GENERALISED SUM CONSTRUCTION OF AUTOMORPHIC BIBDS AND THEIR APPLICATIONS IN EXPERIMENTAL DESIGNS AND TECHNOLOGY

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ABSTRACT. We give more constructions equivalent to “sum construction“, of BIBDs where BIBD is added to a BIBD that is automorphic to it is presented. Moreover we present their applications in technology

INTRODUCTION

Wilson (1975) came up with block spreading method for \( t = 2 \) and for prime power index. Let \( v \) be a positive integer \( v \geq 2 \) and let \( q \) be a prime power. Suppose that there exists a \( S_q(2,k,v) \) design satisfying \( q \geq v + 1 \). Then there exists a group divisible design (GDD) of group type \( (q^d)^v \) with block size \( k \) and index one, whenever \( d \geq \binom{v}{2} \). This method has application in the construction of Steiner \( 2 \)-designs. Blanchard (1995a, 1995b, 1995c) generalizes Wilson’s results for \( t \geq 2 \) as follows: (The “block spreading” method for \( t \geq 2 \) and for prime power index). Let \( v \) and \( t \) be positive integers, \( 2 \leq t \leq v \), and let \( q \) be a prime power. Then there exists a number \( q_0 = q_0(t,v) \) such that for any \( S_q(t,k,v) \) design satisfying \( q \geq q_0 \), there is a \( t \)-GDD of group type \( (q^d)^v \) with block size \( k \) and index one whenever \( d \geq \binom{v}{2} \). More so, Mohacsy and Chaudhuri (2001, 2003) generalizes Blanchard’s construction for general index. (The “block spreading” method for \( t \geq 2 \) and general index). Let \( v, t \) and \( \lambda \) be a positive integers \( 2 \leq t \leq v \). Then there exists a number \( q_0 = q_0(t,v) \) such that for any \( S_\lambda(t,k,v) \) design with prime power decomposition \( \lambda = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n \) satisfying \( \lambda_i \geq q_0 \); \( 1 \leq i \leq n \); there is a \( t \)-GDD of group type \( (\lambda^d)^v \) with block size \( k \) and index one whenever \( d \geq \binom{v}{2} \). This generalized “block spreading” construction has several application such as constructing new Steiner \( 3 \)-designs and new group divisible \( t \)-designs with index one. Limitation of this method is that the bounds on \( d \) are too large.

Table 3.3.3.12. Case 2; for \( x = 3 \) and \( c = 6 \) the possible cases of \( 3 \) – \((v,k,c) \) design

<table>
<thead>
<tr>
<th>( \lambda_2 )</th>
<th>( k )</th>
<th>( \lambda_1 )</th>
<th>( v )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>6</td>
<td>33</td>
<td>12</td>
<td>66</td>
</tr>
<tr>
<td>24</td>
<td>9</td>
<td>87</td>
<td>30</td>
<td>290</td>
</tr>
<tr>
<td>51</td>
<td>18</td>
<td>411</td>
<td>138</td>
<td>3151</td>
</tr>
</tbody>
</table>

Table 3.3.3.13. Case 2; for \( x = 6 \) and \( c = 6 \) the possible cases of \( 3 \) – \((v,k,c) \) design

<table>
<thead>
<tr>
<th>( \lambda_2 )</th>
<th>( k )</th>
<th>( \lambda_1 )</th>
<th>( v )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>4</td>
<td>42</td>
<td>8</td>
<td>84</td>
</tr>
<tr>
<td>30</td>
<td>6</td>
<td>126</td>
<td>22</td>
<td>462</td>
</tr>
<tr>
<td>42</td>
<td>8</td>
<td>258</td>
<td>44</td>
<td>1419</td>
</tr>
</tbody>
</table>

Also for \( c = 7 \) and \( x = 7 \) we get the following table:
Table 3.3.3.14. Case 2; for $x = 7$ and $c = 7$ the possible cases of $3 - (v,k,c)$ design

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$\nu$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>4</td>
<td>49</td>
<td>8</td>
<td>98</td>
</tr>
<tr>
<td>35</td>
<td>6</td>
<td>147</td>
<td>22</td>
<td>539</td>
</tr>
<tr>
<td>42</td>
<td>7</td>
<td>217</td>
<td>32</td>
<td>992</td>
</tr>
</tbody>
</table>

Lastly, for $c = 8, x = 4$ and $x = 8$ we also get the following tables respectively.

