A Novel (G'/G) - Expansion Method and its Application to the Space-Time Fractional Symmetric Regularized Long Wave (SRLW) Equation

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Abstract. In this work, we use the fractional complex transformation which converts nonlinear fractional partial differential equation to nonlinear ordinary differential equation. A fractional novel (G'/G)-expansion method is used to look for exact solutions of nonlinear evolution equation with the aid of symbolic computation. To check the validity of the method we choose the space-time fractional symmetric regularized long wave (SRLW) equation and as a result, many exact analytical solutions are obtained including hyperbolic function solutions, trigonometric function solutions, and rational solutions. The performance of the method is reliable, useful and gives more new general exact solutions than the existing methods.

1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. Fractional Calculus, which is the field of mathematical analysis dealing with the investigation and applications of integrals and derivatives of arbitrary order, has attracted in recent years a considerable interest in many disciplines. It has been found that the behavior of many physical systems can be more properly defined by using the fractional theory. In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering [1]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by differential equations of fractional order [2-6]. Although the fractional calculus was invented by Newton and Leibniz over three centuries ago, it only became a hot topic recently owing to the development of the computer and its exact description of many real-life problems. A physical interpretation of the fractional calculus was given in [7]. With the development of symbolic computation software, like-Maple, many numerical and analytical methods to search for exact solutions of NLEEs have attracted more attention. As a result, the researchers developed and established many methods, for example, Cole-Hopf transformation [8], Tanh-function method [9-13], Inverse scattering transform method [14], Hirota method [15], Backlund transform method [16], Variational iteration method [17, 18], Exp-function method [19-23], Extended tanh-method [24-26], Homogeneous balance method [27, 28] and F-expansion method [29, 30] are used for searching the exact solutions.

Lately, Wang et al. [31] introduced a direct and concise method, called (G'/G)-expansion method and demonstrated that it is a powerful method for seeking analytic solutions of NLEEs. For additional references see the articles [32-37]. In order to establish the efficiency and diligence of (G'/G)-expansion method and to extend the range of applicability, further research has been carried out by several researchers. For instance, Zhang et al. [38] made a generalization of (G'/G)-expansion method for the evolution equations with variable coefficients. Zhang et al. [39] also presented an improved (G'/G)-expansion method to seek more general traveling wave solutions.

The (G'/G)-expansion method and the transformed rational function method used by Ma [40, 41] have a common idea. That is, we firstly put the given NLEE into the corresponding ordinary differential equation (ODE), and then ODE can be transformed into a system of algebraic polynomials with the determining constants. By the solutions of the ordinary differential equation, we can obtain the exact traveling solutions and rational solution of the nonlinear evolution equations. In this article, we will apply novel (G'/G)-expansion method introduced by Alam et al. [42] to solve the space-time fractional partial differential equation in the sense of modified Riemann-Liouville

derivative by Jumarie [43]. To illustrate the originality, consistency and advantages of the method, we will apply it to the space-time fractional SRLW equation and abundant new families of exact solutions are found.

The Jumarie's modified Riemann-Liouville derivative [43] of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, n \geq 1. \end{cases} \quad (1)$$

Some important properties of Jumarie's derivative are:

$$D_t^\alpha f(t) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \quad (2)$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \quad (3)$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha. \quad (4)$$

2. Description of the Method

Suppose that a fractional partial differential equation in the independent variables say, x and t is given by

$$S(u, u_x, u_t, D_x^\alpha u, D_t^\alpha u, \dots) = 0, \quad 0 < \alpha \leq 1, \quad (5)$$

where $D_x^\alpha u$ and $D_t^\alpha u$ are Jumarie's modified Riemann-Liouville derivatives of u , $u(x, t)$ is an unknown function, S is a polynomial in u and its various partial derivatives including fractional derivatives in which the highest order derivatives and nonlinear terms are involved.

The main steps of the method are as follows:

Step 1: Li et al. [44] proposed a fractional complex transformation to convert fractional partial differential equations into ordinary differential equations (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The traveling wave variable

$$u(x, t) = u(\xi), \quad \xi = L \frac{x^\alpha}{\Gamma(1+\alpha)} + V \frac{t^\alpha}{\Gamma(1+\alpha)} + \xi_0, \quad (6)$$

where L, V and ξ_0 are arbitrary constants with $L, V \neq 0$, permits us to reduce Eq. (5) to an ordinary differential equation of integer order in the form

$$P(u, u', u'', u''', \dots) = 0, \quad (7)$$

where the superscripts stand for the ordinary derivatives with respect to ξ

Step 2. Integrate Eq. (7) term by term one or more times if possible, yields constant(s) of integration which can be calculated later.

