Advanced Trends in Mathematics
ISSN: 2394-532X, Vol. 1, pp 33-40
doi:10.18052/www.scipress.com/ATMath.1.33

Online: 2014-12-01

ON THE CAUCHY PROPERTY IN METRIC SPACES

G. T. RAWAL and N. B. OKELO

School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga,
University of Science and Technology, P. O. Box 210-40601, Bondo-Kenya

ABSTRACT: In this note the concept of p-quasi-cone metric space is discussed. New results on points which are fixed in the metric space are given

INTRODUCTION

In this paper, we have introduced the concept of p-quasi-cone metric space for$p\geq 1$ which is a generalization of quasi- metric spaces where $p=1$. Also we have proved some new fixed point results in p-quasi-metric spaces using a comparison functionand the normality of cone which generalize the results of Raja and Vaezpour [1]. We give some basic notations in the section below

PRELIMINARIES

Definition 1. [3] Let E be a real Banach space and P be a subset of E. P is called a cone if and only if

(i) P is closed, $P\neq\emptyset$, $P\neq\{0\}$;

(ii) for all $x, y\in P \Rightarrow \alpha x+\beta y\in P$, where $\alpha, \beta\in R^{+}$;

(iii) $x\in P$and $-x\in P\Rightarrow x=0$.

For a given cone $P\subset E$, we can define a partial ordering with respect to P by $x\leq y$ if and only If $y-x\varepsilon P$. The notation $x<y$ will stand for $x\leq y$ and $x\neq y$, while $x\geq y$ will stand for $y-x\varepsilon$ int P, where intPdenotes the interior of P. The cone Pis called normal if there is a number $k>0$ such that $0\leq x\leq y\Rightarrow\|x\|\leq k\|y\|$, for all$x, y\in E$. The least positive k satisfying this is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded above is convergent; that is if x_n is a sequence such that $x_1\leq x_1\leq x_2\leq\cdots\leq x_n\leq\cdots\leq y$, for some $y\varepsilon E$, then there is $x\varepsilon E$ such that $\|x_n-x\|\rightarrow 0$ as$n\rightarrow\infty$ Equivalently, the cone P is regular if every sequence which is bounded below is convergent.

Definition2. Let X be a nonempty set. Suppose the mapping $d: X\times X\rightarrow E$ satisfies

(i) $0\leq d(x, y)$ for all $x, y\in P$, and $d(x, y)=0$ iff$x=y$;

(ii) $d(x, y)=d(y, x)$for all $x, y\in X$;

(iii) $d(x, y)\leq d(x, z)+d(z, y)$, for all $x, y, z\in X$.

Then, d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Definition3. [2] Let X be a nonempty set. Suppose the mapping q: $X\times X\rightarrow E$ satisfies

(i) $0\leq q(x, y)$ for all $x, y\in X$,

(ii) $q(x, y)=0$ if and only if $x=y$;

(iii) $q(x, y)\leq q(x, z)+q(z, y)$ for all $x, y, z\in X$.

Then q is called a *quasi-cone* metric on X, and (X, q) is called a *quasi-cone metric space*.
Now, we state our definition which is more general than quasi -cone metric space.

Definition4. Let X be a nonempty set and p≥ 1. Suppose the mapping $q_p\ X\times X\to E$: satisfies

(i) $0 \leq q_p(x, y)$ for all $x, y, \in X$,

(ii) $q_p(x, y) = 0$ if and only if $x = y$,

(iii) $q_p(x, z) \leq p(q_p(x, y) + q_p(y, z))$ for all $x, y, z \in X$.

Then q_p is called a p-quasi-cone metric on X, and (X, q_p) is called a p-quasi-cone metricspace.

Example1. Let $X = (0, \infty)$, $E = R^2$, $P = \{(x, y), x, y \in R^+\}$ and $q_1 : X\times X\to E$ defined by

$$q_1(x,y) = \begin{cases} (x-y, \alpha(x-y)), x > y \\ (0,0), x < y \end{cases}$$

where $\alpha \in R^+$.

Remark1. Note that any cone metric space is a p-quasi cone metric space.
Now we introduce the appropriate generalization in p-quasi-cone metric spaces by considering the established notions in quasi-metric spaces.