Table 3.3.3.15. Case 2; for $x = 4$ and $c = 8$ the possible cases of $3 - (v,k,c)$ design

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$\nu$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>4</td>
<td>16</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>44</td>
<td>12</td>
<td>88</td>
</tr>
<tr>
<td>28</td>
<td>8</td>
<td>88</td>
<td>23</td>
<td>253</td>
</tr>
<tr>
<td>44</td>
<td>12</td>
<td>224</td>
<td>57</td>
<td>1064</td>
</tr>
</tbody>
</table>

Table 3.3.3.16. Case 2; for $x = 8$ and $c = 8$ the possible cases of $3 - (v,k,c)$ design

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$\nu$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>4</td>
<td>56</td>
<td>8</td>
<td>112</td>
</tr>
<tr>
<td>40</td>
<td>6</td>
<td>168</td>
<td>22</td>
<td>616</td>
</tr>
<tr>
<td>56</td>
<td>8</td>
<td>344</td>
<td>44</td>
<td>1892</td>
</tr>
<tr>
<td>88</td>
<td>12</td>
<td>888</td>
<td>112</td>
<td>8288</td>
</tr>
</tbody>
</table>

3.3.4. CASE III, $x \neq y$ and $x > 1, y > 1$

In this case (3.3.1.3) is

$$\lambda_1 = \frac{x}{y} (v - 1) \quad \Rightarrow \quad v = \frac{y\lambda_1 + x}{x}$$

$$\lambda_2 = \frac{x}{y} (k - 1) \quad \Rightarrow \quad k = \frac{y\lambda_2 + x}{x}$$

Where $x$ and $y$ are positive integers and $\lambda_2$ and $y$ are divisible by $x$.

Now substituting these values of $v$ and $k$ in:

$$\lambda_2 = \frac{v(v-2)}{k-2}$$

We get $\lambda_1, v, k$ and $k$ as follows:

$$\lambda_1 = \frac{y\lambda_2^2 - x\lambda_2 + cx}{cy}$$

$$v = \frac{y\lambda_2^2 - x\lambda_2 + 2cx}{cx}$$

And

$$k = \frac{y\lambda_2 + x}{x}$$

Using $b = \frac{\lambda_1 v}{k}$ for this to be $3 - (v,k,1)$ design then

$$\frac{(y\lambda_2^2 - x\lambda_2 + cx)(y\lambda_2^2 - x\lambda_2 + 2cx)}{cy} \equiv 0 \mod \left(\frac{y\lambda_2 + x}{x}\right)$$

That is

$$\frac{(y\lambda_2^2 - x\lambda_2 + cx)(y\lambda_2^2 - x\lambda_2 + 2cx)}{c^2y(y\lambda_2 + x)}$$

is positive integer.
Expanding and dividing we will obtain
\[ y\lambda_3^2 - 3x\lambda_2^2 + \frac{\lambda_2^2(3cxy + 4x^2)}{y} - \frac{(6cx^2y + 4x^3)}{y^2} \text{rem} \frac{2c^2x^2y^2 + 6cx^2y + 4x^4}{y^2} \]
That is:
\[ y\lambda_3^2 - 3x\lambda_2^2 + \frac{\lambda_2^2(3cxy + 4x^2)}{y} - \frac{(6cx^2y + 4x^3)}{y^2} + \frac{2c^2x^2y^2 + 6cx^3y + 4x^4}{c^2y^3(y\lambda_2 + x)} \]
Under this case and using similar method we find \(3 - (v, k, c)\) exists if:
\[ \frac{2c^2x^2y^2 + 6cx^3y + 4x^4}{c^2y^3(y\lambda_2 + x)} = \frac{2x^2(c^2 + \frac{3cx}{y} + \frac{2x^2}{y^2})}{c^2y(y\lambda_2 + x)} \]
is an integer.
Take \(c = 5, x = 5\) and \(y = 2\) in this case there is only one non-trivial \(3 - (v, k, 5)\) which we get as follows:
Equation (3.3.4.2) in this case is
\[ \frac{y\lambda_2 + x}{75} \]
And \(\lambda_2\) would take the values 5 or 35 with corresponding values of \(k, \lambda_1, v\) and \(b\) as in the table below.

