FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS OF TYPE (R) IN 2 – METRIC SPACES

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ABSTRACT. The aim of this paper is to introduce the concept of compatible mappings of type (R) in 2-metric spaces and to prove a coincidence point theorem and a fixed point theorem for compatible mappings of type (R) in 2-metric spaces.

1. Introduction

Rohen Singh, M.R & Shambhu introduced the concept of compatible mappings of type (C) by combining the definition of compatible mappings and compatible mapping of type (P) and later on it is renamed as compatible mappings of type (R) [9]. Later it is extensively developed by Rohen and many others. In the last decade a number of authors [11], [12], [13] have studied the aspects of compatible mappings of type (R).

Gahler initially investigated the concept of 2-metric in a series of papers [2], [3]. Later many authors [1] [5] [8] [16] have studied the aspects of fixed point theory in the setting of 2-metric paces in the last 50 years. They have been motivated by various concepts already known for metric spaces and have thus introduced analogous of various concepts in the framework of the 2-metric spaces, Murthy-Chang Cho-Sharma [7] introduced the concept of compatible pairs of self-mappings of type (A) in a 2-metric space and proved several common fixed points theorems. Pathak Chang & Cho [8] introduced the mappings of type (P) in 2-metric space and proved fixed point theorem in 2-metric spaces.

2. Compatible Mappings of type (R)

Here the concept of compatible mappings of type (R) is introduced in 2-metric spaces and shown that these mappings are equivalent to compatible mappings and compatible mappings of type (A) under some conditions. Now, we shall state some definitions.

Definition (2.1): A sequence \( \{x_n\} \) in a 2-metric space \((X,d)\) is said to be convergent to a point \( x \in X \) denoted by

\[
\lim_{n \to \infty} x_n = X \quad \text{if} \quad \lim_{n \to \infty} d(x_n, x, z) = 0 \quad \text{for all} \quad z \in X.
\]

The Point \( x \) is called the limit of the sequence \( \{x_n\} \) in \( X \).

Definition (2.2): A sequence \( \{x_n\} \) in a 2-metric space \((X, d)\) is called a Cauchy sequence if

\[
\lim_{n \to \infty} d(x_n, x, z) = 0 \quad \text{for all} \quad z \in X.
\]

Definition (2.3): A 2-metric space \((X, d)\) is said to be complete if every Cauchy sequence in \( X \) is convergent.
**Definition (2.4):** Let $S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself the mappings $S$ and $T$ are said to be compatible if

$$
\lim_{n \to \infty} d(STx_n, TSx_n z) = 0,
$$

for all $z \in X$ whenever $\{x_n\}$ is a sequence in $X$ such that

$$
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t
$$

for some $t \in X$.

**Definition (2.5):** Let $S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself. The mappings $S$ and $T$ are said to be compatible if type (A) if

$$
\lim_{n \to \infty} d(TSx_n, SSx_n z) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(STx_n, TTx_n z) = 0 \quad \text{for all} \quad z \in X
$$

whenever $\{x_n\}$ is a sequence in $X$ such that

$$
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some} \quad t \in X.
$$

**Definition (2.6):** Let $S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself the mappings $S$ and $T$ are said to be compatible of type (P) if

$$
\lim_{n \to \infty} d(SSx_n, TTx_n z) = 0 \quad \text{for all} \quad z \in X
$$

whenever $\{x_n\}$ is a sequence in $X$ such that

$$
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some} \quad t \in X.
$$

**Definition (2.7):** Let $S$ and $T$ be mappings from a 2-metric space $(X, d)$ into itself the mappings $S$ and $T$ are said to be compatible of type (R) if

$$
\lim_{n \to \infty} d(STx_n, TSx_n z) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(SSx_n, TTx_n z) = 0
$$

for all $z \in X$ whenever $\{x_n\}$ is a sequence in $X$ such that

$$
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some} \quad t \in X.
$$

**Definition (2.8):** A 2-metric space is a set of $X$ with real valued function $d$ on $X \times X \times X$. Satisfying the following conditions.

1. $(M_1)$ for distinct point $x, y \in X$, there exist a point $z \in X$ such that $d(x, y, z) \neq 0$
2. $(M_2)$ $d(x, y, z) = 0$ if at least two of $x, y, z$ are equal.
3. $(M_3)$ $d(x, y, z) = d(x, z, y) = d(y, z, x)$
4. $(M_4)$ $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

The function $d$ is called a 2-metric for the space $X$ and $(X, d)$ is called a 2-metric space.