Definition 5. Let (X, q_p) be a p-quasi-cone metric space. A sequence $\{x_n\}$ in X is called

(i) *p*-bi Cauchy if for each $c\in$ int P, there is $n_0\in N$ such that $q_p(x_n, x_m) < c$ for all $m, n \geq n_0$.

(ii) *p*-right (left) Cauchy if for each $c\in$ int P, there is $n_0\in N$such that $q_p(x_n, x_m) < c$ ($q_p(x_m, x_n) < c$ resp.) for all $n \geq m \geq n_0$;

(iii) *p*-weakly right (left) Cauchy if for each $c\in$ int P, there is $n_0\in N$ such that $q_p(x_n, x_{n0}) < c$ ($q_p(x_{n0}, x_n) < c$ resp.) for all $n \geq n_0$;

(iv) *p*-right (left) q_p-Cauchy if for each $c\in$ int P, there exist $x\in X$ and $n_0\in N$ such that $q_p(x_n, x) < c$, ($q_p(x, x_n) < c$ resp.) for all $n \geq n_0$.

Remark1. These notions in p-quasi-cone metric space are related in this way:

(i) *p*-bi-Cauchy ⇒*p*-right (left) Cauchy ⇒*p*-weakly right (left) Cauchy ⇒*p*-right (left) q_p-Cauchy

(ii) a sequence is *p*-bi-Cauchy if and only if it is both *p*- left and *p*-right Cauchy.

We use the notion of *p*-right Cauchy in this paper.

Definition6. Let (X, q_p) be a p-quasi-cone metric space. Let $\{x_n\}_{n\varepsilon N}$ be a sequence in X. We say that the sequence $\{x_n\}_{n\in N}$ *p-right converges* to $x\in X$ if $q_p(x, x_n)\to 0$. We denote this by $\lim_{n\to\infty} x_n$ or $x_n\to x$.

Definition7. A p-quasi-cone metric space (X, q_p) is called p-right complete if every p-right Cauchy sequence in X converges.

Definition 8. Let (X, q_p) be a p-quasi-cone metric space. A function T : $X\to X$ is called

(i) *continuous* if for any *p*-right convergent sequence $\{x_n\}_{n\in N}$ in X with $\lim_{n\to\infty} x_n$, the sequence $\{Tx_n\}_{n\in N}$ is right convergent and $\lim_{n\to\infty} Tx_n =$.

(ii) *Contractive* if there exist some $h\in [0, 1]$ such that $q_p(Tx, Ty) \leq h q_p(x, y), \forall x, y \in X$ and if h = 1, then T is *non-expansive*.

Definition9. Let $O(x) = \{x, Tx, T^2x, \ldots\}$ where $x\in X$. The set $O(x)$ is called *orbit* of x.

Definition10. Let $M\subseteq X$. $\delta(M) = \sup\{ q_p(x, y), x, y\in M\}$ is called *diameter* of M.

The orbit $O(x)$ is called bounded if there exist a $c\in P$, $\delta(O(x)) \leq c$.

Results

In this section, we prove some fixed point results in p-quasi-cone metric space. Firstly we have given a theorem which is a generalization of [1]due to p-quasi-cone metric space. In this theorem we don't use the normality of cone and we don't take the function T : $X \to X$continuous.

Definition11. A p-quasi-cone metric space (X, qp)is Hausdorff if for each pair $x1$, $x2$ of distinct points of X, there exist neighborhoods V_1,$V_2$of x_1, x_2 respectively, they are disjoint.

Definition12.[1] The functionφ : $P \to P$ which satisfies the following conditions

1. $\forall t \in P, \varphi(t) < t$,
2. $\forall t_1, t_2 \in P, t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$,
3. $\lim_{n\to\infty} \|\varphi^n(t)\| = 0$, $t \in P$,

is called a comparison function.

Now we state a fixed point theorem using a comparison function.