Step 3: Assume that the solution of Eq. (7) can be expressed as:

$$u(\xi) = \sum_{i=-m}^m \alpha_i (k + \Phi(\xi))^i, \quad (8)$$

Where

$$\Phi(\xi) = \frac{G'(\xi)}{G(\xi)}. \quad (9)$$

Herein α_{-m} or α_m may be zero, but both of them can not be zero simultaneously. α_i ($i = 0, \pm 1, \pm 2, \dots, \pm N$) and k are constants to be determined later and $G = G(\xi)$ satisfies the second order nonlinear ordinary differential equation:

$$GG'' = AGG' + BG^2 + C(G')^2, \quad (10)$$

where prime denotes the derivative with respect ξ ; A , B and C are real constants.

The Cole-Hopf transformation $\Phi(\xi) = \ln(G(\xi))_{\xi} = \frac{G'(\xi)}{G(\xi)}$ reduces the Eq. (10) into the Riccati equation:

$$\Phi'(\xi) = B + A\Phi(\xi) + (C-1)\Phi^2(\xi). \quad (11)$$

Eq. (11) has individual twenty five solutions (see Zhu, [45] for details).

Step 4: The value of the positive integer m can be determined by balancing the highest order linear terms with the nonlinear terms of the highest order come out in Eq. (7).

Step 5: Substitute Eq. (8) including Eqs. (9) and (10) into Eq. (7), we obtain polynomials in $\left(k + \frac{G'(\xi)}{G(\xi)}\right)^i$ and $\left(k + \frac{G'(\xi)}{G(\xi)}\right)^{-i}$, ($i = 0, 1, 2, \dots, N$). Collect each coefficient of the resulted polynomials to zero, yields an over-determined set of algebraic equations for α_i ($i = 0, \pm 1, \pm 2, \dots, \pm N$), k , L and V .

Step 6: Suppose the value of the constants can be obtained by solving the algebraic equations obtained in step 4. The values of the constants together with the solutions of Eq. (10) yield abundant exact traveling wave solutions of the nonlinear evolution equation (5).

3. Application of the Method to the Space-Time Fractional SRLW Equation

We consider the following space-time fractional SRLW equation

$$D_t^{2\alpha}u + D_x^{2\alpha}u + uD_t^{\alpha}(D_x^{\alpha}u) + D_x^{\alpha}uD_t^{\alpha}u + D_t^{2\alpha}(D_x^{2\alpha}u) = 0, \quad 0 < \alpha \leq 1, \quad (12)$$

which arises in several physical applications including ion sound waves in plasma. This equation is symmetrical with respect to x and t . It arises in many nonlinear problems of mathematical physics and applied mathematics. Periodic wave solutions of SRLW have been given by using the Exp function method [46] and (G'/G) -expansion method [47]. Alzaidy [48] applied fractional sub-equation method to space-time fractional symmetric regularized long wave (SRLW) equation and find hyperbolic function solutions, trigonometric function solutions, and rational solutions. By the use of Eq. (4), Eq. (12) is converted into an ordinary differential equation and after integrating twice, we obtain

$$(L^2 + V^2)u + \frac{1}{2}LVu^2 + L^2V^2u'' + C_1 = 0, \quad (13)$$

where C_1 is an integration constant.

Considering the homogeneous balance between u'' and u^2 in Eq. (13), we obtain $m = 2$. Therefore, the trial solution becomes

$$u(\xi) = \alpha_{-2}(k + \Phi(\xi))^{-2} + \alpha_{-1}(k + \Phi(\xi))^{-1} + \alpha_0 + \alpha_1(k + \Phi(\xi)) + \alpha_2(k + \Phi(\xi))^2. \quad (14)$$

Using Eq. (14) into Eq. (13), left hand side is converted into polynomials in $\left(k + \frac{G'(\xi)}{G(\xi)}\right)^i$, and

$\left(k + \frac{G'(\xi)}{G(\xi)}\right)^{-i}$, ($i = 0, 1, 2, \dots, N$). Equating the coefficients of same power of the resulted polynomials to zero, we obtain a set of algebraic equations (which are omitted for the sake of simplicity) for $\alpha_0, \alpha_1, \alpha_2, \alpha_{-1}, \alpha_{-2}, k, C_1, L$ and V . Solving the over-determined set of algebraic equations by using the symbolic computation software, such as Maple 13, we obtain the following solution sets:

Set 1:

$$\begin{aligned}\alpha_2 &= -12LV(C-1)^2, \alpha_1 = -12LV(-2C^2k + 4Ck + AC - A - 2k), \\ \alpha_0 &= -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L}, \\ V &= V, L = L, k = k, \alpha_{-1} = 0, \alpha_{-2} = 0,\end{aligned}\quad (15)$$

