hat{t}, \hat{s}} \in \vee q\hat{\nu} \text{ [Since } \hat{\mu}, \hat{\nu} \text{ both are } (\in, \in \vee q)\text{-interval-valued fuzzy dot d-ideals of X]}$$

$$\Rightarrow x_{\hat{t}, \hat{s}} \in \vee q(\hat{\mu} \cap \hat{\nu})$$

Similarly we can prove that

$$x_{\hat{t}}, y_{\hat{s}} \in (\hat{\mu} \cap \hat{\nu}) \Rightarrow (x * y)_{\hat{t}, \hat{s}} \in \vee q(\hat{\mu} \cap \hat{\nu})$$

Hence the proof.

Theorem 0.47 If $\hat{\mu}$ is a (q, q) -interval-valued fuzzy dot d-ideal of a d algebra X then it is also $an(\in, \in)$ -interval-valued fuzzy dot d-ideal of X.

Proof Let $\hat{\mu}$ be an (q, q) -interval-valued fuzzy dot d-ideal of a d-algebra X, to prove $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of X. let $x \in X$ such that $x_{\hat{t}} \in \hat{\mu} \Rightarrow \hat{\mu}(x) \geq \hat{t}$

$$\Rightarrow \hat{\mu}(x) + \hat{\delta} > \hat{t} \text{ where } \hat{\delta} \text{ is a arbitrary small positive interval in } D[0 \ 1]$$

$$\Rightarrow \hat{\mu}(x) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1} \Rightarrow (x)_{\hat{1} + \hat{\delta} - \hat{t}} q\hat{\mu} \Rightarrow (0)_{\hat{1} + \hat{\delta} - \hat{t}} q\hat{\mu} \Rightarrow \hat{\mu}(0) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1} \Rightarrow \hat{\mu}(0) + \hat{\delta} > \hat{t}$$

$$\Rightarrow \hat{\mu}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$$

$$\text{Therefore } x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$$

Again let $x, y \in X$ such that

$$(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \text{ therefore } \hat{\mu}(x * y) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$$

$$\Rightarrow \hat{\mu}(x * y) + \hat{\delta} > \hat{t}, \hat{\mu}(y) + \hat{\delta} > \hat{s} \text{ where } \hat{\delta} \text{ is a arbitrary small positive interval in } D[0 \ 1]$$

$$\Rightarrow \hat{\mu}(x * y) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1}, \hat{\mu}(y) + \hat{\delta} - \hat{s} + \hat{1} > \hat{1}$$

$$\Rightarrow (x * y)_{\hat{1} + \hat{\delta} - \hat{t}} q\hat{\mu} \text{ and } (y)_{\hat{1} + \hat{\delta} - \hat{s}} q\hat{\mu}$$

$$\Rightarrow (x)_{(\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s})} q\hat{\mu} \text{ [since } \hat{\mu} \text{ is } (q, q)\text{-interval-valued fuzzy dot d-ideal of X]}$$

$$\Rightarrow \hat{\mu}(x) + (\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s}) > \hat{1}$$

$$\Rightarrow \hat{\mu}(x) + (\hat{1} - \hat{t}).(\hat{1} - \hat{s}) \geq \hat{1} \text{ taking } \hat{\delta} = \hat{0}$$

$$\Rightarrow \hat{\mu}(x) + \hat{1} - \hat{t} - \hat{s} + \hat{t}.\hat{s} \geq \hat{1}$$

$$\Rightarrow \hat{\mu}(x) \geq \hat{s} + \hat{t} - \hat{t}.\hat{s}$$

$$\Rightarrow \hat{\mu}(x) \geq \hat{t}.\hat{s} + \hat{t}.\hat{s} - \hat{t}.\hat{s} \text{ [Since } \hat{t} \geq \hat{t}.\hat{s} \text{ and } \hat{s} \geq \hat{t}.\hat{s}]}$$

$$\Rightarrow \hat{\mu}(x) \geq \hat{t}.\hat{s}$$

$$\Rightarrow x_{\hat{t}, \hat{s}} \in \hat{\mu}$$

Again let

$$x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \text{ therefore } \hat{\mu}(x) \geq \hat{t}, \hat{\mu}(y) \geq \hat{s}$$

$$\Rightarrow \hat{\mu}(x) + \hat{\delta} > \hat{t}, \hat{\mu}(y) + \hat{\delta} > \hat{s} \text{ where } \hat{\delta} \text{ is a arbitrary small positive interval in } D[0 \ 1]$$

