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ON THE CAUCHY PROPERTY IN METRIC SPACES

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ABSTRACT: In this note the concept of p-quasi-cone metric space is discussed. New results on points which are fixed in the metric space are given

INTRODUCTION

In this paper, we have introduced the concept of p-quasi-cone metric space for$p\geq 1$ which is a generalization of quasi- metric spaces where p=1. Also we have proved some new fixed point results in p-quasi-metric spaces using a comparison functionand the normality of cone which generalize the results of Raja and Vaezpour [1]. We give some basic notations in the section below

PRELIMINARIES

Definition 1. [3] Let E be a real Banach space and P be a subset of E. P is called a cone if and only if

(i) P is closed, $P\neq\emptyset$, $P\neq \{0\}$;

(ii) for all $x, y\in P \Rightarrow \alpha x + \beta y\in P$, where $\alpha, \beta\in R^{+}$;

(iii) $x \in P$and $-x \in P\Rightarrow x = 0$.

For a given cone $P\subset E$, we can define a partial ordering with respect to P by $x\leq y$ if and only If $y-x\varepsilon P$. The notation $x<y$ will stand for $x\leq y$ and $x\neq y$, while $x \geq y$ will stand for $y-x\varepsilon$ int P, where intPdenotes the interior of P. The cone Pis called normal if there is a number $k> 0$ such that $0 \leq x\leq y\Rightarrow ||x|| \leq k||y||$, for all$x, y\in E$. The least positive k satisfying this is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded above is convergent; that is if x_n is a sequence such that $x_1\leq x_1\leq x_2\leq\cdots\leq x_n\leq\cdots\leq y$, for some $y\varepsilon E$, then there is $x\varepsilon E$ such that $||x_n-x|| \to 0$ as$n\to\infty$ Equivalently, the cone P is regular if every sequence which is bounded below is convergent.

Definition2. Let X be a nonempty set. Suppose the mapping $d : X\times X\to E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y\in P$, and $d(x, y) = 0$ iff$x=y$;

(ii) $d(x, y) = d(y, x)$for all $x, y\in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z\in X$.

Then, d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Definition3. [2] Let X be a nonempty set. Suppose the mapping q: $X\times X\to E$ satisfies

(i) $0 \leq q(x, y)$ for all $x, y\in X$,

(ii) $q(x, y)$ =0 if and only if $x = y$;

(iii) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

Then q is called a *quasi-cone* metric on X, and (X, q) is called a *quasi-cone metric space*.
Now, we state our definition which is more general than quasi -cone metric space.

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Definition4. Let X be a nonempty set and p≥ 1. Suppose the mapping q_p $X \times X \to E$: satisfies

(i) $0 \leq q_p(x, y)$ for all $x, y, \in X$,

(ii) $q_p(x, y) = 0$ if and only if $x = y$,

(iii) $q_p(x, z) \leq p(q_p(x, y) + q_p(y, z))$ for all $x, y, z \in X$.

Then q_p is called a p-quasi-cone metric on X, and (X, q_p) is called a p-quasi-cone metricspace.

Example1. Let $X = (0, \infty)$, $E = R^2$, $P = \{(x, y), x, y \in R^+\}$ and $q_1 : X \times X \to E$ defined by

$$q_1(x, y) = \begin{cases} (x - y, \alpha(x - y)), x > y \\ (0, 0), x < y \end{cases}$$

where $\alpha \in R^+$.

Remark1. Note that any cone metric space is a p-quasi cone metric space.
Now we introduce the appropriate generalization in p-quasi-cone metric spaces by considering the established notions in quasi-metric spaces.

Definition 5. Let (X, q_p) be a p-quasi-cone metric space. A sequence $\{x_n\}$ in X is called

(i) p-bi Cauchy if for each $c \in$ int P, there is $n_0 \in N$ such that $q_p(x_n, x_m) < c$ for all $m, n \geq n_0$.

(ii) p-right (left) Cauchy if for each $c \in$ int P, there is $n_0 \in N$such that $q_p(x_n, x_m) < c$ ($q_p(x_m, x_n) < c$ resp.) for all $n \geq m \geq n_0$;

(iii) p-weakly right (left) Cauchy if for each $c \in$ int P, there is $n_0 \in N$ such that $q_p(x_n, x_{n_0}) < c$ ($q_p(x_{n_0}, x_n) < c$ resp.) for all $n \geq n_0$;

(iv) p-right (left) q_p-Cauchy if for each $c \in$ int P, there exist $x \in X$ and $n_0 \in N$ such that $q_p(x_n, x) < c$, ($q_p(x, x_n) < c$ resp.) for all $n \geq n_0$.