**Table 3.3.4.1.** Case 2; for \(x = 5, y = 2\) and \(c = 5\) the possible cases of \(3 - (v,k,c)\) design

<table>
<thead>
<tr>
<th>(\lambda_2)</th>
<th>(k)</th>
<th>(\lambda_1)</th>
<th>(v)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>35</td>
<td>15</td>
<td>230</td>
<td>93</td>
<td>1426</td>
</tr>
</tbody>
</table>

Also for \(c = 7, x = 7\) and \(y = 2\), we get the following table:

**Table 3.3.4.2.** Case 2; for \(x = 7, y = 2\) and \(c = 7\) the possible cases of \(3 - (v,k,c)\) design

<table>
<thead>
<tr>
<th>(\lambda_2)</th>
<th>(k)</th>
<th>(\lambda_1)</th>
<th>(v)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>3</td>
<td>7</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>21</td>
<td>7</td>
<td>56</td>
<td>17</td>
<td>136</td>
</tr>
</tbody>
</table>

### 3.4 CONSTRUCTION OF 4 – (v, k, 1)

In this section, the same technique that has been used to construct \(3 - (v, k, 1)\) is applied.

Here \(t = 4\) and \(\lambda_t = 1\)

From:
\[ \lambda_t(v - (t - 1)) = \lambda_{t-1}(k - (t - 1)) \]

We have:
\[ \lambda_3 = \frac{v - 3}{k - 3} \] (3.4.1)
\[
\lambda_2 = \frac{\nu - 2}{k - 2}
\]
(3.4.2)

And:
\[
\frac{\lambda_1}{\lambda_2} = \frac{\nu - 1}{k - 1}
\]
(3.4.3)

Now:
\[
\frac{\lambda_2}{\lambda_3} = \frac{\nu - 2}{k - 2}
\]

This can be written as:
\[
\lambda_2 = \alpha (\nu - 2)
\]

And:
\[
\lambda_3 = \alpha (k - 2)
\]
(3.4.4)

Given \(\lambda_1, \lambda_2, \lambda_3, \nu - 2, \nu - 1, k - 2, \text{and} \ k - 1\) are all integers, \(\alpha\) is a rational number which we will represent by \(\frac{x}{y}\) where \(x\) and \(y\) are positive integers.

3.4.1. Case 1 \(x = 1\)

Then (3.4.4) becomes:
\[
y\lambda_2 = \nu - 2, \quad \Rightarrow \nu = y\lambda_2 + 2
\]

And:
\[
y\lambda_3 = k - 2, \quad \Rightarrow k = y\lambda_3 + 2
\]

From (3.4.1) we have now:
\[
\lambda_3 = \frac{y\lambda_2 - 1}{y\lambda_3 - 1}
\]

\[
\lambda_2 = \frac{y\lambda_3^2 - \lambda_3 + 1}{y}
\]

Similarly from (3.4.3) we obtain:
\[
\lambda_1 = \frac{y\lambda_2^2 + \lambda_2}{y\lambda_3 + 1}
\]

\[
\lambda_1 = \frac{y^2\lambda_3^4 - 2y\lambda_3^3 + 3y\lambda_3^2 + \lambda_3^2 - 3\lambda_3 + 2}{y\lambda_3 + 1}
\]

Which implies:
\[
\nu = y\lambda_3^2 - \lambda_3 + 3
\]
\[
k = y\lambda_3 + 2
\]
For this design to be $4 - (v, k, 1)$ and from $bk = v\lambda_1$ it means

$$\frac{(y^2\lambda_3^4 - 2y\lambda_3^3 + 3y\lambda_3^2 + \lambda_3^2 - 3\lambda_3 + 2)(y\lambda_3^2 - \lambda_3 + 3)}{(y\lambda_3 + 1)} \equiv 0 \mod (y\lambda_3 + 2)$$

That is

$$\frac{(y^2\lambda_3^4 - 2y\lambda_3^3 + 3y\lambda_3^2 + \lambda_3^2 - 3\lambda_3 + 2)(y\lambda_3^2 - \lambda_3 + 3)}{(y\lambda_3 + 1)(y\lambda_3 + 2)}$$

