

## On the Exact Solutions of Couple Stress Fluids

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**Abstract:** Exact solutions of the momentum equations of couple stress fluid are investigated. Making use of stream function, the two-dimensional flow equations are transformed into non-linear compatibility equation, and then it is linearized by vorticity function. Stream functions and velocity distributions are discussed for various flow situations.

### 1 Introduction

The basic governing equations for the flow of a fluid of Newtonian type are the Navier-Stokes equations. This set of non-linear partial differential equations has no general solution, and only a limited number of exact solutions can be found in literature. On the other hand, exact solutions are very important not only they represent the basic flow phenomena but also solutions obtained by various techniques can be attested by these basic solutions. For situations where flow of fluid is non-Newtonian in nature, it becomes rather difficult to obtain exact solutions. This difficulty occurs because of the highly non-linear terms in the viscous part of the flow equations.

Most of the fluids in nature do not obey the linear relationship between the stress and the rate of strain. Such fluids represent the non-Newtonian class of fluids. Several models have been presented to explain the behavior of these fluids. In the theory of non-Newtonian fluids, the couple stress fluid introduced by Stokes [1] has gained considerable attention and has been widely studied by researchers [2-7]. Couple stress fluid has been the subject of interest, due to its numerous industrial and scientific applications such as extrusion of polymer fluids, solidification of liquid crystals and animal bloods.

In this paper, we investigate exact solutions of the couple stress fluid motion for different flow situations. Solutions are found by using appropriate conditions for the different cases of flow. Expressions for the stream functions and velocity distributions are derived for two-dimensional flow behind a grid; flow above a porous plate; flow due to stretching plate, and the corner flow.

### 2 Governing Equations

The equations of motion governing the flow of a couple stress fluid are given by [1]

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho \left[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\nabla p + \mu \nabla^2 \mathbf{V} - \eta \nabla^4 \mathbf{V} + \mathbf{f}, \tag{2}$$

where  $\mathbf{V}$  is the velocity vector,  $\rho$  the constant density,  $\mu$  the dynamic viscosity,  $p$  the pressure,  $\eta$  the couple stress parameter,  $\mathbf{f}$  the force vector,  $\nabla^2$  denotes the Laplacian operator.

The  $x$  and  $y$  - components of the equations of steady motion in the absence of body forces become

$$P_x - \rho v \omega = -\mu \omega_y + \eta [\omega_{xxy} + \omega_{yyy}] \quad (3)$$

$$P_y + \rho u \omega = \mu \omega_x - \eta [\omega_{xyy} + \omega_{xxx}] \quad (4)$$

Where

$$\omega = v_x - u_y \quad (5)$$

is the vorticity, and

$$P = p + \frac{1}{2} \rho (u^2 + v^2), \quad (6)$$

Eliminating pressure from equations (3) and (4) and by applying the integrability condition  $P_{xy} = P_{yx}$ , we get

$$\rho (u \omega_x + v \omega_y) = \mu \nabla^2 \omega - \eta \nabla^4 \omega. \quad (7)$$

Let us introduce a stream function  $\psi(x, y)$  defined as,

$$u = \psi_y, \quad v = -\psi_x. \quad (8)$$

Using (8) the equation of continuity (1) is satisfied identically and equation (7) takes the form

$$\rho \{ \psi, \nabla^2 \psi \} = -\mu \nabla^4 \psi + \eta \nabla^6 \psi, \quad (9)$$

in which

$$\omega = -\nabla^2 \psi, \quad \nabla^4 = \nabla^2 \cdot \nabla^2, \quad \nabla^6 = \nabla^2 \cdot \nabla^2 \cdot \nabla^2$$

and

$$\{ \psi, \nabla^2 \psi \} = \psi_x (\nabla^2 \psi)_y - \psi_y (\nabla^2 \psi)_x$$

is the Poisson bracket.