The following proposition show that Definition 2.4 and 2.5 are equivalent under some conditions.

**Proposition (2.9):** Let $S$ and $T$ be continuous mappings of a 2-metric space $(X, d)$ into itself. If $S$ and $T$ are compatible, then they are compatible of type (A).

**Proposition (2.10):** Let $S$ and $T$ be compatible mappings of type (A) from a 2-metric space $(X, d)$ into itself. If one of $S$ and $T$ is continuous, then $S$ and $T$ are compatible.

**Proposition (2.11):** Let $S$ and $T$ be continuous mappings from a 2-metric space $(X, d)$ into itself. Then $S$ and $T$ are compatible if and only if they are compatible of type (A).

The following proposition show that Definition 2.5, 2.6 and 2.7 are equivalent under some conditions.

**Proposition (2.12):** Let $S$ and $T$ be continuous mappings of a 2-metric space $(X, d)$ into itself. Then $S$ and $T$ are compatible if and only if they are compatible of type (R).
Suppose that the mappings $S$ and $T$ are compatible. By definition of 2–metric space, we have
\[
d(STx_n, TSx_n, z) < d(STx_n, TSx_n, SSx_n) + d(STx_n, SSx_n, z) + d(SSx_n, TSx_n, z)\]
\[
< d(STx_n, TSx_n, SSx_n) + d(STx_n, SSx_n, z) + d(SSx_n, TTx_n, z) + d(TTx_n, TSx_n, z).
\]

Letting $n \to \infty$, since $S$ and $T$ are compatible and continuous, we have
\[
\lim_{n \to \infty} d(STx_n, TSx_n, z) = 0 \text{ for all } z \in X.
\]

Similarly, we have
\[
d(SSx_n, TTx_n, z) \geq d(SSx_n, TTx_n, STx_n) + d(SSx_n, STx_n, z) + d(STx_n, TTx_n, z)\]
\[
\geq d(SSx_n, TTx_n, STx_n) + d(SSx_n, STx_n, z) + d(STx_n, TTx_n, TSx_n)\]
\[
+ d(SSx_n, TTx_n, z) + d(TTx_n, z).
\]

Letting $n \to \infty$, since $S$ and $T$ are compatible and continuous, we have
\[
\lim_{n \to \infty} d(SSx_n, TTx_n, z) = 0 \text{ for all } z \in X.
\]

Conversely suppose that $S$ and $T$ are compatible type (R). Then by the definition of compatible mapping of type (R), we have \(\lim_{n \to \infty} d(STx_n, TSx_n, z) = 0\) for all \(z \in X\). This is the condition of compatible of $S$ and $T$. So it is proof.

Proposition (2.13): Let $S$ and $T$ be compatible mappings of type (A) from a 2-metric space $(X, d)$ into itself. If one of $S$ and $T$ is continuous, then $S$ and $T$ are compatible of type (R).

Proof: Let \(\{x_n\}\) be a sequence in $X$ such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t\) for some $t \in X$.

Suppose that $S$ and $T$ are compatible type (A). By the definition of 2–metric space, we have
\[
d(STx_n, TSx_n, z) < d(STx_n, TSx_n, SSx_n) + d(STx_n, SSx_n, z) + d(SSx_n, TSx_n, z)\]

Letting $n \to \infty$, we have
\[
\lim_{n \to \infty} d(STx_n, TSx_n, z) = 0 \text{ for all } z \in X.
\]

Again, in the same way, we have
\[
d(SSx_n, TTx_n, z) < d(SSx_n, TTx_n, STx_n) + d(SSx_n, STx_n, z) + d(STx_n, TTx_n, z)\]
\[
\text{Letting } n \to \infty, \text{ we obtain } \lim_{n \to \infty} d(SSx_n, TTx_n, z) = 0 \text{ for all } z \in X.
\]

Therefore, $S$ and $T$ are compatible mappings of type (R).

As a direct consequence of Propositions 2.11, 2.12 and 2.13, we have the followings

Proposition (2.14): Let $S$ and $T$ be continuous mappings of a 2-metric spaces $(X, d)$ into itself. Then,

i) $S$ and $T$ are compatible of type (A) if and only if they are compatible of type (R).

ii) $S$ and $T$ are compatible if and only if they are compatible of type (R).