$$\Rightarrow \hat{\mu}(x) + \hat{\delta} - \hat{t} + \hat{1} > \hat{1}, \hat{\mu}(y) + \hat{\delta} - \hat{s} + \hat{1} > \hat{1}$$

$$\Rightarrow (x)_{\hat{1} + \hat{\delta} - \hat{t}} q\hat{\mu} \text{ and } (y)_{\hat{1} + \hat{\delta} - \hat{s}} q\hat{\mu}$$

$$\Rightarrow (x * y)_{(\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s})} q\hat{\mu} \text{ [since } \hat{\mu} \text{ is } (q, q)\text{-interval-valued fuzzy dot d-ideal of X]}$$

$$\Rightarrow \hat{\mu}(x * y) + (\hat{1} + \hat{\delta} - \hat{t}).(\hat{1} + \hat{\delta} - \hat{s}) > \hat{1}$$

$$\Rightarrow \hat{\mu}(x * y) + (\hat{1} - \hat{t}).(\hat{1} - \hat{s}) \geq \hat{1} \text{ taking } \hat{\delta} = \hat{0}$$

$$\Rightarrow \hat{\mu}(x * y) + \hat{1} - \hat{t} - \hat{s} + \hat{t}.\hat{s} \geq \hat{1}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{s} + \hat{t} - \hat{t}.\hat{s}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t}.\hat{s} + \hat{t}.\hat{s} - \hat{t}.\hat{s} \text{ [Since } \hat{t} \geq \hat{t}.\hat{s} \text{ and } \hat{s} \geq \hat{t}.\hat{s}]}$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t} \cdot \hat{s}$$

$$\Rightarrow (x * y)_{\hat{t} \cdot \hat{s}} \in \hat{\mu}$$

Hence $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of X.

Remark 0.48 *The converse of Theorem 0.47 is not true as shown in following Example.*

Example 0.49 *Consider d-algebra X and $\hat{\mu}$ as in Example 0.11, then $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of X. But $\hat{\mu}$ is not an (q, q) -interval-valued fuzzy dot d-ideal of X because if $x = c, y = a, x * y = c * a = c, t = 0.7, s = 0.4$ then $\mu(x * y) + t = 0.35 + 0.7 > 1, \mu(y) + s = 0.7 + 0.4 > 1$ but $\mu(x) + t \cdot s = 0.35 + 0.7 \times 0.4 = 0.35 + 0.28 < 1$ i.e $x_{\hat{t} \cdot \hat{s}} \bar{q} \hat{\mu}$.*

Theorem 0.50 *A fuzzy subset $\hat{\mu}$ of a d-algebra X is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X iff*

$$(i) \hat{\mu}(0) \geq rmin\{\hat{\mu}(x), [0.5, 0.5]\}$$

$$(ii) \hat{\mu}(x) \geq rmin\{\hat{\mu}(x * y) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x * y) \cdot \hat{\mu}(y)\}$$

$$(iii) \hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x) \cdot \hat{\mu}(y)\}$$

Proof Let $\hat{\mu}$ be an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X

$$(i) \text{ Assume } \hat{\mu}(0) < rmin\{\hat{\mu}(x), [0.5, 0.5]\}$$

SubCase I: If $\hat{\mu}(x) < [0.5, 0.5]$, then $rmin\{\hat{\mu}(x), [0.5, 0.5]\} = \hat{\mu}(x)$

$$\Rightarrow \hat{\mu}(0) < \hat{\mu}(x) \Rightarrow \hat{\mu}(0) < \hat{t} < \hat{\mu}(x) \text{ for some } (0, 0) < \hat{t} < (0.5, 0.5)$$

$$\Rightarrow x_{\hat{t}} \in \hat{\mu} \text{ and } 0_{\hat{t}} \bar{\in} \hat{\mu} \text{ also } \hat{\mu}(0) + \hat{t} < \hat{1} \text{ i.e } 0_{\hat{t}} \bar{q} \hat{\mu} \Rightarrow 0_{\hat{t}} \bar{\in} \vee q \hat{\mu} \text{ Which is a contradiction}$$

SubCase II: If $\hat{\mu}(x) \geq [0.5, 0.5]$ i.e $x_{[0.5, 0.5]} \in \hat{\mu}$, then $rmin\{\hat{\mu}(x), [0.5, 0.5]\} = [0.5, 0.5]$

$$\hat{\mu}(0) < rmin\{\hat{\mu}(x), [0.5, 0.5]\} = [0.5, 0.5]$$

$$\Rightarrow 0_{[0.5, 0.5]} \bar{\in} \hat{\mu} \text{ and } \hat{\mu}(0) + [0.5, 0.5] < [0.5, 0.5] + [0.5, 0.5] = \hat{1} \text{ i.e } 0_{[0.5, 0.5]} \bar{q} \hat{\mu}$$

$$\Rightarrow 0_{[0.5, 0.5]} \bar{\in} \vee q \hat{\mu} \text{ Which is again a contradiction, therefore}$$