Equation (9) is a non-linear partial differential equation of an incompressible couple stress fluid. In order to linearize (9) vorticity is assumed of the form

$$\omega = \psi + A_1 x + A_2 y, \quad (10)$$

where  $A_1, A_2$  are constants. Substitution from (10) into (9) yields

$$\nabla^2 \psi + \frac{1}{\nu \beta} (A_2 \psi_x - A_1 \psi_y) = 0, \quad (11)$$

where  $\nu = \frac{\mu}{\rho}$  and  $\beta = 1 - \frac{\eta}{\mu}$ . Introduce

$$H = \psi + A_1 x + A_2 y, \quad (12)$$

which is a linear partial differential equation in  $H$ , and results in a number of known exact solutions for different values of  $A_1, A_2$ . Using the method of separation of variables and by setting

$$H = f_1(x)f_2(y) \tag{14}$$

equation (13) results in

$$f_1''(x) + \frac{A_2}{\nu\beta} f_1'(x) = Cf_1(x), \tag{15}$$

$$f_2''(y) - \frac{A_1}{\nu\beta} f_2'(y) = -Cf_2(y), \tag{16}$$

in which C is the separation constant,  $f_1(x)$  and  $f_2(x)$  are unknown functions to be determined. We now investigate the significance of these equations in some special cases.

### 3 Exact Solutions for different Flow Geometries

In this part of the paper, we investigate some exact solutions of the couple stress fluid for various flow situations. For different values of the constants  $A_1$ ,  $A_2$  and  $C$  equations (15) and (16) will turn out exact solutions for some physical flow situations.

#### 3.1 Laminar Flow behind Two Dimensional Grid

We consider the flow of an incompressible couple stress fluid behind a two-dimensional grid with grid spacing  $l$ . To obtain a desired solution for this geometry, we assume:  $A_1 = 0$ ,  $A_2 = -V$ , and  $C = \sigma^2 > 0$  where  $V$  is the uniform velocity and  $\sigma$  is a constant. Introducing the above values of  $A_1$  and  $A_2$  and in equations (15) and (16), we find that

$$f_1''(x) - \frac{V}{\nu\beta} f_1'(x) - \sigma^2 f_1(x) = 0, \tag{17}$$

$$f_2''(y) + \sigma^2 f_2(y) = 0, \tag{18}$$

with solutions given below

$$f_1(x) = B_1 e^{(m_1 x)} + B_2 e^{(m_2 x)}, \tag{19}$$

$$f_2(y) = B_3 \cos(\sigma y) + B_4 \sin(\sigma y), \tag{20}$$

in which  $B_i (i = 1-4)$  are integration constants and

$$m_{1,2} = \frac{1}{2} \left\{ \frac{V}{\nu\beta} \pm \sqrt{\left(\frac{V}{\nu\beta}\right)^2 + 4\sigma^2} \right\}. \tag{21}$$

Substituting from equations (19) and (20) in equation (14), and assuming bounded solutions, we get

$$H = M \sin(\sigma y) \exp \left[ \left\{ \frac{V}{\sigma\nu\beta} - \sqrt{\left(\frac{\alpha V}{\sigma\nu\beta}\right)^2 + 4} \right\} \frac{\alpha x}{2} \right], \tag{22}$$

in which the arbitrary constant  $A_2 A_4 = M$  is to be determined. Making use of (12) and introducing non-dimensional parameters

$$\begin{aligned} x^* &= \frac{x}{l}, & y^* &= \frac{y}{l}, & u^* &= \frac{u}{V}, & v^* &= \frac{v}{V}, \\ \psi^* &= \frac{\psi}{Vl}, & M^* &= \frac{M}{Vl}, & P &= \frac{\rho V^*}{2}, & \sigma &= \frac{2\pi}{l} \end{aligned} \tag{23}$$

in (22), we get

$$\psi^* = y^* + M^* \sin(2\pi y^*) \exp(\gamma x^*), \quad (24)$$

where  $\gamma = \frac{(2R_e - \sqrt{(R_e)^2 + 16\pi^2 \beta^2})}{2\beta}$  and  $R_e = \frac{V}{\nu}$  is the Reynolds number. On setting  $\beta \rightarrow 1$  we obtain the solution behind a two dimensional grid obtained by Kovaszny for the viscous fluid [8]. With the help of equation (8) the velocity components  $u^*$  and  $v^*$  are given as

$$u^* = 1 + 2\pi M^* \cos(2\pi y^*) \exp(\gamma x^*), \quad (25)$$

$$v^* = \gamma M^* \sin(2\pi y^*) \exp(\gamma x^*). \quad (26)$$

The constant  $M^*$  can be determined by setting the stagnation point of the flow where  $(u^*, v^*) = (0, 0)$ . Suppose it is at  $(x^*, y^*) = (0, 0)$  then simplification yields  $M^* = -\frac{1}{2\pi}$ .