Theorem1. Let (X, qp) be a p-right complete Hausdorff p-quasi-cone metric space and let$T : X \to X$ be a function that satisfies the nonlinear contraction condition:

$$q_p(T(x),T(y)) \le \varphi\Big(\max\{q_p(x,y), q_p(T(x),x), q_p(T(y),y), q_p(T(x),y), q_p(x,T(y)\}\Big) \tag{1}$$

for all x, y $\in$X, where φ : P→P is a comparison function.

Let $x0 \in X$ such that $O(x0)$is bounded. Then T has a unique fixed point$x^* \in X$ and the sequence $\{Tnx0\}n \in N$is right convergent tox^*.

Proof. We have that the orbit $O(x0)$is bounded. So $\delta(O(x0)) \le c \in P$.We prove now that

$q_p(T^{n+1}(x_0), T^n(x_0)) \le \varphi^n(c)$.

To prove this we use induction method.

For n=1,we have

$$\begin{aligned} q_p(T^2x_0, Tx_0) &= q_p(T(T(x_0)), T(x_0)) \\ &\le \varphi(\max\{q_p(Tx_0, x_0), q_p(T^2x_0, Tx_0), \\ &\quad q_p(Tx_0, x_0), q_p(T^2x_0, x_0), q_p(Tx_0, Tx_0)\}) \\ &= \varphi(\max\{q_p(Tx_0, x_0), q_p(T^2x_0, Tx_0), q_p(T^2x_0, x_0)\}). \end{aligned}$$

Due to $O(x_0)$ is bounded, we have that:

$\max\{q_p(Tx_0, x_0), q_p(T^2x_0, Tx_0), q_p(T^2x_0, x_0)\} \le c$.

So $q_p(T^2x_0, Tx_0) \le \varphi^1(c)$.

Suppose it is true for

$k < n$, $q_p(T^{k+1}x_0, T^kx_0) \le \varphi^k(c)$.

Let it prove for $n>k$.

$$\begin{aligned} q_p(T^{n+1}x_0, T^nx_0) &= q_p(T(T^nx_0), T(T^{n-1}x_0)) \\ &\le \varphi(\max\{q_p(T^nx_0, T^{n-1}x_0), q_p(T^{n+1}x_0, T^nx_0), \\ &\quad q_p(T^nx_0, T^{n-1}x_0), q_p(T^{n+1}x_0, T^{n-1}x_0), q_p(T^nx_0, T^nx_0)\}) \\ &= \varphi(\max\{q_p(T^nx_0, T^{n-1}x_0), q_p(T^{n+1}x_0, T^nx_0), q_p(T^{n+1}x_0, T^{n-1}x_0)\}). \end{aligned}$$

Case 1.

$$\max\{q_p(T^n x_0, T^{n-1}x_0), q_p(T^{n+1}x_0, T^n x_0), q_p(T^{n+1}x_0, T^{n-1}x_0)\}$$
$$= q_p(T^n x_0, T^{n-1}x_0)$$

$$q_p(T^{n+1}x_0, T^n x_0) \le \varphi(q_p(T^n x_0, T^{n-1}x_0)) \le \varphi(\varphi^{n-1}(c) = \varphi^n(c).$$

Case2.

$$\max\{q_p(T^n x_0, T^{n-1}x_0), q_p(T^{n+1}x_0, T^n x_0), q_p(T^{n+1}x_0, T^{n-1}x_0)\}$$
$$= q_p(T^{n+1}x_0, T^n x_0)$$
$$q_p(T^{n+1}x_0, T^n x_0) \le \varphi(q_p(T^{n+1}x_0, T^n x_0))$$

which is absurd because $\varphi(t) < t, \ \forall\ t \in P$.