Remark1. These notions in p-quasi-cone metric space are related in this way:

(i) p-bi-Cauchy $\Rightarrow p$-right (left) Cauchy $\Rightarrow p$-weakly right (left) Cauchy $\Rightarrow p$-right (left) q_p-Cauchy

(ii) a sequence is p-bi-Cauchy if and only if it is both p- left and p-right Cauchy.

We use the notion of p-right Cauchy in this paper.

Definition6. Let (X, q_p) be a p-quasi-cone metric space. Let $\{x_n\}_{n \in N}$ be a sequence in X. We say that the sequence $\{x_n\}_{n \in N}$ *p-right converges* to $x \in X$ if $q_p(x, x_n) \to 0$. We denote this by $\lim_{n \to \infty} x_n$ or $x_n \to x$.

Definition7. A p-quasi-cone metric space (X, q_p) is called p-right complete if every p-right Cauchy sequence in X converges.

Definition 8. Let (X, q_p) be a p-quasi-cone metric space. A function T : $X \to X$ is called

(i) *continuous* if for any p-right convergent sequence $\{x_n\}_{n \in N}$ in X with $\lim_{n \to \infty} x_n$, the sequence $\{Tx_n\}_{n \in N}$ is right convergent and $\lim_{n \to \infty} Tx_n =$.

(ii) *Contractive* if there exist some $h \in [0, 1]$ such that $q_p(Tx, Ty) \leq h q_p(x, y), \forall x, y \in X$ and if h = 1, then T is *non-expansive*.

Definition9. Let $O(x) = \{x, Tx, T^2x, \ldots\}$ where $x \in X$. The set $O(x)$ is called *orbit* of x.

Definition10. Let $M \subseteq X$. $\delta(M) = \sup\{ q_p(x, y), x, y \in M\}$ is called *diameter* of M.

The orbit $O(x)$ is called bounded if there exist a $c \in P$, $\delta(O(x)) \leq c$.

Results

In this section, we prove some fixed point results in p-quasi-cone metric space. Firstly we have given a theorem which is a generalization of [1]due to p-quasi-cone metric space. In this theorem we don't use the normality of cone and we don't take the function T : $X\to X$continuous.

Definition11. A p-quasi-cone metric space $(X, q_p)$is Hausdorff if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods V_1,$V_2$of x_1, x_2 respectively, they are disjoint.

Definition12.[1] The function$\varphi : P\to P$ which satisfies the following conditions

1. $\forall t \in P, \varphi(t) < t$,
2. $\forall t_1, t_2 \in P, t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$,
3. $\lim_{n\to\infty} \|\varphi^n(t)\| = 0$, $t \in P$,

is called a comparison function.

Now we state a fixed point theorem using a comparison function.

Theorem1. Let (X, q_p) be a p-right complete Hausdorff p-quasi-cone metric space and let$T : X \to X$ be a function that satisfies the nonlinear contraction condition:

$$q_p(T(x),T(y)) \le \varphi\Big(\max\big\{q_p(x,y), q_p(T(x),x), q_p(T(y),y), q_p(T(x),y), q_p(x,T(y))\big\}\Big) \tag{1}$$

for all x, y $\in$X, where φ : P$\to$P is a comparison function.

Let $x_0\in X$ such that $O(x_0)$is bounded. Then T has a unique fixed point$x^*\in X$ and the sequence $\{T^nx_0\}_{n\in N}$is right convergent tox^*.

Proof. We have that the orbit $O(x_0)$is bounded. So $\delta(O(x_0)) \le c\in P$.We prove now that $q_p(T^{n+1}(x_0),T^n(x_0)) \le \varphi^n(c)$.

To prove this we use induction method.

For n=1,we have

$$\begin{aligned} q_p(T^2x_0,Tx_0) &= q_p(T(T(x_0)),T(x_0)) \\ &\le \varphi(\max\{q_p(Tx_0,x_0), q_p(T^2x_0,Tx_0), \\ &\quad q_p(Tx_0,x_0), q_p(T^2x_0,x_0), q_p(Tx_0,Tx_0)\}) \\ &= \varphi(\max\{q_p(Tx_0,x_0), q_p(T^2x_0,Tx_0), q_p(T^2x_0,x_0)\}). \end{aligned}$$

Due to $O(x_0)$ is bounded, we have that:

$\max\{q_p(Tx_0,x_0), q_p(T^2x_0,Tx_0), q_p(T^2x_0,x_0)\} \le c$.

So $q_p(T^2x_0,Tx_0) \le \varphi^1(c)$.

Suppose it is true for

$k < n$, $q_p(T^{k+1}x_0,T^kx_0) \le \varphi^k(c)$.

Let it prove for $n>k$.

$$\begin{aligned} q_p(T^{n+1}x_0,T^nx_0) &= q_p(T(T^nx_0),T(T^{n-1}x_0)) \\ &\le \varphi(\max\{q_p(T^nx_0,T^{n-1}x_0), q_p(T^{n+1}x_0,T^nx_0), \\ &\quad q_p(T^nx_0,T^{n-1}x_0), q_p(T^{n+1}x_0,T^{n-1}x_0), q_p(T^nx_0,T^nx_0)\}) \\ &= \varphi(\max\{q_p(T^nx_0,T^{n-1}x_0), q_p(T^{n+1}x_0,T^nx_0), q_p(T^{n+1}x_0,T^{n-1}x_0)\}). \end{aligned}$$

Case 1.

$$\max\{q_p(T^nx_0,T^{n-1}x_0),q_p(T^{n+1}x_0,T^nx_0),q_p(T^{n+1}x_0,T^{n-1}x_0)\} = q_p(T^nx_0,T^{n-1}x_0)$$

$$q_p(T^{n+1}x_0,T^nx_0)\le \varphi(q_p(T^nx_0,T^{n-1}x_0))\le \varphi(\varphi^{n-1}(c)=\varphi^n(c).$$

Case2.

$$\max\{q_p(T^nx_0,T^{n-1}x_0),q_p(T^{n+1}x_0,T^nx_0),q_p(T^{n+1}x_0,T^{n-1}x_0)\} = q_p(T^{n+1}x_0,T^nx_0)$$

$$q_p(T^{n+1}x_0,T^nx_0)\le \varphi(q_p(T^{n+1}x_0,T^nx_0))$$

which is absurd because $\varphi(t) < t,\ \forall\ t \in P$.

Case3.

$$\max\{q_p(T^nx_0,T^{n-1}x_0),q_p(T^{n+1}x_0,T^nx_0),q_p(T^{n+1}x_0,T^{n-1}x_0)\} = q_p(T^{n+1}x_0,T^{n-1}x_0)$$

$$\begin{aligned}q_p(T^{n+1}x_0,T^{n-1}x_0) &= q_p(T(T^nx_0),T(T^{n-2}x_0))\\ &\le \varphi(\max\{q_p(T^nx_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^nx_0),\\ &\quad q_p(T^{n-1}x_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^{n-2}x_0),q_p(T^nx_0,T^{n-1}x_0)\})\end{aligned}$$

Case3/1.

$$\max\{q_p(T^nx_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^nx_0), q_p(T^{n-1}x_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^{n-2}x_0),q_p(T^nx_0,T^{n-1}x_0)\} = q_p(T^{n+1}x_0,T^nx_0)$$

Case 3/2.

$$\max\{q_p(T^nx_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^nx_0), q_p(T^{n-1}x_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^{n-2}x_0),q_p(T^nx_0,T^{n-1}x_0)\} = q_p(T^{n-1}x_0,T^{n-2}x_0)$$

These two cases are trivial.

Case3/3.

$$\max\{q_p(T^nx_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^nx_0), q_p(T^{n-1}x_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^{n-2}x_0),q_p(T^nx_0,T^{n-1}x_0)\} = q_p(T^nx_0,T^{n-1}x_0)$$

$$q_p(T^nx_0,T^{n-1}x_0)\le \varphi^{n-1}(c) \Rightarrow$$

$$\begin{aligned}q_p(T^{n+1}x_0,T^nx_0) &\le \varphi(q_p(T^{n+1}x_0,T^{n-1}x_0))\\ &\le \varphi(\varphi(q_p(T^nx_0,T^{n-1}x_0)))\\ &= \varphi^{n+1}(c) = \varphi(\varphi^n(c)) < \varphi^n(c)\end{aligned}$$

Case3/4.

$$\max\{q_p(T^nx_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^nx_0), q_p(T^{n-1}x_0,T^{n-2}x_0),q_p(T^{n+1}x_0,T^{n-2}x_0),q_p(T^nx_0,T^{n-1}x_0)\} = q_p(T^nx_0,T^{n-2}x_0)$$

Case3/5.

$$\max\{q_p(T^n x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^n x_0),$$
$$q_p(T^{n-1}x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^{n-2}x_0), q_p(T^n x_0, T^{n-1}x_0)\}$$
$$= q_p(T^{n+1}x_0, T^{n-2}x_0)$$

The cases 3/4 and 3/5 can be proved in the same iterative manner.So it is true that for $n \geq 1 \Rightarrow q_p(T^{n+1}x_0, T^n x_0) \leq \varphi^n(c)$.

We have that $\lim_{n\to\infty} \|\varphi^n(c)\| =$

$$\Leftrightarrow (\forall \varepsilon > 0)(\exists n_0 \in N)(\forall n \in N, n > n_0 \Rightarrow \left\|\varphi^n(c)\right\| < \frac{\varepsilon}{2pK^2}),$$

where K is the normal constant of cone. So for $n > n_0$, we have:

$$\left\|q_p(T^{n+1}x_0, T^n x_0)\right\| \leq K\left\|\varphi^n(c)\right\| < K\frac{\varepsilon}{2pK^2} = \frac{\varepsilon}{2pK} \qquad (1)$$

$$\alpha + \beta = \chi. \qquad (1)$$

We show now that the sequence $\{T^n x_0\}_{n\in N}$is right Cauchy. For this we prove that

$\left\|q_p(T^n\right.$for every $k \in N$ and$n > n_0$ (*)

We prove it by induction method.

For k=2. Since

$$q_p(T^{n+2}x_0, T^n x_0) \leq p(q_p(T^{n+2}x_0, T^{n+1}x_0) + q_p(T^{n+1}x_0, T^n x_0))$$

we have

$$\left\|q_p(T^{n+2}x_0, T^n x_0)\right\| \leq pK(\left\|q_p(T^{n+2}x_0, T^{n+1}x_0)\right\| + \left\|q_p(T^{n+1}x_0, T^n x_0)\right\|)$$

We now that $q_p(T^{n+2}x_0, T^{n+1}x_0) \leq \varphi^{n+1}(c) \leq \varphi^n(c)$, so

$$\left\|q_p(T^{n+2}x_0, T^{n+1}x_0)\right\| < \frac{\varepsilon}{2pK}.$$

$$\begin{aligned}\left\|q_p(T^{n+2}x_0, T^n x_0)\right\| &\leq pK(\left\|q_p(T^{n+2}x_0, T^{n+1}x_0)\right\| + \left\|q_p(T^{n+1}x_0, T^n x_0)\right\|) \\ &< pK(\frac{\varepsilon}{2pK} + \frac{\varepsilon}{2pK}) = \varepsilon\end{aligned}$$

Suppose it is true for l<k. So $\left\|q_p(T^{n+l}x_0, T^n x_0)\right\| < \varepsilon$.

Let prove(*) for k

$$\begin{aligned}\left\|q_p(T^{n+k}x_0, T^n x_0)\right\| &\leq pK(\left\|q_p(T^{n+k}x_0, T^{n+k-1}x_0)\right\| + \left\|q_p(T^{n+k-1}x_0, T^n x_0)\right\|) \\ &\leq pK\left\|q_p(T^{n+k}x_0, T^{n+k-1}x_0)\right\| + pK^2\left\|\varphi^n(c)\right\| \\ &< pK\frac{\varepsilon}{2pK} + pK^2\frac{\varepsilon}{2pK^2} = \varepsilon\end{aligned}$$

So we proved that the sequence $\{T^n x_0\}_{n\in N}$is right Cauchy. We see that since by the space is complete and Hausdorff, we have $\lim_{n\to\infty} T^n x_0 =$.

Now we prove that $x^*$is a fixed point of$T : X\to X$.

$$\begin{aligned} q_p(Tx^*, T^n x_0) &= q_p(Tx^*, T(T^{n-1}x_0)) \\ &\leq \varphi(\max\{q_p(x^*, T^{n-1}x_0), q_p(Tx^*, x^*), \\ &\quad q_p(T^n x_0, T^{n-1}x_0), q_p(Tx^*, T^{n-1}x_0), q_p(x^*, T^n x_0)\}) \end{aligned}$$

Taking the limit of both sides when$n\to\infty$,we have

$$q_p(Tx^*, x^*) \leq \varphi(q_p(Tx^*, x^*)) < q_p(Tx^*, x^*),$$

So $\quad q_p(Tx^*, x^*) = 0 \Rightarrow Tx^* = x^*$.

Now we prove the uniqueness of the fixed point for T. Suppose there is another point y^* that$Ty^* = y^*$.

$q_p(x^*, y^*) = q_p(Tx^*, Ty^*) \leq \varphi(q_p(x^*, y^*)) < q_p(x^*, y^*)$.