Set 2:

$$\begin{aligned}\alpha_0 &= -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L}, \\ \alpha_{-1} &= -12LV(2Bk - 2C^2k^3 + AB - 2k^3 - 3Ak^2 - A^2k + 3ACk^2 + 4Ck^3 - 2BCK), \\ \alpha_{-2} &= -12LV(C^2k^4 - 2Bk^2 + B^2 + k^4 + 2Ak^3 + A^2k^2 - 2ACK^3 - 2Ck^4 - 2ABk + 2BCK^2), \\ V &= V, L = L, k = k, \alpha_2 = 0, \alpha_1 = 0, \\ C_1 &= -\frac{L^3V^3}{2}(16B^2C^2 + 16B^2 + 8A^2B - 8A^2BC + A^4 - 32B^2C - 1) + LV + \frac{V^3}{2L} + \frac{L^3}{2V},\end{aligned}\quad (16)$$

where k, L, V, A, B and C are arbitrary constants

Set 3:

$$\begin{aligned}\alpha_2 &= -12LV(C-1)^2, \alpha_0 = -LV(8BC - 2A^2 - 8B) - \frac{L}{V} - \frac{V}{L}, k = \frac{A}{2(C-1)}, \alpha_1 = 0, \alpha_{-1} = 0, \\ \alpha_{-2} &= -\frac{3LV}{4(C-1)^2}(16B^2C^2 - 8A^2BC - 32B^2C + 16B^2 + 8A^2B + A^4), V = V, L = L, \\ C_1 &= -L^3V^3(128B^2C^2 + 128B^2 + 8A^4 - 256B^2C - 64A^2BC + 64A^2B) + LV + \frac{V^3}{2L} + \frac{L^3}{2V},\end{aligned}\quad (17)$$

where V, L, A, B and C are arbitrary constants.

Substituting Eqs. (15)-(17) into Eq. (14), we obtain

$$\begin{aligned}u_1(\xi) &= -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ &\quad - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times (k + (G'/G)) \\ &\quad - 12LV(C-1)^2 \times (k + (G'/G))^2.\end{aligned}\quad (18)$$

$$\begin{aligned}u_2(\xi) &= -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ &\quad - 12LV(2Bk - 2C^2k^3 + AB - 2k^3 - 3Ak^2 - A^2k + 3ACk^2 + 4Ck^3 - 2BCK) \\ &\quad \times (k + (G'/G))^{-1} - 12LV(C^2k^4 - 2Bk^2 + B^2 + k^4 + 2Ak^3 + A^2k^2 - 2ACK^3 \\ &\quad - 2Ck^4 - 2ABk + 2BCK^2) \times (k + (G'/G))^{-2}.\end{aligned}\quad (19)$$

$$\begin{aligned}u_3(\xi) &= -LV(8BC - 2A^2 - 8B) - \frac{L}{V} - \frac{V}{L} - 12LV(C-1)^2 \times \left(\frac{A}{2(C-1)} + (G'/G)\right)^2 \\ &\quad - \frac{3LV}{4(C-1)^2}(16B^2C^2 - 8A^2BC - 32B^2C + 16B^2 + 8A^2B + A^4) \\ &\quad \times \left(\frac{A}{2(C-1)} + (G'/G)\right)^{-2}.\end{aligned}\quad (20)$$

Where $\xi = L \frac{x^\alpha}{\Gamma(1+\alpha)} + V \frac{t^\alpha}{\Gamma(1+\alpha)} + \xi_0$; k, L, V, A, B and C are arbitrary constants.

Substituting the solutions $G(\xi)$ of the Eq. (10) into Eq. (18) and simplifying, we obtain the following solutions:

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$\begin{aligned} u_1^1(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta} \tanh(\sqrt{\Delta}\xi/2)) \right\} \\ & - 12LV(C-1)^2 \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta} \tanh(\sqrt{\Delta}\xi/2)) \right\}^2, \end{aligned} \quad (21)$$

$$\begin{aligned} u_1^2(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta} \coth(\sqrt{\Delta}\xi/2)) \right\} \\ & - 12LV(C-1)^2 \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta} \coth(\sqrt{\Delta}\xi/2)) \right\}^2, \end{aligned} \quad (22)$$

$$\begin{aligned} u_1^3(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi) \pm i \operatorname{sech}(\sqrt{\Delta}\xi))) \right\} \\ & - 12LV(C-1)^2 \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi) \pm i \operatorname{sech}(\sqrt{\Delta}\xi))) \right\}^2, \end{aligned} \quad (23)$$

$$\begin{aligned} u_1^4(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta}(\coth(\sqrt{\Delta}\xi) \pm \operatorname{csch}(\sqrt{\Delta}\xi))) \right\} \\ & - 12LV(C-1)^2 \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta}(\coth(\sqrt{\Delta}\xi) \pm \operatorname{csch}(\sqrt{\Delta}\xi))) \right\}^2, \end{aligned} \quad (24)$$

