# Generalized $\eta$-Einstein 3-dimensional trans-Sasakian manifold

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**Abstract.** In the present note we have introduced a new concept called generalized $\eta$-Einstein manifold in a 3-dimensional trans-Sasakian manifold and have given some preliminary ideas about the same. In the next section some geometric properties about this manifold have been deduced. Finally we have justified the existence of such a manifold by citing an example.

## Introduction

In 1985, J.A. Oubina introduced a new class of almost contact manifold namely trans-Sasakian manifold [4]. Many geometers in [1], [3], [6], [2] have studied the structure of trans-Sasakian manifold and obtained many results on it.

Definition(1.1) A 3-dimensional trans-Sasakian manifold M is said to be an $\eta$-Einstein manifold if Ricci tensor satisfies the relation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1}$$

where *a, b* are smooth functions.
Now we introduce the following definition:
Definition(1.2) A 3-dimensional trans-Sasakian manifold M is said to be an generalized $\eta$-Einstein manifold if the following relation holds on M

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\Omega(X,Y), \tag{2}$$

where *a; b; c* are smooth functions and $\Omega(X,Y) = g(\phi X,Y)$. If $c = 0$ then the manifold reduces to an $\eta$-Einstein manifold.

## Preliminaries

Let M be an 3-dimensional almost contact metric manifold with almost contact metric structure $(\phi, \xi, \eta)$, where $\phi$ is a tensor of type (1; 1), $\xi$ is a vector field, $\eta$ is a 1-form satisfying following properties :

$$\eta(\xi) = 1, \tag{3}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{4}$$

$$\eta \circ \phi = 0, \quad \phi\xi = 0. \tag{5}$$

Then M admits a Riemannian metric g which satisfies,

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{6}$$

for all vector fields X; Y on M. We can also have the following relations

$$g(X,\xi) = \eta(X), \quad \nabla_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \quad \Omega(X,Y) = g(X,\phi Y) = -g(Y,\phi X) = \Omega(Y,X). \tag{7}$$

Definition(2.1) An almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called a trans-Sasakian structure if

$$(\nabla_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi X]$$

for functions $\alpha$ and $\beta$ on M of type $(\alpha; \beta)$, where $\nabla$ is the Levi-Civita connection on M. In a 3-dimensional trans-Sasakian manifold we also have

$$\Omega(X,Y) = \frac{1}{2}[(\nabla_X \eta)Y + (\nabla_Y \eta)X]. \tag{8}$$

Now we recall the definition of Weyl Conformal Curvature tensor[5], which satisfy the following relation

$$C(X,Y)Z = R(X,Y)Z - S(Y,Z)X + S(X,Z)Y - g(Y,Z)QX + g(X,Z)QY + \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{9}$$

Since the Conformal Curvature tensor in a 3-dimensional trans-Sasakian manifold vanishes so

$$R(X,Y)Z = S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \tag{10}$$

where R;Q and r denotes the curvature tensor of type (1; 3), Ricci operator and scalar curvature of M respectively.

**Results of this paper**

In this section we have proved some results related to 3-dimensional trans-Sasakian manifold and generalized $\eta$-Einstein 3-dimensional trans-Sasakian manifold.

Theorem (3.1) In a 3-dimensional trans-Sasakian manifold $\xi$ is an eigen vector of the Ricci tensor S corresponding to the eigen value (a+b).

**Proof :** Taking Y = $\xi$ in (1:2) we obtain

$$S(X,\xi) = ag(X,\xi) + b\eta(X)\eta(\xi) + c\Omega(X,\xi). \tag{11}$$

Now using (2:1) and (2:5) we get

$$S(X,\xi) = (a+b)\eta(X). \tag{12}$$

Therefore from equation (3:2) $\xi$ is an eigen vector of the Ricci tensor S corresponding to the eigen values a + b.

In view of the above theorem we can state our next theorem

    **Corollary :** The scalar (a + b) is the Ricci curvature in the direction of the generator $\xi$.

**Proof :** Now putting X = $\xi$ in (3:2) we obtain,

$$S(\xi,\xi) = (a+b).$$

Hence the proof.

**Theorem (3.3)** If the ricci curvature tensor S of the type (0; 2) of a 3-dimensional trans-Sasakian manifold is non-vanishing and satisfies the relation

$$S(Y,Z)S(X,W) - S(X,Z)S(Y,W) = \rho[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + g(\phi X,W)g(Y,Z),$$

where $\rho$ is non-zero scalar, then the manifold is a generalized $\eta$-Einstein manifold. **Proof :** In a trans-Sasakian manifold we have

$$S(X,\xi) = [2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - (\phi X)\alpha - (2n-1)(X\beta). \tag{13}$$

Using (2:3) and assuming $\beta$ to be a constant function, we calculate on putting $X = \xi$

$$S(\xi,\xi) = 2n(\alpha^2 - \beta^2). \tag{14}$$

Now taking $Y = \xi$ in (3:3) and using (3:5) we get

$$2n(\alpha^2 - \beta^2)S(X,W) - S(X,\xi)S(\xi,W) = \rho[g(\xi,\xi)g(X,W) - g(X,\xi)g(\xi,W)] + g(\phi X,W)g(\xi,\xi).$$