Case3.

$$\max\{q_p(T^n x_0, T^{n-1}x_0), q_p(T^{n+1}x_0, T^n x_0), q_p(T^{n+1}x_0, T^{n-1}x_0)\}$$
$$= q_p(T^{n+1}x_0, T^{n-1}x_0)$$

$$q_p(T^{n+1}x_0, T^{n-1}x_0) = q_p(T(T^n x_0), T(T^{n-2}x_0))$$
$$\le \varphi(\max\{q_p(T^n x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^n x_0),$$
$$q_p(T^{n-1}x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^{n-2}x_0), q_p(T^n x_0, T^{n-1}x_0)\})$$

Case3/1.

$$\max\{q_p(T^n x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^n x_0),$$
$$q_p(T^{n-1}x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^{n-2}x_0), q_p(T^n x_0, T^{n-1}x_0)\}$$
$$= q_p(T^{n+1}x_0, T^n x_0)$$

Case 3/2.

$$\max\{q_p(T^n x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^n x_0),$$
$$q_p(T^{n-1}x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^{n-2}x_0), q_p(T^n x_0, T^{n-1}x_0)\}$$
$$= q_p(T^{n-1}x_0, T^{n-2}x_0)$$

These two cases are trivial.

Case3/3.

$$\max\{q_p(T^n x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^n x_0),$$
$$q_p(T^{n-1}x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^{n-2}x_0), q_p(T^n x_0, T^{n-1}x_0)\}$$
$$= q_p(T^n x_0, T^{n-1}x_0)$$
$$q_p(T^n x_0, T^{n-1}x_0) \le \varphi^{n-1}(c) \Rightarrow$$
$$q_p(T^{n+1}x_0, T^n x_0) \le \varphi(q_p(T^{n+1}x_0, T^{n-1}x_0))$$
$$\le \varphi(\varphi(q_p(T^n x_0, T^{n-1}x_0)))$$
$$= \varphi^{n+1}(c) = \varphi(\varphi^n(c)) < \varphi^n(c)$$

Case3/4.

$$\max\{q_p(T^n x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^n x_0),$$
$$q_p(T^{n-1}x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^{n-2}x_0), q_p(T^n x_0, T^{n-1}x_0)\}$$
$$= q_p(T^n x_0, T^{n-2}x_0)$$

Case3/5.

$$\max\{q_p(T^n x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^n x_0),$$
$$q_p(T^{n-1}x_0, T^{n-2}x_0), q_p(T^{n+1}x_0, T^{n-2}x_0), q_p(T^n x_0, T^{n-1}x_0)\}$$
$$= q_p(T^{n+1}x_0, T^{n-2}x_0)$$

The cases 3/4 and 3/5 can be proved in the same iterative manner.So it is true that for $n \geq 1 \Rightarrow q_p(T^{n+1}x_0, T^n x_0) \leq \varphi^n(c)$.

We have that $\lim_{n\to\infty} \|\varphi^n(c)\| =$

$$\Leftrightarrow (\forall \varepsilon > 0)(\exists n_0 \in N)(\forall n \in N, n > n_0 \Rightarrow \left\|\varphi^n(c)\right\| < \frac{\varepsilon}{2pK^2}),$$

where K is the normal constant of cone. So for $n > n_0$, we have:

$$\left\|q_p(T^{n+1}x_0, T^n x_0)\right\| \leq K\left\|\varphi^n(c)\right\| < K\frac{\varepsilon}{2pK^2} = \frac{\varepsilon}{2pK} \qquad (1)$$

$$\alpha + \beta = \chi. \qquad (1)$$

We show now that the sequence $\{T^n x_0\}_{n\in N}$is right Cauchy. For this we prove that

$\|q_p(T^n$for every $k \in N$ and$n > n_0$ (*)

We prove it by induction method.

For k=2. Since

$$q_p(T^{n+2}x_0, T^n x_0) \leq p(q_p(T^{n+2}x_0, T^{n+1}x_0) + q_p(T^{n+1}x_0, T^n x_0))$$

we have

$$\left\|q_p(T^{n+2}x_0, T^n x_0)\right\| \leq pK(\left\|q_p(T^{n+2}x_0, T^{n+1}x_0)\right\| + \left\|q_p(T^{n+1}x_0, T^n x_0)\right\|)$$

We now that $q_p(T^{n+2}x_0, T^{n+1}x_0) \leq \varphi^{n+1}(c) \leq \varphi^n(c)$, so

$$\left\|q_p(T^{n+2}x_0, T^{n+1}x_0)\right\| < \frac{\varepsilon}{2pK}.$$

$$\begin{aligned}\left\|q_p(T^{n+2}x_0, T^n x_0)\right\| &\leq pK(\left\|q_p(T^{n+2}x_0, T^{n+1}x_0)\right\| + \left\|q_p(T^{n+1}x_0, T^n x_0)\right\|)\\ &< pK(\frac{\varepsilon}{2pK} + \frac{\varepsilon}{2pK}) = \varepsilon\end{aligned}$$

Suppose it is true for $l<k$. So $\left\|q_p(T^{n+l}x_0, T^n x_0)\right\| < \varepsilon$.