$$\begin{aligned} u_1^5(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{1}{4(C-1)}(2A + \sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi/4) + \coth(\sqrt{\Delta}\xi/4))) \right\} \\ & - 12LV(C-1)^2 \times \left\{ k - \frac{1}{4(C-1)}(2A + \sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi/4) + \coth(\sqrt{\Delta}\xi/4))) \right\}^2, \end{aligned} \quad (25)$$

$$\begin{aligned} u_1^6(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm \sqrt{\Delta(F^2 + H^2)} - F\sqrt{\Delta} \cosh(\sqrt{\Delta}\xi)}{F \sinh(\sqrt{\Delta}\xi) + B} \right\} \right] \\ & - 12LV(C-1)^2 \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm \sqrt{\Delta(F^2 + H^2)} - F\sqrt{\Delta} \cosh(\sqrt{\Delta}\xi)}{F \sinh(\sqrt{\Delta}\xi) + B} \right\} \right]^2, \end{aligned} \quad (26)$$

$$\begin{aligned}
u_1^7(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& -12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm\sqrt{\Delta(F^2 + H^2)} + F\sqrt{\Delta} \cosh(\sqrt{\Delta}\xi)}{F\sinh(\sqrt{\Delta}\xi) + B} \right\} \right] \\
& -12LV(C-1)^2 \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm\sqrt{\Delta(F^2 + H^2)} + F\sqrt{\Delta} \cosh(\sqrt{\Delta}\xi)}{F\sinh(\sqrt{\Delta}\xi) + B} \right\} \right]^2,
\end{aligned} \tag{27}$$

where F and H are real constants.

$$\begin{aligned}
u_1^8(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& -12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k + \frac{2B \cosh(\sqrt{\Delta}\xi/2)}{\sqrt{\Delta} \sinh(\sqrt{\Delta}\xi/2) - A \cosh(\sqrt{\Delta}\xi/2)} \right\} \\
& -12LV(C-1)^2 \times \left\{ k + \frac{2B \cosh(\sqrt{\Delta}\xi/2)}{\sqrt{\Delta} \sinh(\sqrt{\Delta}\xi/2) - A \cosh(\sqrt{\Delta}\xi/2)} \right\}^2,
\end{aligned} \tag{28}$$

$$\begin{aligned}
u_1^9(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& -12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k + \frac{2B \sinh(\sqrt{\Delta}\xi/2)}{\sqrt{\Delta} \cosh(\sqrt{\Delta}\xi/2) - A \sinh(\sqrt{\Delta}\xi/2)} \right\} \\
& -12LV(C-1)^2 \times \left\{ k + \frac{2B \sinh(\sqrt{\Delta}\xi/2)}{\sqrt{\Delta} \cosh(\sqrt{\Delta}\xi/2) - A \sinh(\sqrt{\Delta}\xi/2)} \right\}^2,
\end{aligned} \tag{29}$$

$$\begin{aligned}
u_1^{10}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& -12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k + \frac{2B \cosh(\sqrt{\Delta}\xi)}{\sqrt{\Delta} \sinh(\sqrt{\Delta}\xi) - A \cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}} \right\} \\
& -12LV(C-1)^2 \times \left\{ k + \frac{2B \cosh(\sqrt{\Delta}\xi)}{\sqrt{\Delta} \sinh(\sqrt{\Delta}\xi) - A \cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}} \right\}^2,
\end{aligned} \tag{30}$$

$$\begin{aligned}
u_1^{11}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& -12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k + \frac{2B \sinh(\sqrt{\Delta}\xi)}{\sqrt{\Delta} \cosh(\sqrt{\Delta}\xi) - A \sinh(\sqrt{\Delta}\xi) \pm \sqrt{\Delta}} \right\} \\
& -12LV(C-1)^2 \times \left\{ k + \frac{2B \sinh(\sqrt{\Delta}\xi)}{\sqrt{\Delta} \cosh(\sqrt{\Delta}\xi) - A \sinh(\sqrt{\Delta}\xi) \pm \sqrt{\Delta}} \right\}^2.
\end{aligned} \tag{31}$$

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$\begin{aligned}
 u_1^{12}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
 & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k + \frac{1}{2(C-1)}(-A + \sqrt{-\Delta} \tan(\sqrt{-\Delta}\xi/2)) \right\} \\
 & - 12LV(C-1)^2 \times \left\{ k + \frac{1}{2(C-1)}(-A + \sqrt{-\Delta} \tan(\sqrt{-\Delta}\xi/2)) \right\}^2,
 \end{aligned} \tag{32}$$

$$\begin{aligned}
 u_1^{13}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
 & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{-\Delta} \cot(\sqrt{-\Delta}\xi/2)) \right\} \\
 & - 12LV(C-1)^2 \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{-\Delta} \cot(\sqrt{-\Delta}\xi/2)) \right\}^2,
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 u_1^{14}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
 & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{1}{2(C-1)}(-A + \sqrt{-\Delta}(\tan(\sqrt{-\Delta}\xi) \pm \sec(\sqrt{-\Delta}\xi))) \right\} \\
 & - 12LV(C-1)^2 \times \left\{ k - \frac{1}{2(C-1)}(-A + \sqrt{-\Delta}(\tan(\sqrt{-\Delta}\xi) \pm \sec(\sqrt{-\Delta}\xi))) \right\}^2,
 \end{aligned} \tag{34}$$