Is a positive integer

Expanding and dividing we obtain

$$\frac{y^2\lambda_3^5 - 5y\lambda_3^4}{(y\lambda_3 + 1)} + \frac{\lambda_3^3(6y^2 + 13y)}{y(y\lambda_3 + 1)} - \frac{\lambda_3^2(24y^2 + 27y)}{y^2(y\lambda_3 + 1)} + \frac{\lambda_3(11y^3 + 54y^2 + 54y)}{y^3(y\lambda_3 + 1)}$$

Now (3.4.1.2) will be integer when

$$\frac{6y^4 + 66y^3 + 216y^2 + 216y}{y^4(y\lambda_3 + 1)(y\lambda_3 + 2)}$$

Is an integer.

Now (3.4.1.3) will be integer an only when $y$ is 1 and 2

For $y = 2$

$$\frac{6y^4 + 66y^3 + 216y^2 + 216y}{y^4(y\lambda_3 + 1)(y\lambda_3 + 2)} = \frac{120}{2\lambda_3^2 + 6\lambda_3 + 2}$$

In this case (3.4.1.3) is not an integer.

For $y = 1$ (3.4.1.3) is

$$504 \div (\lambda_3^2 + 3\lambda_3 + 2)$$

Thus (3.4.1.3) will be integer if $\lambda_3$ takes any of the following values 2 and 5

The table below gives corresponding values of $\lambda_1, \lambda_2, k, v$ and $b$.

**Table 3.4.1.1.** Case 1; for $y = 1$ the possible cases of $4 - (v, k, 1)$ design

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$k$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>21</td>
<td>77</td>
<td>7</td>
<td>23</td>
<td>253</td>
</tr>
</tbody>
</table>

The first $4 - (5, 4, 1)$ design is trivial. The $4 - (23, 7, 1)$ is the only non trivial. This $4 - (v, k, 1)$ is then identified with the following BIB designs: $2 - (5, 4, 3)$ and $2 - (23, 7, 21)$

3.4.1. Case 2, $y = 1$

In this case (3.4.4) is

$$\lambda_1 = x(v - 2), \quad \Rightarrow v = \frac{\lambda_1 + 2 x}{x}$$
And

\[ \lambda_3 = x(k - 2), \quad \Rightarrow k = \frac{\lambda_3 + 2x}{x} \]

From (3.4.1) we have now

\[ \lambda_3 = \frac{\lambda_3 - x}{\lambda_3 - x} \]

\[ \lambda_2 = \lambda_3^2 - x \lambda_3 + x \]

Similarly from (3.4.3) we obtain

\[ \lambda_1 = \frac{\lambda_2^2 + x \lambda_2}{\lambda_3 + x} \]

\[ \lambda_1 = \frac{\lambda_3^4 - 2x \lambda_3^3 + 3x \lambda_3^2 + x^2 \lambda_3^2 - 3x^2 \lambda_3 + 2x^2}{\lambda_3 + x} \]

This implies

\[ v = \frac{\lambda_3^2 - x \lambda_3 + 3x}{x} \]

\[ k = \frac{\lambda_3 + 2x}{x} \]

For this design to be 4 - (v, k, 1) and from bk = v \lambda_1 it means

\[ \frac{\lambda_3^4 - 2x \lambda_3^3 + 3x \lambda_3^2 + x^2 \lambda_3^2 - 3x^2 \lambda_3 + 2x^2}{(\lambda_3 + x)}(\lambda_3 + x)(\lambda_3 + 2x) \equiv 0 \quad \text{mod}(\lambda_3 + 2x) \]

That is

\[ \frac{(\lambda_3^4 - 2x \lambda_3^3 + 3x \lambda_3^2 + x^2 \lambda_3^2 - 3x^2 \lambda_3 + 2x^2)(v \lambda_3^2 - x \lambda_3 + 3x)}{(\lambda_3 + x)(\lambda_3 + 2x)} \]

Is a positive integer

Using similar argument as before (3.4.2.1) will be integer if

\[ \frac{6x^5 + 258x^4 + 24x^5 + 216x^6}{(\lambda_3 + x)(\lambda_3 + 2x)} \]

Is an integer.