Let prove(*) for k

$$\begin{aligned}\left\|q_p(T^{n+k}x_0, T^n x_0)\right\| &\leq pK(\left\|q_p(T^{n+k}x_0, T^{n+k-1}x_0)\right\| + \left\|q_p(T^{n+k-1}x_0, T^n x_0)\right\|)\\ &\leq pK\left\|q_p(T^{n+k}x_0, T^{n+k-1}x_0)\right\| + pK^2\left\|\varphi^n(c)\right\|\\ &< pK\frac{\varepsilon}{2pK} + pK^2\frac{\varepsilon}{2pK^2} = \varepsilon\end{aligned}$$

So we proved that the sequence $\{T^n x_0\}_{n\in N}$is right Cauchy. We see that since by the space is complete and Hausdorff, we have $\lim_{n\to\infty} T^n x_0 =$.

Now we prove that $x^*$is a fixed point of$T : X\to X$.

$$\begin{aligned} q_p(Tx^*, T^n x_0) &= q_p(Tx^*, T(T^{n-1}x_0)) \\ &\le \varphi(\max\{q_p(x^*, T^{n-1}x_0), q_p(Tx^*, x^*), \\ &\quad q_p(T^n x_0, T^{n-1}x_0), q_p(Tx^*, T^{n-1}x_0), q_p(x^*, T^n x_0)\}) \end{aligned}$$

Taking the limit of both sides when$n\to\infty$,we have

$$q_p(Tx^*, x^*) \le \varphi(q_p(Tx^*, x^*)) < q_p(Tx^*, x^*),$$

So $q_p(Tx^*, x^*) = 0 \Rightarrow Tx^* = x^*$.

Now we prove the uniqueness of the fixed point for T. Suppose there is another point y^* that$Ty^* = y^*$.

$q_p(x^*, y^*) = q_p(Tx^*, Ty^*) \le \varphi(q_p(x^*, y^*)) < q_p(x^*, y^*)$.

So we have that

$$q_p(x^*, y^*) = 0 \Rightarrow x^* = y^*.$$

Example2. Let$X = [0, 1]$, $E = R^2$, $P = \{(x, y) \in E, x, y \ge 0\}$.

Define $q(x,y) = \begin{cases} (0,0), & x \ge y \\ (\frac{1}{2}y, y), & x \le y \end{cases}$.

We take the function

$$T : X \to X, Tx = \begin{cases} \frac{x}{8}, & x \in [0, \frac{1}{2}] \\ \frac{1}{10}, & x \in (\frac{1}{2}, 1] \end{cases}$$

and

$$\varphi : P \to P, \varphi(x,y) = (\frac{x}{3}, \frac{y}{3}).$$

Case 1. For every$x, y \in$ [(, $x\le y$we have

$$q(Tx,Ty) = q(\frac{x}{8}, \frac{y}{8}) = (\frac{y}{16}, \frac{y}{8}).$$

$$\max\{q(x,y), q(\frac{x}{8}, x), q(\frac{y}{8}, y), q(\frac{x}{8}, y), q(x, \frac{y}{8})\} = (\frac{y}{2}, y) \Rightarrow \begin{aligned} &\varphi(\max\{q(x,y), q(\frac{x}{8}, x), q(\frac{y}{8}, y), q(\frac{x}{8}, y), q(x, \frac{y}{8})\}) \\ &= \varphi(\frac{y}{2}, y) = (\frac{y}{6}, \frac{y}{3}) \end{aligned}$$

So $q(Tx,Ty) = (\frac{y}{16}, \frac{y}{8}) < (\frac{y}{6}, \frac{y}{3})$

So we are in conditions of theorem.