$$\begin{aligned}
 u_1^{15}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
 & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{-\Delta}(\cot(\sqrt{-\Delta}\xi) \pm \operatorname{csch}(\sqrt{-\Delta}\xi))) \right\} \\
 & - 12LV(C-1)^2 \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{-\Delta}(\cot(\sqrt{-\Delta}\xi) \pm \operatorname{csch}(\sqrt{-\Delta}\xi))) \right\}^2,
 \end{aligned} \tag{35}$$

$$\begin{aligned}
 u_1^{16}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
 & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm \sqrt{-\Delta(F^2 - H^2)} - F\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi)}{F \sin(\sqrt{-\Delta}\xi) + B} \right\} \right] \\
 & - 12LV(C-1)^2 \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm \sqrt{-\Delta(F^2 - H^2)} - F\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi)}{F \sin(\sqrt{-\Delta}\xi) + B} \right\} \right]^2,
 \end{aligned} \tag{36}$$

$$\begin{aligned}
 u_1^{17}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
 & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm \sqrt{-\Delta(F^2 - H^2)} - F\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi)}{F \sin(\sqrt{-\Delta}\xi) + B} \right\} \right] \\
 & - 12LV(C-1)^2 \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm \sqrt{-\Delta(F^2 - H^2)} - F\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi)}{F \sin(\sqrt{-\Delta}\xi) + B} \right\} \right]^2,
 \end{aligned} \tag{37}$$

$$\begin{aligned}
u_1^{18}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm \sqrt{-\Delta(F^2 - H^2)} + F\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi)}{F\sin(\sqrt{-\Delta}\xi) + B} \right\} \right] \\
& - 12LV(C-1)^2 \times \left[k + \frac{1}{2(C-1)} \left\{ -A + \frac{\pm \sqrt{-\Delta(F^2 - H^2)} + F\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi)}{F\sin(\sqrt{-\Delta}\xi) + B} \right\} \right]^2,
\end{aligned} \tag{38}$$

where F and H are real constants such that $F^2 - H^2 > 0$.

$$\begin{aligned}
u_1^{19}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{2B \cos(\sqrt{-\Delta}\xi/2)}{\sqrt{-\Delta} \sin(\sqrt{-\Delta}\xi/2) + A \cos(\sqrt{-\Delta}\xi/2)} \right\} \\
& - 12LV(C-1)^2 \times \left\{ k - \frac{2B \cos(\sqrt{-\Delta}\xi/2)}{\sqrt{-\Delta} \sin(\sqrt{-\Delta}\xi/2) + A \cos(\sqrt{-\Delta}\xi/2)} \right\}^2,
\end{aligned} \tag{39}$$

$$\begin{aligned}
u_1^{20}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k + \frac{2B \sin(\sqrt{-\Delta}\xi/2)}{\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi/2) - A \sin(\sqrt{-\Delta}\xi/2)} \right\} \\
& - 12LV(C-1)^2 \times \left\{ k + \frac{2B \sin(\sqrt{-\Delta}\xi/2)}{\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi/2) - A \sin(\sqrt{-\Delta}\xi/2)} \right\}^2,
\end{aligned} \tag{40}$$

$$\begin{aligned}
u_1^{21}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{2B \cos(\sqrt{-\Delta}\xi)}{\sqrt{-\Delta} \sin(\sqrt{-\Delta}\xi) + A \cos(\sqrt{-\Delta}\xi) \pm \sqrt{-\Delta}} \right\} \\
& - 12LV(C-1)^2 \times \left\{ k - \frac{2B \cos(\sqrt{-\Delta}\xi)}{\sqrt{-\Delta} \sin(\sqrt{-\Delta}\xi) + A \cos(\sqrt{-\Delta}\xi) \pm \sqrt{-\Delta}} \right\}^2,
\end{aligned} \tag{41}$$

$$\begin{aligned}
u_1^{22}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k + \frac{2B \sin(\sqrt{-\Delta}\xi/2)}{\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi/2) - A \sin(\sqrt{-\Delta}\xi/2) \pm \sqrt{-\Delta}} \right\} \\
& - 12LV(C-1)^2 \times \left\{ k + \frac{2B \sin(\sqrt{-\Delta}\xi/2)}{\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi/2) - A \sin(\sqrt{-\Delta}\xi/2) \pm \sqrt{-\Delta}} \right\}^2.
\end{aligned} \tag{42}$$

When $B = 0$ and $A(C - 1) \neq 0$,

$$u_1^{23}(\xi) = -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{Ac_1}{(C-1)\{c_1 + \cosh(A\xi) - \sinh(A\xi)\}} \right\} \\ - 12LV(C-1)^2 \times \left\{ k - \frac{Ac_1}{(C-1)\{c_1 + \cosh(A\xi) - \sinh(A\xi)\}} \right\}^2, \quad (43)$$

(a) $\alpha = 0.25$.