If \( x = 1 \), we get the result of case 1 for \( x = 2 \), (3.4.2.2) becomes

\[ 18768 \]

\[ (\lambda_3 + 2)(\lambda_3 + 4) \]

Thus \( \lambda_3 \) takes any of the following values, 2 and 4. But \( \lambda_3 \) must be greater than 2, hence 2 is not a possibility. We give corresponding values of \( \lambda_1, \lambda_2, v, k \) and \( b \) in the table below.
**Table 3.4.2.1.** Case 2; for $x = 1$ the possible cases of $4 - (v,k,1)$ design

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$k$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>10</td>
<td>20</td>
<td>4</td>
<td>7</td>
<td>35</td>
</tr>
</tbody>
</table>

This $4 - (7,4,1)$ design is trivial. Hence, for $x = 2$ there is no nontrivial $4 - (v,k,1)$ design. For $x = 3$, (3.4.2.2) is

$$\frac{184356}{(\lambda_3 + 3)(\lambda_3 + 6)}$$

And $\lambda_3$ takes only 6 as its value. The corresponding values of $\lambda_1, \lambda_2, v, k$ and $b$ are given in the table below.

**Table 3.4.2.2.** Case 2; for $x = 3$ the possible cases of $4 - (v,k,1)$ design

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$k$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>21</td>
<td>56</td>
<td>4</td>
<td>9</td>
<td>126</td>
</tr>
</tbody>
</table>

This $4 - (9,4,1)$ design is trivial. Hence, also for $x = 3$ there is no nontrivial $4 - (v,k,1)$ design. For $x = 4$, (3.4.2.2) is

$$\frac{975744}{(\lambda_3 + 4)(\lambda_3 + 8)}$$

And $\lambda_3$ takes the following values 8 and 28. The corresponding values of $\lambda_1, \lambda_2, v, k$ and $b$ are given in the table below.

**Table 3.4.2.3.** Case 2; for $x = 4$ the possible cases of $4 - (v,k,1)$ design

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$k$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>36</td>
<td>120</td>
<td>4</td>
<td>11</td>
<td>330</td>
</tr>
<tr>
<td>28</td>
<td>676</td>
<td>14365</td>
<td>9</td>
<td>171</td>
<td>272935</td>
</tr>
</tbody>
</table>

Again we obtain the desired $4 - (v,k,1)$ designs from BIB designs below:

$2 - (11,4,36), 2 - (171,9,676)$

For $x = 5$ and using similar arguments as before, the possible values of $\lambda_3$ are as follows: 10, 15 and 20 which gives the values of $\lambda_1, \lambda_2, v, k$ and $b$ as in the table below.

**Table 3.4.2.4.** Case 2; for $x = 5$ the possible cases of $4 - (v,k,1)$ design

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$k$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>55</td>
<td>220</td>
<td>4</td>
<td>13</td>
<td>715</td>
</tr>
<tr>
<td>15</td>
<td>155</td>
<td>1240</td>
<td>5</td>
<td>33</td>
<td>8184</td>
</tr>
<tr>
<td>20</td>
<td>305</td>
<td>3782</td>
<td>6</td>
<td>63</td>
<td>39711</td>
</tr>
</tbody>
</table>

For $x = 6$, (3.4.2.2) is

$$\frac{556496}{(\lambda_3 + 5)(\lambda_3 + 10)}$$

And $\lambda_3$ takes the following values 12 and 48. The corresponding values of $\lambda_1, \lambda_2, v, k$ and $b$ are given in the table below.

**Table 3.4.2.5.** Case 2; for $x = 6$ the possible cases of $4 - (v,k,1)$ design

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$k$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>121</td>
<td>1721</td>
<td>8</td>
<td>18</td>
<td>21927</td>
</tr>
<tr>
<td>48</td>
<td>401</td>
<td>28084</td>
<td>18</td>
<td>30</td>
<td>657407</td>
</tr>
</tbody>
</table>

Again, we obtain the desired $4 - (v,k,1)$ designs from BIB designs below:

$2 - (23,4,90), 2 - (207,13,1878)$

For $x = 7$, (3.4.2.2) is

$$\frac{6900288}{(\lambda_3 + 6)(\lambda_3 + 12)}$$

And $\lambda_3$ takes the following values 24 and 96. The corresponding values of $\lambda_1, \lambda_2, v, k$ and $b$ are given in the table below.