Therefore we must have $\hat{\mu}(0) \geq rmin\{\hat{\mu}(x), [0.5, 0.5]\}$

$$(ii) \text{ Assume } \hat{\mu}(x) < rmin\{\hat{\mu}(x * y) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x * y) \cdot \hat{\mu}(y)\}$$

choose $[0, 0] \leq \hat{t}, \hat{s} \leq [0, 1]$ such that $\hat{\mu}(x) < \hat{t} \cdot \hat{s} < rmin\{\hat{\mu}(x * y) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x * y) \cdot \hat{\mu}(y)\}$

where $\hat{\mu}(x * y) \geq \hat{t}$ and $\hat{\mu}(y) \geq \hat{s}$

SubCase I: If $\hat{\mu}(x * y) \cdot \hat{\mu}(y) < [0.5, 0.5]$, then $rmin\{\hat{\mu}(x * y) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x * y) \cdot \hat{\mu}(y)\} = \hat{\mu}(x * y) \cdot \hat{\mu}(y)$

$$\Rightarrow \hat{\mu}(x) < \hat{t} \cdot \hat{s} < \hat{\mu}(x * y) \cdot \hat{\mu}(y) < [0.5, 0.5]$$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t} \text{ and } \hat{\mu}(y) \geq \hat{s}$$

$\Rightarrow (x * y)_{\hat{t}} \in \hat{\mu} \text{ and } y_{\hat{s}} \in \hat{\mu} \text{ but } x_{\hat{t} \cdot \hat{s}} \bar{\in} \hat{\mu} \text{ and } \hat{\mu}(x) + \hat{t} \cdot \hat{s} < \hat{1} \text{ and } x_{\hat{t} \cdot \hat{s}} \bar{q} \hat{\mu} \text{ Which contradict the fact that } \hat{\mu} \text{ is an } (\in, \in \vee q)\text{-interval-valued fuzzy dot d-ideal of X}$

SubCase II: If $\hat{\mu}(x * y) \cdot \hat{\mu}(y) \geq [0.5, 0.5]$, then $\hat{\mu}(x) < \hat{t} \cdot \hat{s} < [0.5, 0.5] < \hat{\mu}(x * y) \cdot \hat{\mu}(y)$

$$\Rightarrow \hat{\mu}(x * y) \geq \hat{t} \text{ and } \hat{\mu}(y) \geq \hat{s}$$

$$\Rightarrow (x * y)_{\hat{t}} \in \hat{\mu} \text{ and } y_{\hat{s}} \in \hat{\mu} \text{ but } \hat{\mu}(x) \leq \hat{t} \cdot \hat{s} \text{ and } \hat{\mu}(x) + \hat{t} \cdot \hat{s} < \hat{1} \text{ that is } x_{\hat{t} \cdot \hat{s}} \bar{\in} \hat{\mu} \text{ and } x_{\hat{t} \cdot \hat{s}} \bar{q} \hat{\mu} \text{ that is } x_{\hat{t} \cdot \hat{s}} \bar{\in} \vee q \hat{\mu}$$

Which is again a contradiction, therefore in both subcases

Therefore we must have $\hat{\mu}(x * y) \geq rmin\{\hat{\mu}(x) \cdot \hat{\mu}(y), \hat{1} - \hat{\mu}(x) \cdot \hat{\mu}(y)\}$

(iii) Proof is similar to case (ii) above

Theorem 0.51 *A fuzzy subset $\hat{\mu}$ of d- algebra X is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X and $\hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \forall x \in X$ then $\hat{\mu}$ is also an (\in, \in) -interval-valued fuzzy dot d-ideal of X.*

Proof Let $\hat{\mu}$ be an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X and $\hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \forall x \in X$
Let $x_{\hat{t}} \in \hat{\mu} \Rightarrow \hat{\mu}(x) \geq \hat{t}$

$$\Rightarrow \hat{t} \leq \hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \text{ and also } \hat{\mu}(0) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$$

$$\hat{\mu}(0) + \hat{t} < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] + [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] = [\sqrt{5} - 1, \sqrt{5} - 1] \Rightarrow \hat{\mu}(0) + \hat{t} \not\geq \hat{1}$$

$$\Rightarrow 0_{\hat{t}}\bar{q}\hat{\mu} \text{ therefore } x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}}\bar{q}\hat{\mu}$$

Since $\hat{\mu}$ be an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X , therefore we must have $x_{\hat{t}} \in \hat{\mu} \Rightarrow 0_{\hat{t}} \in \hat{\mu}$

Again let $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu}$

$$\Rightarrow \hat{t} \leq \hat{\mu}(x * y) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \text{ and } \hat{t} \leq \hat{\mu}(y) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$$

$$\Rightarrow \hat{t}.\hat{s} \leq \hat{\mu}(x * y).\hat{\mu}(y) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}].[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \text{ and}$$

$$\Rightarrow \hat{t}.\hat{s} \leq [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}].[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] \text{ and } \hat{\mu} \text{ is also an } (\in, \in)\text{-interval-valued fuzzy dot d-ideal of } X$$

$$\hat{\mu}(x) < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$$

$$\Rightarrow \mu(x) + \hat{t}.\hat{s} < [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}].[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] + [\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}] = \hat{1}$$

$$\Rightarrow \hat{\mu}(x) + \hat{t}.\hat{s} < \hat{1}$$

$$\Rightarrow x_{\hat{t}.\hat{s}}\bar{q}\hat{\mu}$$

therefore $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{\hat{t}.\hat{s}}\bar{q}\hat{\mu}$

since $\hat{\mu}$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X , therefore we must have $x_{\hat{t}.\hat{s}} \in \hat{\mu}$.

Therefore $(x * y)_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow x_{\hat{t}.\hat{s}} \in \hat{\mu}$

Similarly we can prove that $x_{\hat{t}}, y_{\hat{s}} \in \hat{\mu} \Rightarrow (x * y)_{\hat{t}.\hat{s}} \in \hat{\mu}$

Hence $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of X .

Theorem 0.52 *Let $\hat{\mu}$ be a left fuzzy relation on a fuzzy subset $\hat{\sigma}$ of a d-algebra X . If $\hat{\mu}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of $X \times X$ then $\hat{\sigma}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of d-algebra X .*

Proof Suppose that a left fuzzy relation $\hat{\mu}$ on $\hat{\sigma}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of $X \times X$.

To prove that $\hat{\sigma}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of X

$$\text{Let } x, y \in X \text{ since } \hat{\sigma}(x) = \hat{\mu}(x, y) \text{ let } x_{\hat{t}} \in \hat{\sigma} \Rightarrow \hat{\sigma}(x) \geq \hat{t} \Rightarrow \hat{\sigma}(x) = \hat{\mu}(x, y) \geq \hat{t}$$

$$(x, y)_{\hat{t}} \in \hat{\mu} \Rightarrow (0, 0)_{\hat{t}} \in \hat{\mu} \text{ [Since } \hat{\mu} \text{ is an } (\in, \in)\text{-interval-valued fuzzy dot d-ideal of } X \times X \text{]}$$

$$\Rightarrow \hat{\mu}(0, 0) \geq \hat{t} \Rightarrow \hat{\sigma}(0) \geq \hat{t} \Rightarrow 0_{\hat{t}} \in \hat{\sigma}$$

therefore $x_{\hat{t}} \in \hat{\sigma} \Rightarrow 0_{\hat{t}} \in \hat{\sigma}$

Let $x, x', y, y' \in X$

$$\hat{\sigma}(x) = \hat{\mu}(x, y)$$

now Let $(x * x')_{\hat{t}}, (x')_{\hat{s}} \in \hat{\sigma}$

$$\Rightarrow \hat{\sigma}(x * x') \geq \hat{t} \text{ and } \hat{\sigma}(x') \geq \hat{s}$$

Now $\hat{\sigma}(x) = \hat{\mu}(x, y) \geq \hat{\mu}((x, y) * (x', y')).\hat{\mu}(x', y')$ [Since $\hat{\mu}$ is also an interval-valued fuzzy dot d-ideal of $X \times X$, see Theorem 0.33]

$$= \hat{\mu}(x * x', y * y').\hat{\mu}(x', y') = \hat{\sigma}(x * x').\hat{\sigma}(x') \geq \hat{t}.\hat{s}$$

$$\Rightarrow (x)_{\hat{t}.\hat{s}} \in \hat{\sigma}$$

$$(x * x')_{\hat{t}}, (x')_{\hat{s}} \in \hat{\sigma} \Rightarrow (x)_{\hat{t}.\hat{s}} \in \hat{\sigma}$$

Again let $x_{\hat{t}}, x'_{\hat{s}} \in \hat{\sigma} \Rightarrow \hat{\sigma}(x) \geq \hat{t} \text{ and } \hat{\sigma}(x') \geq \hat{s}$

$$\hat{\sigma}(x * x') = \hat{\mu}(x * x', y * y')$$

$$= \hat{\mu}((x, y) * (x', y')) \geq \hat{\mu}(x, y).\hat{\mu}(x', y') \geq \hat{\sigma}(x).\hat{\sigma}(x') \geq \hat{t}.\hat{s}$$

$$(x * x')_{\hat{t}.\hat{s}} \in \hat{\sigma}$$

$$x_{\hat{t}}, x'_{\hat{s}} \in \hat{\sigma} \Rightarrow (x * x')_{\hat{t}.\hat{s}} \in \hat{\sigma}$$

Hence from above $\hat{\sigma}$ is an (\in, \in) -interval-valued fuzzy dot d-ideal of X

4 Homomorphism of d -algebras and $(\in, \in \vee q)$ -interval-valued Fuzzy Dot d -ideals

Definition 0.53 Let X and X' be two d -algebras, then a mapping $f : X \rightarrow X'$ is said to be homomorphism if $f(x * y) = f(x) * f(y) \forall x, y \in X$.

Theorem 0.54 Let X and X' be two d -algebras and $f : X \rightarrow X'$ be a homomorphism. Then $f(0) = 0'$

Proof Let $x \in X$ therefore $f(x) \in X'$. Now $f(0) = f(x * x) = f(x) * f(x) = 0 * 0 = 0'$.

Theorem 0.55 Let $f : X \rightarrow X'$ be an onto homomorphism of d -algebras, $\hat{\nu}$ be a $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal of X' , then the pre-image $f^{-1}(\hat{\nu})$ of $\hat{\nu}$ under f is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal of X .

Proof $f^{-1}(\hat{\nu})$ is defined as $f^{-1}(\hat{\nu})(x) = \hat{\nu}(f(x)) \forall x \in X$

Let $\hat{\nu}$ be an (\in, \in) -interval-valued fuzzy dot d -ideal of X'

Let $x \in X$ such that $x_{\hat{t}} \in f^{-1}(\hat{\nu}) \Rightarrow f^{-1}(\hat{\nu})(x) \geq \hat{t} \Rightarrow \hat{\nu}f(x) \geq \hat{t} \Rightarrow (f(x))_{\hat{t}} \in \hat{\nu} \Rightarrow x'_{\hat{t}} \in \hat{\nu}$ where $f(x) = x'$

$\Rightarrow 0'_{\hat{t}} \in \vee q \hat{\nu}$ [Since $\hat{\nu}$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal of X']

$\Rightarrow [f(0)]_{\hat{t}} \in \vee q \hat{\nu} \Rightarrow \hat{\nu}f(0) \geq \hat{t}$ or $\hat{\nu}f(0) + \hat{t} \geq \hat{1} \Rightarrow f^{-1}(\hat{\nu})(0) \geq \hat{t}$ or $f^{-1}(\hat{\nu})(0) + \hat{t} \geq \hat{1}$

$\Rightarrow 0_{\hat{t}} \in f^{-1}(\hat{\nu})$ or $0_{\hat{t}}qf^{-1}(\hat{\nu})$

$\Rightarrow 0_{\hat{t}} \in \vee q f^{-1}(\hat{\nu})$

Therefore $x_{\hat{t}} \in f^{-1}(\hat{\nu}) \Rightarrow 0_{\hat{t}} \in \vee q f^{-1}(\hat{\nu})$

Let $x, y \in X$ such that $(x * y)_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu})$

$\Rightarrow f^{-1}(\hat{\nu})(x * y) \geq \hat{t}$ and $f^{-1}(\hat{\nu})(y) \geq \hat{s}$

$\Rightarrow \hat{\nu}f(x * y) \geq \hat{t}$ and $\hat{\nu}f(y) \geq \hat{s}$

$\Rightarrow (f(x * y))_{\hat{t}} \in \hat{\nu}$ and $(f(y))_{\hat{s}} \in \hat{\nu}$

$\Rightarrow (f(x) * f(y))_{\hat{t}} \in \hat{\nu}$ and $(f(y))_{\hat{s}} \in \hat{\nu}$ [Since f is homomorphism]