Next, we give some properties of compatible mappings of type (R) in 2 metric spaces.

Proposition (2.15): Let $S$ and $T$ be compatible mappings of type (R) from a 2-metric space $(X, d)$ into itself in \(S t = T t\) for some $t \in X$. Then \(S T t = S S t = T T t = T S t\).
Proof: Suppose that \( \{x_n \} \) is a sequence in \( X \) defined by \( x_n = t, n = 1, 2 \ldots \), and \( S^t = T^t \). Then we have
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = St.
\]
Since \( S \) and \( T \) are compatible mappings of type (R), we have
\[
d(St, TT^2z) = \lim_{n \to \infty} d(SS^2x_n, TT^2x_n, z) = 0.
\]
Hence we have
\[
SS^t = TT^t. \quad \text{Therefore,} \quad ST^t = SS^t = TT^t = TS^t.
\]

Proposition (2.16): Let \( S \) and \( T \) be compatible mappings of type (R) from a 2-metric space \( (X, d) \) to itself, suppose
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some} \quad t \in X.
\]
Then we have the following:

i) \[
\lim_{n \to \infty} TTx_n = St \quad \text{if} \quad S \quad \text{is continuous at} \quad t.
\]

ii) \[
\lim_{n \to \infty} SSx_n = Tt \quad \text{if} \quad T \quad \text{is continuous at} \quad t.
\]

iii) \( ST^t = Tst \) and \( ST^t = Tt \) if \( S \) and \( T \) are continuous at \( t \).

Proof: i) Suppose that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some} \quad t \in X. \) Since \( S \) is continuous, we have
\[
\lim_{n \to \infty} STx_n = St. \quad \text{By definition of 2-metric we have}
\]
\[
d(TTx_n, St, z) < d(TTx_n, St, SSx_n) + d(TTx_n, SSx_n, z) + d(SSx_n, z)
\]
Therefore, since \( S \) and \( T \) are compatible mappings of type (R), we have
\[
\lim_{n \to \infty} TTx_n = St.
\]

ii) The proof of \( \lim_{n \to \infty} SSx_n = Tt \) follows on the similar lines as argued in (i).

iii) Since \( t \) is continuous at \( t \), we have \( TTx_n = Tt. \) By (i), since \( S \) is continuous at \( t \), we also have
\[
\lim_{n \to \infty} TTx_n = St. \quad \text{Hence, by the uniqueness of the limit, we have} \quad St = Tt \quad \text{and so, by proposition (2.14),} \quad ST^t = TSt.
\]

3. Coincidence Point Theorem

Let \( S \) and \( T \) be mappings from a 2-metric space \( (X, d) \) to itself. A point \( u \in X \) is called a coincidence point of \( S \) and \( T \) if \( Su = Tu \).

Let \( N \) and \( R^+ \) be the sets of all natural numbers and non-negative real numbers, respectively, and \( F \) be the family of mappings \( \Phi \) from \( (R^+)^9 \) to \( R^+ \) such that each \( \Phi \) is upper semi-continuous and non-decreasing in each coordinate variable, and for any \( t > 0, \)
\[
\Phi(t, t, t, 0, t, t, 0, 0, 0, 0) < \beta \quad \text{and} \quad \Phi(t, t, t, t, 0, 0, 0, 0) > \beta t.
\]

Where \( \beta = 1 \) for \( \alpha = 2 \) and \( \beta < 1 \) for \( \alpha < 2. \) \( y(t) = \Phi(t, t, t, t, a_1, t, a_2, t, a_3, t, a_4, t, a_5, t) < t. \)

Where \( y : R^+ \to R^+ \) is a mapping and \( a_1 + a_2 + a_3 + a_4 + a_5 = 7. \)

Let \( A, B, S \) and \( T \) be mappings from a 2-metric space \( (X, d) \) to itself, suppose that.
(3.1) \( A(X) \cup B(X) \subset S(X) \cap T(X) \)