**Table 3.4.2.6.** Case 2; for $x = 7$ the possible cases of $4 - (v,k,1)$ design

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$k$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>242</td>
<td>3582</td>
<td>24</td>
<td>24</td>
<td>483384</td>
</tr>
<tr>
<td>96</td>
<td>962</td>
<td>7442</td>
<td>48</td>
<td>48</td>
<td>3628864</td>
</tr>
</tbody>
</table>

Again, we obtain the desired $4 - (v,k,1)$ designs from BIB designs below:

$2 - (39,4,270), 2 - (255,17,5544)$

For $x = 8$, (3.4.2.2) is

$$\frac{20775856}{(\lambda_3 + 7)(\lambda_3 + 14)}$$

And $\lambda_3$ takes the following values 42 and 168. The corresponding values of $\lambda_1, \lambda_2, v, k$ and $b$ are given in the table below.

**Table 3.4.2.7.** Case 2; for $x = 8$ the possible cases of $4 - (v,k,1)$ design

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$k$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>422</td>
<td>5072</td>
<td>42</td>
<td>42</td>
<td>1014288</td>
</tr>
<tr>
<td>168</td>
<td>1682</td>
<td>11636</td>
<td>84</td>
<td>84</td>
<td>9067344</td>
</tr>
</tbody>
</table>

Again, we obtain the desired $4 - (v,k,1)$ designs from BIB designs below:

$2 - (78,4,540), 2 - (525,25,3675)$

For $x = 9$, (3.4.2.2) is

$$\frac{62326016}{(\lambda_3 + 8)(\lambda_3 + 16)}$$

And $\lambda_3$ takes the following values 84 and 336. The corresponding values of $\lambda_1, \lambda_2, v, k$ and $b$ are given in the table below.

**Table 3.4.2.8.** Case 2; for $x = 9$ the possible cases of $4 - (v,k,1)$ design

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\lambda_2$</th>
<th>$\lambda_1$</th>
<th>$k$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>84</td>
<td>842</td>
<td>9682</td>
<td>84</td>
<td>84</td>
<td>1685184</td>
</tr>
<tr>
<td>336</td>
<td>3362</td>
<td>25444</td>
<td>168</td>
<td>168</td>
<td>425885184</td>
</tr>
</tbody>
</table>

Again, we obtain the desired $4 - (v,k,1)$ designs from BIB designs below:

$2 - (156,4,810), 2 - (1029,51,9075)$

For $x = 10$, (3.4.2.2) is

$$\frac{186978048}{(\lambda_3 + 9)(\lambda_3 + 18)}$$

And $\lambda_3$ takes the following values 120 and 480. The corresponding values of $\lambda_1, \lambda_2, v, k$ and $b$ are given in the table below.
3.4.3. Case 3, \( x \neq y \), \( x > 1, y > 1 \)

In this case (3.4.4) is
\[
y\lambda_2 = x(\nu - 2), \quad \Rightarrow \nu = \frac{y\lambda_2 + 2x}{x}
\]
And
\[
y\lambda_3 = x(k - 2), \quad \Rightarrow k = \frac{y\lambda_2 + 2x}{x}
\]

From (3.4.1) we have now
\[
\lambda_3 = \frac{y\lambda_2 - x}{y\lambda_3 - x}
\]
\[
\lambda_2 = \frac{y\lambda_3^2 - x\lambda_3 + x}{y}
\]

Similarly from (3.4.3) we obtain
\[
\lambda_1 = \frac{y\lambda_2^2 + x\lambda_2}{y\lambda_3 + x}
\]
\[
\lambda_1 = \frac{y^2\lambda_3^4 - 2xy\lambda_3^3 + 3xy\lambda_3^2 + x^2\lambda_3 - 3x^2\lambda_3 + 2x^2}{y^2\lambda_3 + xy}
\]

Which implies
\[
\nu = \frac{y\lambda_3^2 - x\lambda_3 + 3x}{x}
\]
\[
k = \frac{y\lambda_3 + 2x}{x}
\]

For this design to be \( 4 - (\nu, k, 1) \) and from \( bk = \nu\lambda_1 \) it means
\[
\frac{y^2\lambda_3^4 - 2xy\lambda_3^3 + 3xy\lambda_3^2 + x^2\lambda_3 - 3x^2\lambda_3 + 2x^2)(y\lambda_3^2 - x\lambda_3 + 3x)}{(y^2\lambda_3 + xy)} \equiv 0 \mod(y\lambda_3 + 2x)
\]