$\Rightarrow (f(x))_{\hat{t}, \hat{s}} \in \vee q \hat{\nu}$ [Since $\hat{\nu}$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal of X']

$\Rightarrow \hat{\nu}f(x) \geq \hat{t}, \hat{s}$ or $\hat{\nu}f(x) + \hat{t}, \hat{s} \geq \hat{1}$

$\Rightarrow f^{-1}(\hat{\nu})(x) \geq \hat{t}, \hat{s}$ or $f^{-1}(\nu)(x) + \hat{t}, \hat{s} \geq \hat{1}$

$\Rightarrow x_{\hat{t}, \hat{s}} \in f^{-1}(\hat{\nu})$ or $x_{\hat{t}, \hat{s}}qf^{-1}(\hat{\nu})$

$\Rightarrow x_{\hat{t}, \hat{s}} \in \vee q f^{-1}(\hat{\nu})$ Therefore $(x * y)_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu}) \Rightarrow x_{\hat{t}, \hat{s}} \in \vee q f^{-1}(\hat{\nu})$

Again let $x, y \in X$ such that $x_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu})$

$\Rightarrow f^{-1}(\hat{\nu})(x) \geq \hat{t}$ and $f^{-1}(\hat{\nu})(y) \geq \hat{s}$

$\Rightarrow \hat{\nu}f(x) \geq \hat{t}$ and $\hat{\nu}f(y) \geq \hat{s}$

$\Rightarrow (f(x))_{\hat{t}} \in \hat{\nu}$ and $(f(y))_{\hat{s}} \in \hat{\nu}$

$\Rightarrow (f(x) * f(y))_{\hat{t}, \hat{s}} \in \vee q \hat{\nu}$ [Since $\hat{\nu}$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal of X']

$\Rightarrow \hat{\nu}(f(x) * f(y)) \geq \hat{t}, \hat{s}$ or $\hat{\nu}(f(x) * f(y)) + \hat{t}, \hat{s} \geq \hat{1}$

$\Rightarrow \hat{\nu}(f(x * y)) \geq \hat{t}, \hat{s}$ or $\hat{\nu}(f(x * y)) + \hat{t} \geq \hat{1}$ [Since f is homomorphism]

$\Rightarrow f^{-1}(\hat{\nu})(x * y) \geq \hat{t}, \hat{s}$ or $f^{-1}(\hat{\nu})(x * y) + \hat{t}, \hat{s} \geq \hat{1}$

$\Rightarrow (x * y)_{\hat{t}, \hat{s}} \in f^{-1}(\hat{\nu})$ or $(x * y)_{\hat{t}, \hat{s}}qf^{-1}(\hat{\nu})$

$\Rightarrow (x * y)_{\hat{t}, \hat{s}} \in \vee q f^{-1}(\hat{\nu})$

Therefore $x_{\hat{t}}, y_{\hat{s}} \in f^{-1}(\hat{\nu}) \Rightarrow (x * y)_{\hat{t}, \hat{s}} \in \vee q f^{-1}(\nu)$

Hence $f^{-1}(\hat{\nu})$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal of X .

Theorem 0.56 An onto homomorphic image of an $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal with the sup property is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d -ideal.

Proof Let $f : X \rightarrow X'$ be an onto homomorphism of d-algebras, $\hat{\mu}$ be an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X , and the image of $\hat{\mu}$ under f be $f(\hat{\mu})$. To prove $f(\hat{\mu})$ is an $(\in, \in \vee q)$ -fuzzy dot d-ideal of X'

Since $0 \in f^{-1}(0')$ therefore, for any $x', y' \in X'$, let $x_0, y_0 \in X$ such that

$$\hat{\mu}(x_0) = \sup_{z \in f^{-1}(x')} \hat{\mu}(z) \quad \hat{\mu}(y_0) = \sup_{z \in f^{-1}(y')} \hat{\mu}(z)$$

and

$$\hat{\mu}(x_0 * y_0) = \sup_{z \in f^{-1}(x' * y')} \hat{\mu}(z)$$

then, let

$$\begin{aligned} x'_t \in f(\hat{\mu}) &\Rightarrow f(\hat{\mu})(x') \geq \hat{t} \Rightarrow \sup_{z \in f^{-1}(x')} \hat{\mu}(z) = \hat{t} \\ &\Rightarrow \hat{\mu}(x_0) \geq \hat{t} \Rightarrow (x_0)_t \in \hat{\mu} \Rightarrow 0_t \in \vee q \hat{\mu} \end{aligned}$$

[Since $\hat{\mu}$ be an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X]

$$\begin{aligned} &\Rightarrow \hat{\mu}(0) \geq \hat{t} \quad \text{or} \quad \hat{\mu}(0) + \hat{t} \geq \hat{1} \\ &\Rightarrow \sup_{z \in f^{-1}(0')} \hat{\mu}(z) \geq \hat{t} \quad \text{or} \quad \sup_{z \in f^{-1}(0')} \hat{\mu}(z) + \hat{t} \geq \hat{1} \\ &\Rightarrow f(\hat{\mu})(0') \geq \hat{t} \quad \text{or} \quad f(\hat{\mu})(0') + \hat{t} \geq \hat{1} \\ &\Rightarrow 0'_t \in f(\hat{\mu}) \quad \text{or} \quad 0'_t q f(\hat{\mu}) \\ &\Rightarrow 0'_t \in \vee q f(\hat{\mu}) \end{aligned}$$

Therefore $\Rightarrow x'_t \in f(\hat{\mu}) \Rightarrow 0'_t \in \vee q f(\hat{\mu})$ Again, let $(x' * y')_{t, \hat{s}} \in f(\hat{\mu}) \Rightarrow f(\hat{\mu})(x' * y') \geq \hat{t}, f(\hat{\mu})(y') \geq \hat{s}$

$$\begin{aligned} &\Rightarrow \sup_{z \in f^{-1}(x' * y')} \hat{\mu}(z) \geq \hat{t}, \sup_{z \in f^{-1}(y')} \hat{\mu}(z) \geq \hat{s} \\ &\Rightarrow \hat{\mu}(x_0 * y_0) \geq \hat{t}, \hat{\mu}(y_0) \geq \hat{s} \\ &\Rightarrow (x_0 * y_0)_{t, \hat{s}} \in \hat{\mu}, (y_0)_{\hat{s}} \in \hat{\mu} \Rightarrow (x_0)_{t, \hat{s}} \in \vee q \hat{\mu} \end{aligned}$$

[Since $\hat{\mu}$ be an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X ,]

$$\begin{aligned} &\Rightarrow \hat{\mu}(x_0) \geq \hat{t} \cdot \hat{s} \quad \text{or} \quad \hat{\mu}(x_0) + \hat{t} \cdot \hat{s} \geq \hat{1} \\ &\Rightarrow \sup_{z \in f^{-1}(x')} \hat{\mu}(z) \geq \hat{t} \cdot \hat{s} \quad \text{or} \quad \sup_{z \in f^{-1}(x')} \hat{\mu}(z) + \hat{t} \cdot \hat{s} \geq \hat{1} \end{aligned}$$

$$\Rightarrow f(\hat{\mu})(x') \geq \hat{t} \cdot \hat{s} \quad \text{or} \quad f(\hat{\mu})(x') + \hat{t} \cdot \hat{s} \geq \hat{1}$$

$$\Rightarrow (x')_{t, \hat{s}} \in f(\hat{\mu}) \quad \text{or} \quad (x')_{t, \hat{s}} q f(\hat{\mu})$$

$$\Rightarrow (x')_{t, \hat{s}} \in \vee q f(\hat{\mu})$$

$$\Rightarrow (x' * y')_{t, \hat{s}} \in f(\hat{\mu}) \Rightarrow (x')_{t, \hat{s}} \in \vee q f(\hat{\mu}) \text{ Similarly we can prove}$$

$$x'_{t, \hat{s}}, y'_{\hat{s}} \in f(\hat{\mu}) \Rightarrow (x' * y')_{t, \hat{s}} \in \vee q f(\hat{\mu})$$

Hence from above $f(\hat{\mu})$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X' .

Theorem 0.57 Let $f : X \rightarrow X'$ be an onto homomorphism of d-algebras, $\hat{\mu}$ be a fuzzy subset of X' such that $f^{-1}(\hat{\mu})$ is an $(\in, \in \vee q)$ -fuzzy dot d-ideal of X , then $\hat{\mu}$ is also an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X' .

Proof Let $f^{-1}(\hat{\mu})$ be an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X

Let $x', y' \in X'$ since f is onto so there exists $x, y \in X$ such that $f(x) = x', f(y) = y'$ and also f is homomorphism therefore $f(x * y) = f(x) * f(y) = x' * y'$

Let $x'_t \in \hat{\mu}$ where $t \in D[0 1]$

Therefore $\hat{\mu}(x') \geq t \Rightarrow \hat{\mu}(f(x)) \geq t$

$\Rightarrow f^{-1}(\hat{\mu})(x) \geq t$

$\Rightarrow x_t \in f^{-1}(\hat{\mu})$

$\Rightarrow 0_t \in \vee q f^{-1}(\hat{\mu})$ [Since $f^{-1}(\hat{\mu})$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X.]