So we have that

$$q_p(x^*, y^*) = 0 \Rightarrow x^* = y^*.$$

Example2. Let$X = [0, 1]$, $E = R^2$, $P = \{(x, y) \in E, x, y \geq 0\}$.

Define $q(x,y) = \begin{cases} (0,0), & x \geq y \\ (\frac{1}{2}y, y), & x \leq y \end{cases}$.

We take the function

$$T : X \to X, Tx = \begin{cases} \frac{x}{8}, & x \in [0, \frac{1}{2}] \\ \frac{1}{10}, & x \in (\frac{1}{2}, 1] \end{cases}$$

and

$$\varphi : P \to P, \varphi(x,y) = (\frac{x}{3}, \frac{y}{3}).$$

Case 1. For every$x, y \in$ $[($, $x\leq y$we have

$$q(Tx, Ty) = q(\frac{x}{8}, \frac{y}{8}) = (\frac{y}{16}, \frac{y}{8}).$$

$$\max\{q(x,y), q(\frac{x}{8}, x), q(\frac{y}{8}, y), q(\frac{x}{8}, y), q(x, \frac{y}{8})\} = (\frac{y}{2}, y) \Rightarrow \begin{aligned} &\varphi(\max\{q(x,y), q(\frac{x}{8}, x), q(\frac{y}{8}, y), q(\frac{x}{8}, y), q(x, \frac{y}{8})\}) \\ &= \varphi(\frac{y}{2}, y) = (\frac{y}{6}, \frac{y}{3}) \end{aligned}$$

So $\quad q(Tx, Ty) = (\frac{y}{16}, \frac{y}{8}) < (\frac{y}{6}, \frac{y}{3})$

So we are in conditions of theorem.

For every$x, y \in$ $[($, $x\geq y \Rightarrow q_p(Tx, Ty) = 0$, so the theorem is true.

Case2.

For every $x, y \in$ $(\frac{1}{2}$, $x \leq y$ we have

$$q(Tx,Ty) = q(\frac{1}{10},\frac{1}{10}) = (\frac{1}{20},\frac{1}{10}).$$

$$\max\{q(x,y), q(\frac{1}{10},x), q(\frac{1}{10},y), q(\frac{1}{10},y), q(x,\frac{1}{10})\} = (\frac{y}{2},y) \Rightarrow q(Tx,Ty) = (\frac{1}{20},\frac{1}{10}) \leq \varphi(\frac{y}{2},y) = (\frac{y}{6},\frac{y}{3}).$$

So we are in conditions of theorem.

For every $x, y \in$ $(\frac{1}{2}$, $x \geq y \Rightarrow q_p(Tx, Ty) = 0$, so the theorem is true.

Case3.

For every $x \in$ $[$ and $y \in$ $(\frac{1}{2}$. It is clear that the conditions of theorem true in this case.

So, the function $T : X \to X$,

$$Tx = \begin{cases} \frac{x}{4}, & x \in [0,\frac{1}{2}] \\ \frac{1}{10}, & x \in (\frac{1}{2},1] \end{cases}$$

has a fixed point $x = 0$.

This result is a generalization offollowing resultRaja and Vaezpour[1] in cone metric space because our quasi- contraction is better.

Theorem1'. Let (X, qp) be a p-right complete Hausdorff p-quasi-cone metric space and let T : $X \to X$ be a function that satisfies the nonlinear contraction condition:

(2) $q_p(T(x), T(y)) \leq \varphi(q_p(x, y))$ for all $x, y \in X$,

where $\varphi : P \to P$ is a comparison function. Let $x0 \in X$ such that $O(x0)$ is bounded. Then T has a unique fixed point $x^* \in X$ and the sequence $\{Tnx0\}n \in N$ is right convergent to x^*.
The following result is a generalization Theorem of Ciric[2] in metric space.

Theorem2. Let (X, qp) be a p-right complete, Hausdorff, p-quasi-cone metric space and let $T : X \to X$ be a function that satisfies the nonlinear contraction condition:

$$q_p(T(x),T(y)) \leq h\max\{q_p(x,y), q_p(T(x),x), q_p(T(y),y), q_p(T(x),y), q_p(x,T(y))\}$$

for all $x, y \in X$, where $h \in 0$. Let $x0 \in X$ such that $O(x0)$ is bounded. Then T has a unique fixed point $x^* \in X$ and the sequence $\{Tnx0\}n \in N$ is right convergent to x^*.

Proof. If we take $\varphi(t) = ht$ for $t \in P$, we are in condition of Theorem1. So Theorem 2 follows and this completes the proof.

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