That is
\[
\frac{(y^2\lambda_3^4 - 2xy\lambda_3^3 + 3xy\lambda_3^2 + x^2\lambda_3 - 3x^2\lambda_3 + 2x^2)(y\lambda_3^2 - x\lambda_3 + 3x)}{(y^2\lambda_3 + x)(y\lambda_3 + 2x)} \quad (3.4.3.1)
\]

Is a positive integer

Using similar argument as before (3.4.3.1) will be integer if
6x^3y^4 + 216x^5y^2 + 66x^4y^3 + 216x^6y
\frac{y^5(y\lambda_3 + x)(y\lambda_3 + 2x)}{y^5(y\lambda_3 + x)(y\lambda_3 + 2x)}

(3.4.3.2)

Is an integer
For \( x = 3 \) and \( y = 2 \), (3.4.3.2) becomes

17820
\frac{(2\lambda_3 + 3)(2\lambda_3 + 6)}{(2\lambda_3 + 3)(2\lambda_3 + 6)}

In this case there is no non-trivial \( 4 - (v, k, 1) \). We give corresponding values of \( \lambda_1, \lambda_2, v, k \) and \( b \) in the table below.

**Table 3.4.3.1.** Case 3; for \( x = 3 \) and \( y = 2 \) the possible cases of \( 4 - (v, k, 1) \) design

<table>
<thead>
<tr>
<th>( \lambda_3 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_1 )</th>
<th>( k )</th>
<th>( v )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>10</td>
<td>4</td>
<td>6</td>
<td>15</td>
</tr>
</tbody>
</table>

This \( 4 - (6,4,1) \) design is trivial. Hence, for \( x = 3 \) and \( y = 2 \) there is no nontrivial \( 4 - (v, k, 1) \) design

Also for \( x = 5 \) and \( y = 2 \) there is no non-trivial \( 4 - (v, k, 1) \), the following table gives the corresponding values of \( \lambda_1, \lambda_2, v, k \) and \( b \).

**Table 3.4.3.2.** Case 3; for \( x = 5 \) and \( y = 2 \) the possible cases of \( 4 - (v, k, 1) \) design

<table>
<thead>
<tr>
<th>( \lambda_3 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_1 )</th>
<th>( k )</th>
<th>( v )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>35</td>
<td>4</td>
<td>8</td>
<td>70</td>
</tr>
</tbody>
</table>

For \( x = 4 \) and \( y = 3 \) equation (3.4.3.2) becomes

\( \frac{21120}{(3\lambda_3 + 4)(3\lambda_3 + 8)} \)

In this case there is only one non-trivial \( 4 - (v, k, 1) \). The following table gives the corresponding values of \( \lambda_1, \lambda_2, v, k \) and \( b \).

**Table 3.4.3.3.** Case 3; for \( x = 4 \) and \( y = 3 \) the possible cases of \( 4 - (v, k, 1) \) design

<table>
<thead>
<tr>
<th>( \lambda_3 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_1 )</th>
<th>( k )</th>
<th>( v )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>12</td>
<td>30</td>
<td>5</td>
<td>11</td>
<td>66</td>
</tr>
</tbody>
</table>

For \( x = 6 \) and \( y = 3 \) equation (3.4.3.2) becomes

\( \frac{196560}{(3\lambda_3 + 6)(3\lambda_3 + 12)} \)

And the corresponding values of \( \lambda_1, \lambda_2, v, k \) and \( b \) are as shown below.
Table 3.4.3.4. Case 3; for \(x = 6\) and \(y = 3\) the possible cases of \(4 - (v,k,1)\) design

<table>
<thead>
<tr>
<th>(\lambda_3)</th>
<th>(\lambda_2)</th>
<th>(\lambda_1)</th>
<th>(k)</th>
<th>(v)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>10</td>
<td>20</td>
<td>4</td>
<td>7</td>
<td>35</td>
</tr>
<tr>
<td>6</td>
<td>26</td>
<td>91</td>
<td>5</td>
<td>15</td>
<td>273</td>
</tr>
<tr>
<td>8</td>
<td>50</td>
<td>260</td>
<td>6</td>
<td>27</td>
<td>1170</td>
</tr>
<tr>
<td>10</td>
<td>82</td>
<td>574</td>
<td>7</td>
<td>43</td>
<td>3526</td>
</tr>
<tr>
<td>22</td>
<td>442</td>
<td>8177</td>
<td>13</td>
<td>223</td>
<td>140267</td>
</tr>
</tbody>
</table>