$\Rightarrow f^{-1}(\hat{\mu})(0) \geq t$ or $f^{-1}(\hat{\mu})(0) + t \geq \hat{1}$

$\Rightarrow \hat{\mu}(f(0)) \geq t$ or $\hat{\mu}(f(0)) + t \geq \hat{1}$

$\Rightarrow \hat{\mu}(0') \geq t$ or $\hat{\mu}(0') + t \geq \hat{1}$

$\Rightarrow 0'_t \in \hat{\mu}$ or $0'_t q \hat{\mu}$

$\Rightarrow 0'_t \in \vee q \hat{\mu}$

Therefore $x'_t \in \hat{\mu} \Rightarrow 0'_t \in \vee q \hat{\mu}$

Again let $(x' * y')_{t, s} \in \hat{\mu}$ where $t, s \in D[0 1]$

Therefore $\hat{\mu}(x' * y') \geq t$ and $\hat{\mu}(y') \geq s$

$\Rightarrow \hat{\mu}(f(x * y)) \geq t$ and $\hat{\mu}(f(y)) \geq s$

$\Rightarrow f^{-1}(\hat{\mu})(x * y) \geq t$ and $f^{-1}(\hat{\mu})(y) \geq s$

$\Rightarrow (x * y)_t \in f^{-1}(\hat{\mu})$ and $y_s \in f^{-1}(\hat{\mu})$

$\Rightarrow x_{t, s} \in \vee q f^{-1}(\hat{\mu})$ [Since $f^{-1}(\hat{\mu})$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X.]

$\Rightarrow f^{-1}(\hat{\mu})(x) \geq t \cdot s$ or $f^{-1}(\hat{\mu})(x) + t \cdot s \geq \hat{1}$

$\Rightarrow \hat{\mu}(f(x)) \geq t \cdot s$ or $\hat{\mu}(f(x)) + t \cdot s \geq \hat{1}$

$\Rightarrow \hat{\mu}(x') \geq t \cdot s$ or $\hat{\mu}(x') + t \cdot s \geq \hat{1}$

$\Rightarrow x'_{t, s} \in \hat{\mu}$ or $x'_{t, s} q \hat{\mu}$

$\Rightarrow x'_{t, s} \in \vee q \hat{\mu}$

$\Rightarrow (x' * y')_{t, s} \in \hat{\mu} \Rightarrow x'_{t, s} \in \vee q \hat{\mu}$

Again let $x', y' \in X'$ such that $(x')_t, y'_s \in \hat{\mu}$

therefore $\hat{\mu}(x') \geq t$ and $\hat{\mu}(y') \geq s$

$\Rightarrow \hat{\mu}(f(x)) \geq t$ and $\hat{\mu}(f(y)) \geq s$

$\Rightarrow f^{-1}(\hat{\mu})(x) \geq t$ and $f^{-1}(\hat{\mu})(y) \geq s$

$\Rightarrow x_t \in f^{-1}(\hat{\mu})$ and $y_s \in f^{-1}(\hat{\mu})$

$\Rightarrow (x * y)_{t, s} \in \vee q f^{-1}(\hat{\mu})$ [Since $f^{-1}(\hat{\mu})$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X.]

$\Rightarrow f^{-1}(\hat{\mu})(x * y) \geq t \cdot s$ or $f^{-1}(\hat{\mu})(x * y) + t \cdot s \geq \hat{1}$

$\Rightarrow \hat{\mu}(f(x * y)) \geq t \cdot s$ or $\hat{\mu}(f(x * y)) + t \cdot s \geq \hat{1}$

$\Rightarrow \hat{\mu}(f(x) * f(y)) \geq t \cdot s$ or $\hat{\mu}(f(x) * f(y)) + t \cdot s \geq \hat{1}$

$\Rightarrow (x' * y')_{t, s} \in \hat{\mu}$ or $(x' * y')_{t, s} q \hat{\mu}$

$\Rightarrow (x' * y')_{t, s} \in \vee q \hat{\mu}$

$\Rightarrow x'_t, y'_s \in \hat{\mu} \Rightarrow (x' * y')_{t, s} \in \vee q \hat{\mu}$

Hence $\hat{\mu}$ is an $(\in, \in \vee q)$ -interval-valued fuzzy dot d-ideal of X' .

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