For every$x, y \in$ [(, $x\ge y \Rightarrow q_p(Tx, Ty) = 0$, so the theorem is true.

Case2.

For every $x, y\in$ $\left(\frac{1}{2}, x\leq y\right.$ we have

$$q(Tx,Ty)=q(\frac{1}{10},\frac{1}{10})=(\frac{1}{20},\frac{1}{10}).$$

$$\max\{q(x,y),q(\frac{1}{10},x),q(\frac{1}{10},y),q(\frac{1}{10},y),q(x,\frac{1}{10})\}=(\frac{y}{2},y)\Rightarrow q(Tx,Ty)=(\frac{1}{20},\frac{1}{10})\leq\varphi(\frac{y}{2},y)=(\frac{y}{6},\frac{y}{3}).$$

So we are in conditions of theorem.

For every $x, y\in$ $\left(\frac{1}{2}, x\geq y\Rightarrow q_p(Tx, Ty) = 0\right.$, so the theorem is true.

Case3.

For every $x\in$ $[$ and $y\in$ $\left(\frac{1}{2}\right.$. It is clear that the conditions of theorem true in this case.

So, the function $T : X\rightarrow X$,

$$Tx=\begin{cases}\frac{x}{4}, & x\in[0,\frac{1}{2}]\\ \frac{1}{10}, & x\in(\frac{1}{2},1]\end{cases}$$

has a fixed point $x = 0$.

This result is a generalization of following result Raja and Vaezpour[1] in cone metric space because our quasi- contraction is better.

Theorem1'. Let (X, qp) be a p-right complete Hausdorff p-quasi-cone metric space and let $T : X\rightarrow X$ be a function that satisfies the nonlinear contraction condition:

(2) $q_p(T(x), T(y)) \leq\varphi(q_p(x, y))$ for all $x, y\in X$,

where $\varphi : P\rightarrow P$ is a comparison function. Let $x0\in X$ such that $O(x0)$ is bounded. Then T has a unique fixed point $x^*\in X$ and the sequence $\{Tnx0\}n\in N$ is right convergent to x^*.
The following result is a generalization Theorem of Ciric[2] in metric space.

Theorem2. Let (X, qp) be a p-right complete, Hausdorff, p-quasi-cone metric space and let $T : X\rightarrow X$ be a function that satisfies the nonlinear contraction condition:

$$q_p(T(x),T(y))\leq h\max\{q_p(x,y),q_p(T(x),x),\\ q_p(T(y),y),q_p(T(x),y),q_p(x,T(y))\}$$

for all $x, y\in X$, where $h\in 0$. Let $x0\in X$ such that $O(x0)$ is bounded. Then T has a unique fixed point $x^*\in X$ and the sequence $\{Tnx0\}n\in N$ is right convergent to x^*.

Proof. If we take $\varphi(t) = ht$ for $t\in P$, we are in condition of Theorem1. So Theorem 2 follows and this completes the proof.

References

[1] P. Raja and S. M. Vaezpour, Hindawi Publishing Corporation, Fixed Point Theory and Applications, Volume 2008, Article ID 768294, 11 pages

[2] Lj. B. Ciric, A Generalization of Banach's Contraction Principle, Proceedings of the American Mathematical Society, Volume 45, Number 2, August 1974.

[3] L.-G. Haung, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings,. Math.Anal. Appl. 332 (2007) 1468–1476.

[4] F. Shaddad and M. S. M. Noorani,Fixed Point Results in Quasi-Cone Metric Spaces,Hindawi Publishing Corporation, Abstract and Applied Analysis,Volume 2013, Article ID 303626, 7 pages.

[5] Sh. Rezapour and R. Hamlbarani, “Some notes on the Cone metric spaces and fixed point theorems of contractivemappings,” Journal of Mathematical Analysis and Applications,vol. 345, no. 2, pp. 719–724, 2008.

[6] T. Abdeljawad, E. Karapinar, Quasicone metric spaces and generalitions ofCaristi Kirks theorem. Fixed Point Theory Appl., (2009), 9 pages.

[7] A. Sonmez, “Fixed point theorems in partial cone metric spaces,” http://arxiv.org/abs/1101.2741.