(b) $\alpha = 0.50$.

(c) $\alpha = 0.75$.

(d) $\alpha = 1$.

Fig.1. (a)-(d) show the soliton solution of u_1^{23}

$$u_1^{24}(\xi) = -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{A(\cosh(A\xi) + \sinh(A\xi))}{(C-1)\{c_1 + \cosh(A\xi) + \sinh(A\xi)\}} \right\} \\ - 12LV(C-1)^2 \times \left\{ k - \frac{A(\cosh(A\xi) + \sinh(A\xi))}{(C-1)\{c_1 + \cosh(A\xi) + \sinh(A\xi)\}} \right\}^2, \quad (44)$$

where c_1 is an arbitrary constant.

When $A = B = 0$ and $(C - 1) \neq 0$, the solution of Eq. (12) is

$$\begin{aligned} u_1^{25}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(-2C^2k + 4Ck + AC - A - 2k) \times \left\{ k - \frac{1}{(C-1)\xi + c_2} \right\} \\ & - 12LV(C-1)^2 \times \left\{ k - \frac{1}{(C-1)\xi + c_2} \right\}^2, \end{aligned} \quad (45)$$

where c_2 is an arbitrary constant.

Substituting the solutions $G(\xi)$ of the Eq. (10) into Eq. (19) and simplifying, we obtain the following solutions:

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$\begin{aligned} u_2^1(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(2Bk - 2C^2k^3 + AB - 2k^3 - 3Ak^2 - A^2k + 3ACk^2 + 4Ck^3 - 2BCK) \\ & \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta} \tanh(\sqrt{\Delta}\xi/2)) \right\}^{-1} - 12LV(C^2k^4 - 2Bk^2 + B^2 + k^4 + 2Ak^3 + A^2k^2 \\ & - 2ACk^3 - 2Ck^4 - 2ABk + 2BCK^2) \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta} \tanh(\sqrt{\Delta}\xi/2)) \right\}^{-2}, \end{aligned} \quad (46)$$

where $\xi = \frac{Lx^\alpha}{\Gamma(1+\alpha)} + \frac{Vt^\alpha}{\Gamma(1+\alpha)} + \xi_0$; k, L, V, A, B and C are arbitrary constants.

$$\begin{aligned} u_2^2(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(2Bk - 2C^2k^3 + AB - 2k^3 - 3Ak^2 - A^2k + 3ACk^2 + 4Ck^3 - 2BCK) \\ & \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta} \coth(\sqrt{\Delta}\xi/2)) \right\}^{-1} - 12LV(C^2k^4 - 2Bk^2 + B^2 + k^4 + 2Ak^3 \\ & + A^2k^2 - 2ACk^3 - 2Ck^4 - 2ABk + 2BCK^2) \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta} \coth(\sqrt{\Delta}\xi/2)) \right\}^{-2}, \end{aligned} \quad (47)$$

$$\begin{aligned} u_2^3(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\ & - 12LV(2Bk - 2C^2k^3 + AB - 2k^3 - 3Ak^2 - A^2k + 3ACk^2 + 4Ck^3 - 2BCK) \\ & \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi) \pm i \operatorname{sech}(\sqrt{\Delta}\xi))) \right\}^{-1} - 12LV(C^2k^4 - 2Bk^2 + B^2 + k^4 + 2Ak^3 \\ & + A^2k^2 - 2ACk^3 - 2Ck^4 - 2ABk + 2BCK^2) \times \left\{ k - \frac{1}{2(C-1)}(A + \sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi) \pm i \operatorname{sech}(\sqrt{\Delta}\xi))) \right\}^{-2}. \end{aligned} \quad (48)$$

The other families of exact solutions of Eq. (12) are omitted for convenience.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$\begin{aligned}
 u_2^{12}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
 & - 12LV(2Bk - 2C^2k^3 + AB - 2k^3 - 3Ak^2 - A^2k + 3ACk^2 + 4Ck^3 - 2BCK) \\
 & \times \left\{ k + \frac{1}{2(C-1)} \left(-A + \sqrt{-\Delta} \tan(\sqrt{-\Delta}\xi/2) \right) \right\}^{-1} - 12LV(C^2k^4 - 2Bk^2 + B^2 + k^4 + 2Ak^3 \\
 & + A^2k^2 - 2ACk^3 - 2Ck^4 - 2ABk + 2BCK^2) \times \left\{ k + \frac{1}{2(C-1)} \left(-A + \sqrt{-\Delta} \tan(\sqrt{-\Delta}\xi/2) \right) \right\}^{-2},
 \end{aligned} \quad (49)$$

(a) $\alpha = 0.25$.

(b) $\alpha = 0.50$.

(c) $\alpha = 0.75$.

(d) $\alpha = 1$.

Fig. 2 . (a)- (d) show the periodic solution of u_2^{12} .

$$\begin{aligned}
 u_2^{13}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
 & - 12LV(2Bk - 2C^2k^3 + AB - 2k^3 - 3Ak^2 - A^2k + 3ACk^2 + 4Ck^3 - 2BCK) \\
 & \times \left\{ k - \frac{1}{2(C-1)} \left(A + \sqrt{-\Delta} \cot(\sqrt{-\Delta}\xi/2) \right) \right\}^{-1} - 12LV(C^2k^4 - 2Bk^2 + B^2 + k^4 + 2Ak^3 \\
 & + A^2k^2 - 2ACk^3 - 2Ck^4 - 2ABk + 2BCK^2) \times \left\{ k - \frac{1}{2(C-1)} \left(A + \sqrt{-\Delta} \cot(\sqrt{-\Delta}\xi/2) \right) \right\}^{-2},
 \end{aligned} \quad (50)$$

$$\begin{aligned}
u_2^{14}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} \\
& - 12LV(2Bk - 2C^2k^3 + AB - 2k^3 - 3Ak^2 - A^2k + 3ACk^2 + 4Ck^3 - 2BCK) \\
& \times \left\{ k + \frac{1}{2(C-1)} \left(-A + \sqrt{-\Delta} (\tan(\sqrt{-\Delta}\xi) \pm \sec(\sqrt{-\Delta}\xi)) \right) \right\}^{-1} - 12LV(C^2k^4 - 2Bk^2 + B^2 + k^4 + 2Ak^3 \\
& + A^2k^2 - 2ACk^3 - 2Ck^4 - 2ABk + 2BCK^2) \times \left\{ k + \frac{1}{2(C-1)} \left(-A + \sqrt{-\Delta} (\tan(\sqrt{-\Delta}\xi) \pm \sec(\sqrt{-\Delta}\xi)) \right) \right\}^{-2}. \quad (51)
\end{aligned}$$

When $A = B = 0$ and $(C-1) \neq 0$, the solution of Eq. (12) is

$$\begin{aligned}
u_2^{25}(\xi) = & -LV(12C^2k^2 + 12k^2 + 12kA - 12kAC - 24k^2C + 8BC - 8B + A^2) - \frac{L}{V} - \frac{V}{L} - 12LV(2Bk - 2C^2k^3 \\
& + AB - 2k^3 - 3Ak^2 - A^2k + 3ACk^2 + 4Ck^3 - 2BCK) \times \left\{ k - \frac{1}{(C-1)\xi + c_3} \right\}^{-1} - 12LV(C^2k^4 - 2Bk^2 \\
& + B^2 + k^4 + 2Ak^3 + A^2k^2 - 2ACk^3 - 2Ck^4 - 2ABk + 2BCK^2) \times \left\{ k - \frac{1}{(C-1)\xi + c_3} \right\}^{-2}, \quad (52)
\end{aligned}$$

where c_3 is an arbitrary constant.

We can write down the other families of exact solutions of Eq. (12) which are omitted for practicality.

Finally, substituting the solutions $G(\zeta)$ of the Eq. (10) into Eq.(20) and simplifying, we obtain the following solutions:

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$\begin{aligned}
u_3^1(\xi) = & -LV(8BC - 2A^2 - 8B) - \frac{L}{V} - \frac{V}{L} - 12LV(C-1)^2 \times \left(\frac{1}{2(C-1)} (\sqrt{\Delta} \tanh(\sqrt{\Delta} \xi / 2)) \right)^2 \\
& - \frac{3LV}{4(C-1)^2} (16B^2C^2 - 8A^2BC - 32B^2C + 16B^2 + 8A^2B + A^4) \times \left(\frac{1}{2(C-1)} (\sqrt{\Delta} \tanh(\sqrt{\Delta} \xi / 2)) \right)^{-2}, \quad (53)
\end{aligned}$$

where $\xi = \frac{Lx^\alpha}{\Gamma(1+\alpha)} + \frac{Vt^\alpha}{\Gamma(1+\alpha)} + \xi_0$; L, V, A, B and C are arbitrary constants.

$$\begin{aligned}
u_3^2(\xi) = & -LV(8BC - 2A^2 - 8B) - \frac{L}{V} - \frac{V}{L} - 12LV(C-1)^2 \times \left(\frac{1}{2(C-1)} (\sqrt{\Delta} \coth(\sqrt{\Delta} \xi / 2)) \right)^2 \\
& - \frac{3LV}{4(C-1)^2} (16B^2C^2 - 8A^2BC - 32B^2C + 16B^2 + 8A^2B + A^4) \times \left(\frac{1}{2(C-1)} (\sqrt{\Delta} \coth(\sqrt{\Delta} \xi / 2)) \right)^{-2}, \quad (54)
\end{aligned}$$

$$\begin{aligned}
u_3^3(\xi) = & -LV(8BC - 2A^2 - 8B) - \frac{L}{V} - \frac{V}{L} - 12LV(C-1)^2 \times \left(\frac{1}{2(C-1)} \left\{ \sqrt{\Delta} (\tanh(\sqrt{\Delta} \xi) \pm i \operatorname{sech}(\sqrt{\Delta} \xi)) \right\} \right)^2 \\
& - \frac{3LV}{4(C-1)^2} (16B^2C^2 - 8A^2BC - 32B^2C + 16B^2 + 8A^2B + A^4) \times \left(\frac{1}{2(C-1)} \left\{ \sqrt{\Delta} (\tanh(\sqrt{\Delta} \xi) \pm i \operatorname{sech}(\sqrt{\Delta} \xi)) \right\} \right)^{-2}. \quad (55)
\end{aligned}$$

Others families of exact solutions are omitted for the sake of simplicity.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$\begin{aligned}
u_3^{12}(\xi) = & -LV(8BC - 2A^2 - 8B) - \frac{L}{V} - \frac{V}{L} - 12LV(C-1)^2 \times \left(\frac{1}{2(C-1)} (\sqrt{-\Delta} \tan(\sqrt{-\Delta} \xi / 2)) \right)^2 \\
& - \frac{3LV}{4(C-1)^2} (16B^2C^2 - 8A^2BC - 32B^2C + 16B^2 + 8A^2B + A^4) \times \left(\frac{1}{2(C-1)} (\sqrt{-\Delta} \tan(\sqrt{-\Delta} \xi / 2)) \right)^{-2}, \quad (56)
\end{aligned}$$

Fig. 3. (a)- (d) show the singular soliton solution of u_3^{12} .

$$u_3^{13}(\xi) = -LV(8BC - 2A^2 - 8B) - \frac{L}{V} - \frac{V}{L} - 12LV(C-1)^2 \times \left(\frac{1}{2(C-1)} (\sqrt{-\Delta} \cot(\sqrt{-\Delta} \xi / 2)) \right)^2$$

$$- \frac{3LV}{4(C-1)^2} (16B^2C^2 - 8A^2BC - 32B^2C + 16B^2 + 8A^2B + A^4) \times \left(\frac{1}{2(C-1)} (\sqrt{-\Delta} \cot(\sqrt{-\Delta} \xi / 2)) \right)^{-2}, \quad (57)$$

$$u_3^{14}(\xi) = -LV(8BC - 2A^2 - 8B) - \frac{L}{V} - \frac{V}{L} - 12LV(C-1)^2 \times \left(\frac{1}{2(C-1)} \{ \sqrt{-\Delta} (\tan(\sqrt{-\Delta} \xi) \pm \sec(\sqrt{-\Delta} \xi)) \} \right)^2$$

$$- \frac{3LV}{4(C-1)^2} (16B^2C^2 - 8A^2BC - 32B^2C + 16B^2 + 8A^2B + A^4) \times \left(\frac{1}{2(C-1)} \{ \sqrt{-\Delta} (\tan(\sqrt{-\Delta} \xi) \pm \sec(\sqrt{-\Delta} \xi)) \} \right)^{-2}, \quad (58)$$

When $(C-1) \neq 0$ and $A=B=0$, the solution of Eq. (12) is

$$u_3^{25}(\xi) = -LV(8BC - 2A^2 - 8B) - \frac{L}{V} - \frac{V}{L} - 12LV(C-1)^2 \times \left(\frac{A}{2(C-1)} - \frac{1}{(C-1)\xi + c_4} \right)^2$$

$$- \frac{3LV}{4(C-1)^2} (16B^2C^2 - 8A^2BC - 32B^2C + 16B^2 + 8A^2B + A^4) \times \left(\frac{A}{2(C-1)} - \frac{1}{(C-1)\xi + c_4} \right)^{-2}, \quad (59)$$

where c_4 is an arbitrary constant.

Other exact solutions of Eq. (12) are omitted here for convenience.

Remark: We have checked the obtained solutions by putting them back into the original equation and found correct.

5. Conclusions

A novel (G'/G) -expansion method is applied to fractional partial differential equation successfully. As applications, abundant new exact solutions for space-time fractional symmetric regularized long wave (SRLW) equation have been successfully obtained. The nonlinear fractional complex transformation for ζ is very important, which ensures that a certain fractional partial differential equation can be converted into another ordinary differential equation of integer order. The obtained solutions are more general, and many known solutions are only a special case of them. Thus novel (G'/G) -expansion method would be a powerful mathematical tool for solving nonlinear evolution equations.

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