For \(x = 7\) and \(y = 3\) equation (3.4.3.2) becomes

\[
\frac{466480}{(3\lambda_3 + 7)(3\lambda_3 + 14)}
\]

In this case there is only one non-trivial \(4 - (v,k,1)\). The following table gives the corresponding values of \(\lambda_1,\lambda_2, v, k\) and \(b\).

Table 3.4.3.5. Case 3; for \(x = 7\) and \(y = 3\) the possible cases of \(4 - (v,k,1)\) design

<table>
<thead>
<tr>
<th>(\lambda_3)</th>
<th>(\lambda_2)</th>
<th>(\lambda_1)</th>
<th>(k)</th>
<th>(v)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>35</td>
<td>140</td>
<td>5</td>
<td>17</td>
<td>476</td>
</tr>
</tbody>
</table>

For \(x = 8\) and \(y = 3\) equation (3.4.3.2) becomes

\[
\frac{992256}{(3\lambda_3 + 8)(3\lambda_3 + 16)}
\]

Also in this case, there is only one non-trivial \(4 - (v,k,1)\). The following table gives the corresponding values of \(\lambda_1,\lambda_2, v, k\) and \(b\).

Table 3.4.3.6. Case 3; for \(x = 8\) and \(y = 3\) the possible cases of \(4 - (v,k,1)\) design

<table>
<thead>
<tr>
<th>(\lambda_3)</th>
<th>(\lambda_2)</th>
<th>(\lambda_1)</th>
<th>(k)</th>
<th>(v)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>1496</td>
<td>52547</td>
<td>17</td>
<td>563</td>
<td>1740233</td>
</tr>
</tbody>
</table>

For \(x = 9\) and \(y = 3\) equation (3.4.3.2) becomes

\[
\frac{1939140}{(3\lambda_3 + 9)(3\lambda_3 + 18)}
\]

And the corresponding values of \(\lambda_1,\lambda_2, v, k\) and \(b\) are as shown below.

Table 3.4.3.7. Case 3; for \(x = 9\) and \(y = 3\) the possible cases of \(4 - (v,k,1)\) design

<table>
<thead>
<tr>
<th>(\lambda_3)</th>
<th>(\lambda_2)</th>
<th>(\lambda_1)</th>
<th>(k)</th>
<th>(v)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>21</td>
<td>56</td>
<td>4</td>
<td>9</td>
<td>126</td>
</tr>
<tr>
<td>9</td>
<td>57</td>
<td>285</td>
<td>5</td>
<td>21</td>
<td>1197</td>
</tr>
<tr>
<td>15</td>
<td>183</td>
<td>1891</td>
<td>7</td>
<td>63</td>
<td>17019</td>
</tr>
</tbody>
</table>
Conclusions

In note a new recursive technique has been developed for the construction of \(t-(v,k,\lambda_t)\) designs. It has also clearly shown that every \(t\) design is also a BIB \((v,k,\lambda)\) design. Therefore, this construction technique also generates BIBDs. Thus, the study has presented an alternative method that is simpler and unified for the construction of BIBDs that are very important in the experimental designs. As it provides designs for different values of \(k\), unlike many methods that provide designs for a single value of \(k\). Moreso, it provides both Steiner and non-Steiner designs.

Recommendation

Although this study has provided a technique for the construction of \(t\) designs, it is still a bit clear that construction method of \(t\) designs is somehow known in general. In fact, it is not clear how one might construct \(t\) designs with arbitrary block size. We therefore invite researchers to come up with “additive theorems “for this construction to make it general for any value of \(t\) as this may bring in new techniques and ideas. There is also need for obtaining a theorem which would give all values of \(x\) and \(y\) for the case three in this construction in order to see new Steiner \(t\) designs. Lastly, if there is a computer package that could be incorporated in the method to aid in calculations.

REFERENCES: