ON THE CAUCHY PROPERTY IN METRIC SPACES

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ABSTRACT: In this note the concept of p -quasi-cone metric space is discussed. New results on points which are fixed in the metric space are given

INTRODUCTION

In this paper, we have introduced the concept of p -quasi-cone metric space for $p \geq 1$ which is a generalization of quasi-metric spaces where $p=1$. Also we have proved some new fixed point results in p -quasi-metric spaces using a comparison function and the normality of cone which generalize the results of Raja and Vaezpour [1]. We give some basic notations in the section below

PRELIMINARIES

Definition 1. [3] Let E be a real Banach space and P be a subset of E . P is called a cone if and only if

- (i) P is closed, $P \neq \emptyset$, $P \neq \{0\}$;
- (ii) for all $x, y \in P \Rightarrow \alpha x + \beta y \in P$, where $\alpha, \beta \in \mathbb{R}^+$;
- (iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we can define a partial ordering with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \geq y$ will stand for $y - x \in \text{int } P$, where $\text{int } P$ denotes the interior of P . The cone P is called normal if there is a number $k > 0$ such that $0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$, for all $x, y \in E$. The least positive k satisfying this is called the normal constant of P . The cone P is called regular if every increasing sequence which is bounded above is convergent; that is if x_n is a sequence such that $x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n \leq \dots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Equivalently, the cone P is regular if every sequence which is bounded below is convergent.

Definition 2. Let X be a nonempty set. Suppose the mapping $d : X \times X \rightarrow E$ satisfies

- (i) $0 \leq d(x, y)$ for all $x, y \in P$, and $d(x, y) = 0$ iff $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then, d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Definition 3. [2] Let X be a nonempty set. Suppose the mapping $q : X \times X \rightarrow E$ satisfies

- (i) $0 \leq q(x, y)$ for all $x, y \in X$,
- (ii) $q(x, y) = 0$ if and only if $x = y$;
- (iii) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

Then q is called a *quasi-cone metric* on X , and (X, q) is called a *quasi-cone metric space*. Now, we state our definition which is more general than quasi-cone metric space.

Definition4. Let X be a nonempty set and $p \geq 1$. Suppose the mapping $q_p : X \times X \rightarrow E$: satisfies

- (i) $0 \leq q_p(x, y)$ for all $x, y \in X$,
- (ii) $q_p(x, y) = 0$ if and only if $x = y$,
- (iii) $q_p(x, z) \leq p(q_p(x, y) + q_p(y, z))$ for all $x, y, z \in X$.

Then q_p is called a p -quasi-cone metric on X , and (X, q_p) is called a p -quasi-cone metricspace.

Example1. Let $X = (0, \infty)$, $E = R^2$, $P = \{(x, y), x, y \in R^+\}$ and $q_1 : X \times X \rightarrow E$ defined by

$$q_1(x, y) = \begin{cases} (x - y, \alpha(x - y)), & x > y \\ (0, 0), & x < y \end{cases}$$

where $\alpha \in R^+$.

Remark1. Note that any cone metric space is a p -quasi cone metric space.

Now we introduce the appropriate generalization in p -quasi-cone metric spaces by considering the established notions in quasi-metric spaces.

Definition 5. Let (X, q_p) be a p -quasi-cone metric space. A sequence $\{x_n\}$ in X is called

- (i) p -bi Cauchy if for each $c \in \text{int } P$, there is $n_0 \in N$ such that $q_p(x_n, x_m) < c$ for all $m, n \geq n_0$.
- (ii) p -right (left) Cauchy if for each $c \in \text{int } P$, there is $n_0 \in N$ such that $q_p(x_n, x_m) < c$ ($q_p(x_m, x_n) < c$ resp.) for all $n \geq m \geq n_0$;
- (iii) p -weakly right (left) Cauchy if for each $c \in \text{int } P$, there is $n_0 \in N$ such that $q_p(x_n, x_{n_0}) < c$ ($q_p(x_{n_0}, x_n) < c$ resp.) for all $n \geq n_0$;
- (iv) p -right (left) q_p -Cauchy if for each $c \in \text{int } P$, there exist $x \in X$ and $n_0 \in N$ such that $q_p(x_n, x) < c$, ($q_p(x, x_n) < c$ resp.) for all $n \geq n_0$.

Remark1. These notions in p -quasi-cone metric space are related in this way:

- (i) p -bi-Cauchy $\Rightarrow p$ -right (left) Cauchy $\Rightarrow p$ -weakly right (left) Cauchy $\Rightarrow p$ -right (left) q_p -Cauchy
- (ii) a sequence is p -bi-Cauchy if and only if it is both p -left and p -right Cauchy.

We use the notion of p -right Cauchy in this paper.

Definition6. Let (X, q_p) be a p -quasi-cone metric space. Let $\{x_n\}_{n \in N}$ be a sequence in X . We say that the sequence $\{x_n\}_{n \in N}$ p -right converges to $x \in X$ if $q_p(x, x_n) \rightarrow 0$. We denote this by $\lim_{n \rightarrow \infty} x_n$ or $x_n \rightarrow x$.

Definition7. A p -quasi-cone metric space (X, q_p) is called p -right complete if every p -right Cauchy sequence in X converges.

Definition 8. Let (X, q_p) be a p -quasi-cone metric space. A function $T : X \rightarrow X$ is called

- (i) *continuous* if for any p -right convergent sequence $\{x_n\}_{n \in N}$ in X with $\lim_{n \rightarrow \infty} x_n = x$, the sequence $\{Tx_n\}_{n \in N}$ is right convergent and $\lim_{n \rightarrow \infty} Tx_n = Tx$.
- (ii) *Contractive* if there exist some $h \in [0, 1]$ such that $q_p(Tx, Ty) \leq h q_p(x, y), \forall x, y \in X$ and if $h = 1$, then T is *non-expansive*.

Definition9. Let $O(x) = \{x, Tx, T^2x, \dots\}$ where $x \in X$. The set $O(x)$ is called *orbit* of x .

Definition10. Let $M \subseteq X$. $\delta(M) = \sup\{q_p(x, y), x, y \in M\}$ is called *diameter* of M .

The orbit $O(x)$ is called bounded if there exist a $c \in P$, $\delta(O(x)) \leq c$.

Results

In this section, we prove some fixed point results in p -quasi-cone metric space. Firstly we have given a theorem which is a generalization of [1] due to p -quasi-cone metric space. In this theorem we don't use the normality of cone and we don't take the function $T : X \rightarrow X$ continuous.

Definition 11. A p -quasi-cone metric space (X, q_p) is Hausdorff if for each pair x_1, x_2 of distinct points of X , there exist neighborhoods V_1, V_2 of x_1, x_2 respectively, they are disjoint.

Definition 12. [1] The function $\varphi : P \rightarrow P$ which satisfies the following conditions

1. $\forall t \in P, \varphi(t) < t$,
2. $\forall t_1, t_2 \in P, t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$,
3. $\lim_{n \rightarrow \infty} \|\varphi^n(t)\| = 0, t \in P$,

is called a comparison function.

Now we state a fixed point theorem using a comparison function.

Theorem 1. Let (X, q_p) be a p -right complete Hausdorff p -quasi-cone metric space and let $T : X \rightarrow X$ be a function that satisfies the nonlinear contraction condition:

$$q_p(T(x), T(y)) \leq \varphi \left(\max \{ q_p(x, y), q_p(T(x), x), q_p(T(y), y), q_p(T(x), y), q_p(x, T(y)) \} \right) \quad (1)$$

for all $x, y \in X$, where $\varphi : P \rightarrow P$ is a comparison function.

Let $x_0 \in X$ such that $O(x_0)$ is bounded. Then T has a unique fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is right convergent to x^* .

Proof. We have that the orbit $O(x_0)$ is bounded. So $\delta(O(x_0)) \leq c \in P$. We prove now that

$$q_p(T^{n+1}(x_0), T^n(x_0)) \leq \varphi^n(c).$$

To prove this we use induction method.

For $n=1$, we have

$$\begin{aligned} q_p(T^2 x_0, T x_0) &= q_p(T(T x_0), T(x_0)) \\ &\leq \varphi(\max \{ q_p(T x_0, x_0), q_p(T^2 x_0, T x_0), \\ &\quad q_p(T x_0, x_0), q_p(T^2 x_0, x_0), q_p(T x_0, T x_0) \}) \\ &= \varphi(\max \{ q_p(T x_0, x_0), q_p(T^2 x_0, T x_0), q_p(T^2 x_0, x_0) \}). \end{aligned}$$

Due to $O(x_0)$ is bounded, we have that:

$$\max \{ q_p(T x_0, x_0), q_p(T^2 x_0, T x_0), q_p(T^2 x_0, x_0) \} \leq c.$$

So $q_p(T^2 x_0, T x_0) \leq \varphi^1(c)$.

Suppose it is true for

$$k < n, q_p(T^{k+1} x_0, T^k x_0) \leq \varphi^k(c).$$

Let it prove for $n > k$.

$$\begin{aligned} q_p(T^{n+1} x_0, T^n x_0) &= q_p(T(T^n x_0), T(T^{n-1} x_0)) \\ &\leq \varphi(\max \{ q_p(T^n x_0, T^{n-1} x_0), q_p(T^{n+1} x_0, T^n x_0), \\ &\quad q_p(T^n x_0, T^{n-1} x_0), q_p(T^{n+1} x_0, T^{n-1} x_0), q_p(T^n x_0, T^n x_0) \}) \\ &= \varphi(\max \{ q_p(T^n x_0, T^{n-1} x_0), q_p(T^{n+1} x_0, T^n x_0), q_p(T^{n+1} x_0, T^{n-1} x_0) \}). \end{aligned}$$

Case 1.

$$\begin{aligned} & \max\{q_p(T^n x_0, T^{n-1} x_0), q_p(T^{n+1} x_0, T^n x_0), q_p(T^{n+1} x_0, T^{n-1} x_0)\} \\ & = q_p(T^n x_0, T^{n-1} x_0) \end{aligned}$$

$$q_p(T^{n+1} x_0, T^n x_0) \leq \varphi(q_p(T^n x_0, T^{n-1} x_0)) \leq \varphi(\varphi^{n-1}(c)) = \varphi^n(c).$$

Case2.

$$\begin{aligned} & \max\{q_p(T^n x_0, T^{n-1} x_0), q_p(T^{n+1} x_0, T^n x_0), q_p(T^{n+1} x_0, T^{n-1} x_0)\} \\ & = q_p(T^{n+1} x_0, T^n x_0) \\ & q_p(T^{n+1} x_0, T^n x_0) \leq \varphi(q_p(T^{n+1} x_0, T^n x_0)) \end{aligned}$$

which is absurd because $\varphi(t) < t, \forall t \in P$.

Case3.

$$\begin{aligned} & \max\{q_p(T^n x_0, T^{n-1} x_0), q_p(T^{n+1} x_0, T^n x_0), q_p(T^{n+1} x_0, T^{n-1} x_0)\} \\ & = q_p(T^{n+1} x_0, T^{n-1} x_0) \end{aligned}$$

$$\begin{aligned} q_p(T^{n+1} x_0, T^{n-1} x_0) &= q_p(T(T^n x_0), T(T^{n-2} x_0)) \\ &\leq \varphi(\max\{q_p(T^n x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^n x_0), \\ & q_p(T^{n-1} x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^{n-2} x_0), q_p(T^n x_0, T^{n-1} x_0)\}) \end{aligned}$$

Case3/1.

$$\begin{aligned} & \max\{q_p(T^n x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^n x_0), \\ & q_p(T^{n-1} x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^{n-2} x_0), q_p(T^n x_0, T^{n-1} x_0)\} \\ & = q_p(T^{n+1} x_0, T^n x_0) \end{aligned}$$

Case 3/2.

$$\begin{aligned} & \max\{q_p(T^n x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^n x_0), \\ & q_p(T^{n-1} x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^{n-2} x_0), q_p(T^n x_0, T^{n-1} x_0)\} \\ & = q_p(T^{n-1} x_0, T^{n-2} x_0) \end{aligned}$$

These two cases are trivial.

Case3/3.

$$\begin{aligned} & \max\{q_p(T^n x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^n x_0), \\ & q_p(T^{n-1} x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^{n-2} x_0), q_p(T^n x_0, T^{n-1} x_0)\} \\ & = q_p(T^n x_0, T^{n-1} x_0) \\ & q_p(T^n x_0, T^{n-1} x_0) \leq \varphi^{n-1}(c) \Rightarrow \\ & q_p(T^{n+1} x_0, T^n x_0) \leq \varphi(q_p(T^{n+1} x_0, T^{n-1} x_0)) \\ & \leq \varphi(\varphi(q_p(T^n x_0, T^{n-1} x_0))) \\ & = \varphi^{n+1}(c) = \varphi(\varphi^n(c)) < \varphi^n(c) \end{aligned}$$

Case3/4.

$$\begin{aligned} & \max\{q_p(T^n x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^n x_0), \\ & q_p(T^{n-1} x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^{n-2} x_0), q_p(T^n x_0, T^{n-1} x_0)\} \\ & = q_p(T^n x_0, T^{n-2} x_0) \end{aligned}$$

Case3/5.

$$\begin{aligned} & \max\{q_p(T^n x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^n x_0), \\ & q_p(T^{n-1} x_0, T^{n-2} x_0), q_p(T^{n+1} x_0, T^{n-2} x_0), q_p(T^n x_0, T^{n-1} x_0)\} \\ & = q_p(T^{n+1} x_0, T^{n-2} x_0) \end{aligned}$$

The cases 3/4 and 3/5 can be proved in the same iterative manner. So it is true that for $n \geq 1 \Rightarrow q_p(T^{n+1} x_0, T^n x_0) \leq \varphi^n(c)$.

We have that $\lim_{n \rightarrow \infty} \|\varphi^n(c)\| =$

$$\Leftrightarrow (\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \in \mathbb{N}, n > n_0 \Rightarrow \|\varphi^n(c)\| < \frac{\varepsilon}{2pK^2}),$$

where K is the normal constant of cone. So for $n > n_0$, we have:

$$\|q_p(T^{n+1} x_0, T^n x_0)\| \leq K \|\varphi^n(c)\| < K \frac{\varepsilon}{2pK^2} = \frac{\varepsilon}{2pK} \quad (1) \quad \alpha + \beta = \chi. \quad (1)$$

We show now that the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is right Cauchy. For this we prove that

$$\|q_p(T^n x_0, T^k x_0)\| < \varepsilon \text{ for every } k \in \mathbb{N} \text{ and } n > n_0 \quad (*)$$

We prove it by induction method.

For $k=2$. Since

$$\begin{aligned} q_p(T^{n+2} x_0, T^n x_0) & \leq p(q_p(T^{n+2} x_0, T^{n+1} x_0) \\ & + q_p(T^{n+1} x_0, T^n x_0)) \end{aligned}$$

we have

$$\begin{aligned} \|q_p(T^{n+2} x_0, T^n x_0)\| & \leq pK(\|q_p(T^{n+2} x_0, T^{n+1} x_0)\| \\ & + \|q_p(T^{n+1} x_0, T^n x_0)\|) \end{aligned}$$

We now that $q_p(T^{n+2} x_0, T^{n+1} x_0) \leq \varphi^{n+1}(c) \leq \varphi^n(c)$, so

$$\begin{aligned} \|q_p(T^{n+2} x_0, T^n x_0)\| & < \frac{\varepsilon}{2pK} \cdot \\ \|q_p(T^{n+2} x_0, T^n x_0)\| & \leq pK(\|q_p(T^{n+2} x_0, T^{n+1} x_0)\| \\ & + \|q_p(T^{n+1} x_0, T^n x_0)\|) \\ & < pK\left(\frac{\varepsilon}{2pK} + \frac{\varepsilon}{2pK}\right) = \varepsilon \end{aligned}$$

Suppose it is true for $l < k$. So $\|q_p(T^{n+l} x_0, T^n x_0)\| < \varepsilon$.

Let prove(*) for k

$$\begin{aligned} \|q_p(T^{n+k} x_0, T^n x_0)\| & \leq pK(\|q_p(T^{n+k} x_0, T^{n+k-1} x_0)\| \\ & + \|q_p(T^{n+k-1} x_0, T^n x_0)\|) \\ & \leq pK\|q_p(T^{n+k} x_0, T^{n+k-1} x_0)\| + pK^2 \|\varphi^n(c)\| \\ & < pK \frac{\varepsilon}{2pK} + pK^2 \frac{\varepsilon}{2pK^2} = \varepsilon \end{aligned}$$

So we proved that the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is right Cauchy. We see that since by the space is complete and Hausdorff, we have $\lim_{n \rightarrow \infty} T^n x_0 =$.

Now we prove that x^* is a fixed point of $T : X \rightarrow X$.

$$\begin{aligned} q_p(Tx^*, T^n x_0) &= q_p(Tx^*, T(T^{n-1} x_0)) \\ &\leq \varphi(\max\{q_p(x^*, T^{n-1} x_0), q_p(Tx^*, x^*), \\ &\quad q_p(T^{n-1} x_0, T^{n-2} x_0), q_p(Tx^*, T^{n-2} x_0), q_p(x^*, T^{n-1} x_0)\}) \end{aligned}$$

Taking the limit of both sides when $n \rightarrow \infty$, we have

$$q_p(Tx^*, x^*) \leq \varphi(q_p(Tx^*, x^*)) < q_p(Tx^*, x^*),$$

So $q_p(Tx^*, x^*) = 0 \Rightarrow Tx^* = x^*$.

Now we prove the uniqueness of the fixed point for T . Suppose there is another point y^* that $Ty^* = y^*$.

$$q_p(x^*, y^*) = q_p(Tx^*, Ty^*) \leq \varphi(q_p(x^*, y^*)) < q_p(x^*, y^*).$$

So we have that

$$q_p(x^*, y^*) = 0 \Rightarrow x^* = y^*.$$

Example2. Let $X = [0, 1]$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E, x, y \geq 0\}$.

Define $q(x, y) = \begin{cases} (0, 0), & x \geq y \\ (\frac{1}{2}y, y), & x \leq y \end{cases}$.

We take the function

$$T : X \rightarrow X, Tx = \begin{cases} \frac{x}{8}, & x \in [0, \frac{1}{2}] \\ \frac{1}{10}, & x \in (\frac{1}{2}, 1] \end{cases}$$

and

$$\varphi : P \rightarrow P, \varphi(x, y) = (\frac{x}{3}, \frac{y}{3}).$$

Case 1. For every $x, y \in [0, x \leq y]$ we have

$$q(Tx, Ty) = q(\frac{x}{8}, \frac{y}{8}) = (\frac{y}{16}, \frac{y}{8}).$$

$$\begin{aligned} \max\{q(x, y), q(\frac{x}{8}, x), q(\frac{y}{8}, y), q(\frac{x}{8}, y), q(x, \frac{y}{8})\} &= (\frac{y}{2}, y) \Rightarrow \varphi(\max\{q(x, y), q(\frac{x}{8}, x), q(\frac{y}{8}, y), q(\frac{x}{8}, y), q(x, \frac{y}{8})\}) \\ &= \varphi(\frac{y}{2}, y) = (\frac{y}{6}, \frac{y}{3}) \end{aligned}$$

So $q(Tx, Ty) = (\frac{y}{16}, \frac{y}{8}) < (\frac{y}{6}, \frac{y}{3})$

So we are in conditions of theorem.

For every $x, y \in [0, x \geq y] \Rightarrow q_p(Tx, Ty) = 0$, so the theorem is true.

Case2.

For every $x, y \in \left(\frac{1}{2}, x \leq y\right)$ we have

$$q(Tx, Ty) = q\left(\frac{1}{10}, \frac{1}{10}\right) = \left(\frac{1}{20}, \frac{1}{10}\right).$$

$$\max\{q(x, y), q\left(\frac{1}{10}, x\right), q\left(\frac{1}{10}, y\right), q\left(\frac{1}{10}, y\right), q\left(x, \frac{1}{10}\right)\} = \left(\frac{y}{2}, y\right) \Rightarrow q(Tx, Ty) = \left(\frac{1}{20}, \frac{1}{10}\right) \leq \varphi\left(\frac{y}{2}, y\right) = \left(\frac{y}{6}, \frac{y}{3}\right).$$

So we are in conditions of theorem.

For every $x, y \in \left(\frac{1}{2}, x \geq y\right) \Rightarrow q_p(Tx, Ty) = 0$, so the theorem is true.

Case3.

For every $x \in \left[\frac{1}{2}, 1\right]$ and $y \in \left(\frac{1}{2}, 1\right)$. It is clear that the conditions of theorem true in this case.

So, the function $T : X \rightarrow X$,

$$Tx = \begin{cases} \frac{x}{4}, & x \in [0, \frac{1}{2}] \\ \frac{1}{10}, & x \in (\frac{1}{2}, 1] \end{cases}$$

has a fixed point $x = 0$.

This result is a generalization of following result Raja and Vaezpour[1] in cone metric space because our quasi-contraction is better.

Theorem1'. Let (X, q_p) be a p -right complete Hausdorff p -quasi-cone metric space and let $T : X \rightarrow X$ be a function that satisfies the nonlinear contraction condition:

$$(2) q_p(T(x), T(y)) \leq \varphi(q_p(x, y)) \text{ for all } x, y \in X,$$

where $\varphi : P \rightarrow P$ is a comparison function. Let $x_0 \in X$ such that $O(x_0)$ is bounded. Then T has a unique fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is right convergent to x^* .

The following result is a generalization Theorem of Ciric[2] in metric space.

Theorem2. Let (X, q_p) be a p -right complete, Hausdorff, p -quasi-cone metric space and let $T : X \rightarrow X$ be a function that satisfies the nonlinear contraction condition:

$$q_p(T(x), T(y)) \leq h \max\{q_p(x, y), q_p(T(x), x), q_p(T(y), y), q_p(T(x), y), q_p(x, T(y))\}$$

for all $x, y \in X$, where $h \in [0, 1)$. Let $x_0 \in X$ such that $O(x_0)$ is bounded. Then T has a unique fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ is right convergent to x^* .

Proof. If we take $\varphi(t) = ht$ for $t \in P$, we are in condition of Theorem1. So Theorem 2 follows and this completes the proof.

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