(3.2) There exist a \( \phi \in F \) such that

\[
\begin{align*}
d^2(Ax, By, z) & \leq \phi(d^2Sx, Ty, z) d(Sx, Ax, z) d(Ty, By, z) \\
d(Sx, Ty, z) d(Sx, Ax, z) d(Sx, Ty, z) & d(Ty, By, z) \\
d(Sx, Ty, z) d(Sx, By, z) d(Sx, Ty, z) & d(Ty, Ax, z) \\
d(Sx, By, z) d(Ty, Ax, z) & d(Sx, Ax, z) d(Ty, Ax, z) \\
d(Sx, Ax, z) & d(Ty, By, z)
\end{align*}
\]

Then by (3.1) since \( A(X) \subset T(X) \) for any arbitrary \( x_0 \in X \) there exist a point \( x_1 \in X \) such that \( Ax_0 = TX_1 \). Since \( B(X) \subset S(X) \), for this point \( x_1 \), we can choose a point \( x_2 \in X \) such that \( Bx_1 = Sx_2 \) and so on we can define a sequence \( \{y_n\} \) in \( X \) such that

\[
y_{2n} = T_{x_{2n+1}} = A_{x_{2n}} \quad \text{and} \quad y_{2n+1} = S_{x_{2n+2}} = B_{x_{2n+1}}
\]

for every \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

**Lemma (3.1)** For any \( t > 0 \), \( \Phi(t) < t \) if and only if

\[
\lim_{n \to \infty} y^n(t) = 0
\]

where \( y^n \) denotes the composite of \( y \) with itself \( n \)-times.

**Lemma (3.2)** Let \( A, B, S \) and \( T \) be mappings for a 2-metric space \( (X, d) \) into itself satisfying the condition (3.1) and (3.2) then we have the following

1. For every \( n \in \mathbb{N}_0 \), \( d(y_n, y_{n+1}, y_{n+2}) = 0 \)

2. For every \( i, j, k \in \mathbb{N}_0 \), \( d(y_i, y_j, y_k) = 0 \)

Where \( \{y_n\} \) is a sequence in \( X \) defined by condition (3.3)

**Proof:** (1) In (3.2), taking \( x=x_{2n+2}, y=y_{2n+1} \) and \( z=z_{2n} \), we have

\[
\begin{align*}
d^2(y_{2n+2}, y_{2n+1}, y_{2n}) & = d^2(Ax_{2n+2}, By_{2n}, y_{2n}) \\
& \leq \Phi d^2(y_{2n+1}, y_{2n}) \\
d(y_{2n}, y_{2n+1}, y_{2n}) & d(y_{2n}, y_{2n-1}, y_{2n}) \\
d(y_{2n+1}, y_{2n}, y_{2n}) & d(y_{2n+1}, y_{2n+2}, y_{2n}) \\
d(y_{2n+1}, y_{2n}, y_{2n}) & d(y_{2n+1}, y_{2n+2}, y_{2n}) \\
d(y_{2n}, y_{2n+1}, y_{2n}) & d(y_{2n}, y_{2n+2}, y_{2n}) \\
d(y_{2n+1}, y_{2n}, y_{2n}) & d(y_{2n+1}, y_{2n+2}, y_{2n})
\end{align*}
\]
\[
\begin{align*}
&d(y_{2n+1}, y_{2n+2}, y_{2n})
d(y_{2n}, y_{2n+2}, y_{2n})
&d(y_{2n+1}, y_{2n+2}, y_{2n})
d(y_{2n}, y_{2n+2}, y_{2n})
&d(y_{2n+1}, y_{2n+2}, y_{2n})
d(y_{2n}, y_{2n+1}, y_{2n})
= \phi(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)
\leq 0
\end{align*}
\]

And so \(d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0\), Similarly, we have \(d(y_{2n+1}, y_{2n+2}, y_{2n+3}) = 0\). Thus it follows that \(d(y_{2n}, y_{2n+1}, y_{2n}) = 0\) for every \(n \in \mathbb{N}_0\).

(2) For all \(z \in X\), let \(d_n(z) = d(y_n, y_{n+1}, z)\) for \(n = 0, 1, 2, \ldots\). By (1), We have

\[
\begin{align*}
d(y_n, y_{n+2}, z) & \leq d(y_n, y_{n+2}, y_{n+1}) + d(y_{n+1}, y_{n+2}, z) \\
& = d(y_n, y_{n+1}, z) + d(y_{n+1}, y_{n+2}, z) \\
& = d_n(z) + d_{n+1}(z).
\end{align*}
\]

Taking \(x = x_{2n+2}\) and \(y = x_{2n+2}\) in (3,20), we have

\[
\begin{align*}
d_{2n+1}^2(z) &= d^2(y_{2n+2}, y_{2n+1}, z) \\
&= d^2(Ax_{2n+2}, Bx_{2n+1}, z) \\
&= \phi(d^2(y_{2n+1}, y_{2n+2}, z), d(y_{2n+1}, y_{2n+2}, z)d(y_{2n}, y_{2n+1}, z), \\
d(y_{2n+1}, y_{2n+2}, z)d(y_{2n}, y_{2n+1}, z), d(y_{2n+1}, y_{2n+2}, z)d(y_{2n}, y_{2n+1}, z), \\
d(y_{2n+1}, y_{2n+2}, z)d(y_{2n}, y_{2n+1}, z), d(y_{2n+1}, y_{2n+2}, z)d(y_{2n}, y_{2n+1}, z), \\
d(y_{2n+1}, y_{2n+2}, z)d(y_{2n}, y_{2n+1}, z), d(y_{2n+1}, y_{2n+2}, z)d(y_{2n}, y_{2n+1}, z), \\
&= \phi(d_{2n}(z), d_{2n+1}(z) d_{2n}(z), d_{2n}(z), d_{2n+1}(z)) \\
d_{2n}(z), 0 \ d_{2n}(z) (d_{2n}(z)= d_{2n+1}(z)), 0, \\
d_{2n+1}(z) (d_{2n}(z)+ d_{2n+1}(z)), 0, \\
\end{align*}
\]

Now, we shall prove that \(\{d_n(z)\}\) is a non-sequences in \(\mathbb{R}^+\). In fact, suppose that \(d_{n+1}(z) > d_n(z)\) for some \(n\). Then, for some \(\alpha < 2\), \(d_{n+1}(z) = d_n(z) = \alpha d_{n+1}(z)\). Since, \(\phi\) is non-decreasing in each variable and \(\beta < 1\) for some \(\alpha < 2\), by (3.2), we have

\[
\begin{align*}
d_{2n+1}^2(z) & \leq \phi(d_{2n+1}^2(z), d_{2n+1}^2(z), d_{2n+1}^2(z), d_{2n+1}^2(z), 0, \\
&= \beta d_{2n+1}^2(z) \\
&< d_{2n+1}^2(z)
\end{align*}
\]

and
By using the above inequality completes the proof of our claim. Finally, let i, j and k be arbitrary non-negative integers. We may assume that i < j. By definition of 2-metric, we have

\[ d_{2n+2}^2(z) \leq \phi(d_{2n+2}^2(z), d_{2n+2}^2(z), d_{2n+2}^2(z), d_{2n+2}^2(z), \alpha d_{2n+2}^2(z)). \]

\[ 0, 0, 0, \alpha d_{2n+2}^2(z)) \]

\[ \leq \beta(d_{2n+2}^2(z)) \]

\[ < d_{2n+2}^2(z). \]

Hence, for every \( n \in \mathbb{N}_0 \), \( d_n^2(z) \leq \beta d_n^2(z) < d_n^2(z) \), which is a contradiction.

Therefore, \( \{d_n(z)\} \) is a non-increasing sequences in \( \mathbb{R}^+ \).

Now, we claim that \( d_n(y_m) = 0 \) for all non-negative integers \( m, n \).

**CASE 1**: \( n \geq m \). Then we have \( 0 = d_m(y_m) \geq d_n(y_m) \).

**Case 2**: \( n < m \). By definition of 2-metric, we have

\[ d_n(y_m) \leq d_n(y_{m-1}) + d_{m-1}(y_n). \]

\[ \leq d_n(y_{m-1}) + d_n(y_n). \]

\[ \leq d_n(y_{m-1}) \]

By using the above inequality completes the proof of our claim.

Finally, let i, j and k be arbitrary non-negative integers. We may assume that i < j. By definition of 2-metric, we have

\[ d(y_i, y_j, y_k) \leq d_i(y_j) + d(y_{i+1}, y_j, y_k) + d_i(y_k) \]

\[ - d(y_{i+1}, y_j, y_k) \]

Therefore, by repetition of the above inequality, we have

\[ d(y_i, y_j, y_k) \leq d(y_{i+1}, y_j, y_k) \leq \cdots \leq d(y_i, y_j, y_k) = 0 \]

**LEMMA 3.3.** Let A, B, S and T be mappings from a 2-metric space \( (X, d) \) into itself satisfying the conditions (3.1) and (3.2). Then the sequences \( \{y_n\} \) defined by (3.3) is a Cauchy sequences in \( X \).

**Proof.** In the proof of Lemma 3.2, since \( \{d_n(z)\} \) is a non-decreasing of \( \mathbb{R}^+ \), by (3.2), we have

\[ d^2_n(z) = d^2(y_i, y_2, z) \]

\[ = d^2(Bx_1, Ax_2, z) \]

\[ \leq \phi(d^2_0(z), d_0(z) d_0(z) d_0(z), d_0(z), d^2_0(z), 0) \]

\[ d_0(z) d_0(z) + d_1(z), 0, d_0(z) (d_0(z) + d_1(z)), 0 \]

\[ \leq \phi(d^2_0(z), d^2_0(z), d^2_0(z), d^2_0(z), 0, 2 d^2_0(z), 0, 2d^2_0(z), 0) \]

\[ = \gamma(d^2_0(z)). \]

In general, we have \( d^2_n(z) \leq \gamma^n(d^2_0(z)) \), which implies, if \( d_0(z) > 0 \), by Lemma 3.1, we have

\[ \lim_{n \to \infty} d^2_n(z) \leq \lim_{n \to \infty} \gamma^n(d^2_0(z)) = 0 \]
Therefore, it follows that \( \lim_{n \to \infty} d_n(z) = 0 \). For \( d_0(z)=0 \), since \( \{d_n(z)\} \) is non-increasing, we have clearly \( \lim_{n \to \infty} d_n(z)=0 \).

Now, we shall prove that \( \{y_n\} \) is a Cauchy sequence in \( X \). Since, \( \lim d_n(z)=0 \), it is sufficient to show that a subsequence \( \{y_{2n}\} \) of \( \{y_n\} \) is not a Cauchy sequence in \( X \). Suppose that the sequence is not \( X \). Then there exists a number \( \varepsilon >0 \) and strictly increasing sequences \( \{m_k\}, \{n_k\} \) of positive integers such that \( k \leq n_k < m_k \).

\[
(3.4) \quad d(y_{2n_k}, y_{2m_k}) \geq \varepsilon \quad \text{and} \quad d(y_{2n_k}, y_{2m_k-2}, z) < \varepsilon \quad \text{for all} \quad k = 1, 2, \ldots.
\]

By lemma 3.2 and by definition of 2-metric, we have

\[
d(y_{2n_k}, y_{2m_k}, z) - d(y_{2n_k}, y_{2m_k-2}, z) \leq d(y_{2m_k-2}, y_{2m_k}, z)
\]

\[
\leq d(y_{2m_k-2}, z) + d(y_{2m_k-2}, z)
\]

Since \( \{d(y_{2n_k}, y_{2m_k}, z) - \varepsilon \} \) and \( \{\varepsilon - (y_{2n_k}, y_{2m_k-2}, z)\} \) are sequences in \( R^+ \) and \( \lim d_n(z)=0 \), we have

\[
(3.5) \quad \lim_{n \to \infty} d(y_{2n_k}, y_{2m_k}, z) = \varepsilon \quad \text{and} \quad \lim_{n \to \infty} d(y_{2n_k}, y_{2m_k-2}, z) = \varepsilon
\]

Note that, by definition of 2-metric,

\[
(3.6) \quad |d(x,y,a) - d(x,y,b)| \leq d(a, b, y) \quad \text{for all} \quad x, y, a, b \in X.
\]

Taking \( x = y_{2n_k}, y = z, z = y_{2m_k-2}, a = y_{2m_k} \) in (3.6) and using Lemma 3.2 and (3.5) we have

\[
(3.7) \quad \lim_{n \to \infty} d(y_{2n_k}, y_{2m_k-2}, z) = \varepsilon
\]

Once again, by using Lemma 3.2, (3.5) and (3.6), we have

\[
(3.8) \quad \lim_{n \to \infty} d(y_{2m_k-1}, y_{2m_k}, z) = \varepsilon \quad \text{and} \quad \lim_{n \to \infty} d(y_{2m_k-1}, y_{2m_k}, z) = \varepsilon
\]

Thus, by (3.2) we have

\[
d^2(y_{2n_k}, y_{2m_k+1}, z) = d^2(Ax_{2m_k}, Bx_{2m_k+1}, z)
\]

\[
\leq \phi \left( d^2(y_{2m_k-1}, y_{2m_k-2}, z), d(y_{2m_k-1}, y_{2m_k-2}, z) \right) d(y_{2m_k-1}, y_{2m_k+1}, z)
\]

\[
d(y_{2m_k-1}, y_{2m_k}, z), d(y_{2m_k-1}, y_{2m_k}, z), d(y_{2m_k-1}, y_{2m_k}, z), d(y_{2m_k-1}, y_{2m_k}, z)
\]

\[
d(y_{2m_k-1}, y_{2m_k+1}, z), d(y_{2m_k}, y_{2m_k}, z), d(y_{2m_k}, y_{2m_k}, z), d(y_{2m_k}, y_{2m_k}, z)
\]

\[
d(y_{2m_k-1}, y_{2m_k+1}, z), d(y_{2m_k}, y_{2m_k}, z), d(y_{2m_k}, y_{2m_k+1}, z)
\]

Using (3.4), (3.5), (3.6) and (3.7) since \( \phi \in F \) we have

\[
\varepsilon^2 \leq \phi \left( \varepsilon^2, 0, 0, \varepsilon^2, \varepsilon^2, 0, 0 \right) \leq \gamma \left( \varepsilon^2 \right) < \varepsilon^2
\]

as \( n \to \infty \) in (3.9), which is a contradiction. Therefore, \( \{y_{2n}\} \) is a Cauchy sequence in \( X \).

4. OUR MAIN THEOREM

**Theorem (3.1):** Let \( A, B, S \) and \( T \) in mappings from a 2-metric space \( (X,d) \) into itself satisfying the condition (3.1), (3.2) and (3.10) \( S(X) \cap T(X) \) is a complete subspace of \( X \) then the pairs \( A,S \) and \( B,T \) have a coincidence point in \( X \).
Similarly, we can show that \( v \) is a coincidence point of \( B \) and \( T \).

5. COMMON FIXED POINT THEOREM

Theorem (4.1): Let \( A, B, S \) and \( T \) be mappings from a 2-metric space \((X, d)\) into itself satisfying the condition (3.1) (3.2) (3.10) and the followings

(4.1) The pairs \( A, S \) and \( B, T \) are compatible mappings of type (R).
(4.2) The pairs \( A, S \) and \( B, T \) are continuous at their coincidence points.

Then \( A, B \) and \( T \) have a unique common fixed point in \( X \).

Proof: By Lemma (3.3) the sequence \( \{y_n\} \) defined by the condition (3.3) is a Cauchy sequence in \( S(X) \cap T(X) \) since \( S(X) \cap T(X) \) is a complete subspace of \( X \{y_n\} \) converges to a point \( w \) in \( S(X) \cap T(X) \). On the other hand since the subsequences \( \{y_{2n}\} \) and \( \{y_{2n+1}\} \) of \( \{y_n\} \) are also Cauchy sequence in \( (X) \cap T(X) \). They also converge to the same limit \( w \). Hence there exist two points \( u, v \) in \( X \) such that \( Su = w \) and \( Tv = w \) respectively condition (3.2) we have

\[
\begin{align*}
  d^2(Au, Bu_{2n+1}, z) &\leq \phi \left( d^2(Su, Tx_{2n+1}, z) d(Su, Au, z) d(Tu_{2n+1}, Bu_{2n+1}, z) \right) \\
  d(Su, Tx_{2n+1}, z) d(Su, Au, z) d(Su, Tx_{2n+1}, z) d(Tu_{2n+1}, Bu_{2n+1}, z) \\
  d(Su, Au, z) d(Tx_{2n+1}, Au, z) d(Su, Au, z) d(Tx_{2n+1}, Bu_{2n+1}, z)
\end{align*}
\]

that is

\[
\begin{align*}
  d^2(Au, y_{2n+1}, z) &\leq \phi \left( d^2(Su, y_{2n}, z) d(Su, Au, z) d(y_{2n}, y_{2n+1}, z) \right) \\
  d(Su, y_{2n}, z) d(Su, Au, z) d(Su, y_{2n}, z) d(y_{2n}, y_{2n+1}, z) \\
  d(Su, y_{2n}, z) d(Su, y_{2n+1}, z) d(Su, Au, z) d(y_{2n}, Au, z) \\
  d(Su, y_{2n+1}, z) d(y_{2n}, Au, z) d(Su, Au, z) d(y_{2n}, Au, z) \\
  d(Su, Au, z) d(y_{2n}, y_{2n+1}, z)
\end{align*}
\]

Since

\[
\lim_{n \to \infty} d_n (z) = 0 \quad \text{Litting} \quad n \to \infty \text{ in lemma (3.2) we have}
\]

\[
d^2(Au, w, z) < d^2(w, Au, z)
\]

Which is a contradiction. Hence \( Au = w = Su \), that is \( u \) is a coincidence of \( A \) and \( S \).

Similarly, we can show that \( v \) is a coincidence point of \( B \) and \( T \).

5. COMMON FIXED POINT THEOREM

Theorem (4.1): Let \( A, B, S \) and \( T \) be mappings from a 2-metric space \((X, d)\) into itself satisfying the condition (3.1) (3.2) (3.10) and the followings

(4.1) The pairs \( A, S \) and \( B, T \) are compatible mappings of type (R).
(4.2) The pairs \( A, S \) and \( B, T \) are continuous at their coincidence points.

Then \( A, B \) and \( T \) have a unique common fixed point in \( X \).

Proof: By the coincidence point theorem there exist two points \( u, v \) in \( X \) such that \( Au = Su = w \) and \( Bv = Tv = w \) respectively, since \( A \) and \( S \) are compatible mappings of type (R) \( ASu = SSu = SAu = AAu \), which implies that \( Aw = Sw \). Similarly since \( B \) and \( T \) are compatible mappings of type (R) we have \( Bw = Tw \). Now we prove that \( Aw = w \). If \( Aw \neq w \) then by condition (3.2) we have

\[
\begin{align*}
  d^2(Aw, Bu_{2n+1}, z) &\leq \phi \left( d^2(Sw, Tx_{2n+1}, z) \right) \\
  d^2(Sw, Aw, z) d(Tu_{2n+1}, Bu_{2n+1}, z) d(Su, Tx_{2n+1}, z) d(Su, Au, z) \\
  d(Sw, Tx_{2n+1}, z) d(Tx_{2n+1}, Bu_{2n+1}, z) d(Sw, Tx_{2n+1}, z) d(Su, Bu_{2n+1}, z) \\
  d(Sw, Tx_{2n+1}, z) d(Aw, Tx_{2n+1}, z) d(Sw, Bu_{2n+1}, z) d(Tu_{2n+1} Au, z) \\
  d(Su, Au, z) d(Tx_{2n+1}, Aw, z) d(Su, Au, z) d(Tx_{2n+1}, Bu_{2n+1}, z)
\end{align*}
\]
\[ d^2(Aw, y_{2n+1}z) \leq \phi d^2(Sw, y_{2n}z)d(Sw, Aw, z)d(y_{2n}, y_{2n+1}z) \]
\[ d(Sw, y_{2n}z)d(Sw, Aw, z)d(y_{2n}, y_{2n+1}z) \]
\[ d^2(Sw, y_{2n+1}z)d(y_{2n}, y_{2n+1}z)d(y_{2n}, Awz) \]
\[ d^2(y_{2n}, y_{2n+1}z) \]

Letting \( n \to \infty \), we have
\[ d^2(Aw, w, z) \leq \phi d^2(Sw, w, z)0,00d^2(Sw, w, z) \]
\[ d(Sw, w, z)d(w, Aw, z) d^2(Aw, w, z) 0,0 \]
\[ \phi (d^2(Aw, w, z)0,00, d^2(Aw, w, z)d^2(Aw, w, z) \]
\[ d^2(Aw, w, z) 00 < d^2(Aw, w, z) \]

Which is a contradiction. Hence we have \( Aw = Sw \), Similarly we have \( Bw = w = Tw \). This means that \( w \) is a common fixed point of \( A, B, S \) and \( T \). The uniqueness of the fixed point \( w \) follows easily from the condition (3.2). This completes the proof.

REFERENCES