After some steps of calculations we have

$$S(X,W) = \frac{\rho}{2n(\alpha^2 - \beta^2)}g(X,W) + [2n(\alpha^2 - \beta^2) - \rho]\eta(X)\eta(W) + \frac{1}{2n(\alpha^2 - \beta^2)}g(\phi X,W). \tag{15}$$

Taking $a = \frac{\rho}{2n(\alpha^2-\beta^2)}, b = 2n(\alpha^2 - \beta^2) - \rho$ and $c = \frac{1}{2n(\alpha^2-\beta^2)}$ we obtain

$$S(X,W) = ag(X,W) + b\eta(X)\eta(W) + c\Omega(X,W). \tag{16}$$

Since $S \neq 0$, so $a,b,c$ are all not zero . So the proof is complete.

Now we state our next theorem

**Theorem (3.4)** In a generalized $\eta$-Einstein 3-dimensional trans-Sasakian manifold, the curvature tensor of type (1; 3) satisfies the following properties (3:6); (3:7); (3:8) $\forall X,Y,Z \in \xi^\perp$ and the 2-dimensional distribution is orthogonal to the distribution $\xi$.

**Proof :** Putting (1:2) in (2:9) we calculate

$$\begin{aligned} R(X,Y)Z &= (2a - \frac{r}{2})[g(Y,Z)X - g(X,Z)Y] + b[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi \\ &\quad -g(X,Z)\eta(Y)\xi] + c[Xg(\phi Y,Z) - Yg(\phi X,Z) + \phi Xg(Y,Z) - \phi Yg(X,Z)] \end{aligned} \tag{17}$$

Putting $Y = \xi$ in the above equation we get

$$R(X,\xi)Z = (\frac{r}{2} - 2a)[g(X,Z)\xi] - b[g(X,Z)\xi] - c\xi[g(\phi X,Z)], \tag{18}$$

$\forall X,Y,Z \in \xi^\perp$. Now putting $Z = \xi$ we compute

$$R(X,\xi)\xi = 0, \tag{19}$$

$\forall X,Y,Z \in \xi^\perp$. Therefore the proof is done.
From (3:8) we can conclude the following corollary:
**Corollary :** In a 3-dimensional generalized $\eta$-Einstein 3-dimensional trans-Sasakian manifold, the sectional curvature of the plane determined by X,$\eta$ is zero.

**Example of a 3-dimensional generalized (eta)-Einstein trans-Sasakian manifold**

Let us consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$ where (x; y; z) are the standard coordinate in $R^3$. The vector fields

$$e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e^{z-ax} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_i, e_j) = 1, \quad for \ i = j,$$

$$g(e_i, e_j) = 0, \quad for \ i \neq j,$$

Here i and j runs from 1 to 3. Let η be the 1-form defined by η(X) = g(X,e3); for any vector field X tangent to M. Let ϕ be the (1,1) tensor field defined by

$$\phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0$$

Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3,$$

for any vector fields Z;W tangent to $\tilde{M}$. Thus for $e_3 = \xi, M(\phi, \xi, \eta, g)$ forms an almost contact structure.

Let $\nabla$ be the Levi-Civita connection on $\tilde{M}$ with respect to the metric g. Then the followings can be obtained

$$[e_1, e_2] = -ae^z e_2, \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2$$

and On taking e3 =ξ and using Koszul's formula for the metric g, we calculate

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1,$$

$$\nabla_{e_2} e_1 = ae^z e_2, \quad \nabla_{e_2} e_2 = ae^z e_1 - e_3, \quad \nabla_{e_2} e_3 = -e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

It can be shown that M is a trans-Sasakian manifold of type (1; 0). Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_1, e_2)e_2 = (1 - a^2 e^{2z})e_1$$

$$R(e_2, e_3)e_2 = -ae^z e_1 - e_3, \quad R(e_1, e_3)e_1 = -e_3, \quad R(e_1, e_2)e_1 = -(1 - a^2 e^{2z})e_2$$

$$R(e_2, e_1)e_1 = -e_2$$

From the above, we can calculate the non-vanishing components of Ricci tensor S as follows :

$$S(e_1, e_1) = -a^2 e^{2z}, S(e_2, e_2) = -a^2 e^{2z}, S(e_3, e_3) = -2$$

Therefore we can write the Ricci tensor components as

$$S(e_1, e_1) = ag(e_1, e_1) + b\eta(e_1)\eta(e_1) + c\Omega(e_1, e_1)$$

$$S(e_2, e_2) = ag(e_2, e_2) + b\eta(e_2)\eta(e_2) + c\Omega(e_2, e_2)$$

$$S(e_3, e_3) = ag(e_3, e_3) + b\eta(e_3)\eta(e_3) + c\Omega(e_3, e_3),$$

where $a = 1 - 2a^2 e^{2z}, b = -(1 + -a^2 e^{2z}), c = 1 - a^2 e^{2z}$. Thus the manifold under consideration is a generalized η-Einstein trans-Sasakian manifold.

**References**

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