$$u^* = 1 + \cos(2\pi y^*) \exp(\gamma x^*), \quad (27)$$

$$v^* = -\frac{\gamma}{2\pi} \sin(2\pi y^*) \exp(\gamma x^*). \quad (28)$$

### 3.2 Reverse Flow above a Plate

Consider steady flow of an incompressible couple stress fluid above a plate with suction at  $y = 0$ . In order to study the behavior of the fluid above the plate it is assumed that  $A_1=0$ ,  $A_2=V$  and  $C=-\sigma^2 < 0$ , where  $V$  is the uniform velocity and  $\sigma$  is a constant. Employing these values in equations (15) and (16), we obtain

$$f_1''(x) + \frac{V}{\nu\beta} f_1'(x) + \sigma^2 f_1(x) = 0, \quad (29)$$

$$f_2''(y) - \sigma^2 f_2(y) = 0. \quad (30)$$

Both these equations yield solutions of the form

$$f_1(x) = E_1 e^{(n_1 x)} + E_2 e^{(n_2 x)}, \quad (31)$$

$$f_2(y) = E_3 e^{(\sigma y)} + E_4 e^{(-\sigma y)}, \quad (32)$$

Where  $E_i (i=1-4)$  are arbitrary constants and

$$n_{1,2} = \frac{1}{2} \left\{ -\frac{V}{\nu\beta} \pm \sqrt{\left(\frac{V}{\nu\beta}\right)^2 - 4\sigma^2} \right\}. \quad (33)$$

Therefore, with the help of equation (14) bounded solution for H becomes

$$H = E \exp \left[ -\sigma y - \left\{ \frac{V}{\nu\beta} + \sqrt{\left(\frac{V}{\nu\beta}\right)^2 - 4\sigma^2} \right\} \frac{x}{2} \right], \quad (34)$$

Where  $E = E_2 E_4$  is a constant to be determined and  $\sigma = \frac{2\pi}{l}$ , where  $l$  is the length of the plate. The relation in equation (12) with the help of (23) yields a non-dimensional stream function of the form

$$\psi^* = -y^* + E \exp \left[ -\left( 2\pi y^* + \frac{x}{2} \right) \right], \quad (35)$$

where  $\mathcal{G} = \left[ \frac{-2\gamma/\beta + \sqrt{(2\gamma/\beta)^2 - 16\pi^2}}{2} \right]$  and  $E^* = \frac{E}{\nu} > 0$ , represents suction at the plate. The velocity components are given by

$$u^* = -1 - 2\pi E^* \exp \left[ - (2\pi y^* + \mathcal{G} x^*) \right], \tag{36}$$

$$v^* = \mathcal{G} E^* \exp \left[ - (2\pi y^* + \mathcal{G} x^*) \right], \tag{37}$$

For  $\beta \rightarrow 1$  these solutions are compatible with the solutions given by Lin and Tobak [9].

### 3.3 Flow due to Stretching Plate

We consider the two-dimensional boundary layer flow of a couple stress fluid above a stretching surface placed at  $y = 0$ . The application of an external force along the  $x$  - direction results in stretching of the plate and hence the flow. The boundary conditions for the flow situation are:

$$\begin{aligned} u(x,0) &= \lambda x, & u(0,y) &= 0, \\ v(x,0) &= 0, & v(x,\infty) &= -V, \end{aligned} \tag{38}$$

where  $\lambda$  is the stretching parameter. For flow due to stretching of the plate, we assume  $A_1 = -V$ ,  $A_2 = 0 = C$  in (15) and (16), we find

$$f_1''(x) = 0, \tag{39}$$

$$f_2''(y) + \frac{V}{\nu\beta} f_2'(y) = 0. \tag{40}$$

Solutions to both these equations are

$$f_1(x) = E_5 x + E_6, \tag{41}$$

$$f_2(y) = E_7 + E_8 \exp \left( - \frac{V}{\nu\beta} y \right), \tag{42}$$

in which  $E_i (i = 5 - 8)$  are integration constants. Substituting from (41) and (42) and taking help of (12) and (14) it is obtained

$$\psi = Vx + a_1 + a_2 x + a_3 \exp \left( - \frac{V}{\nu\beta} y \right) + a_4 x \exp \left( - \frac{V}{\nu\beta} y \right), \tag{43}$$

Where  $a_i, i = 1 - 4$  are redefined constants of equations (41) and (42) to be determined. With the help of boundary conditions (38) and non-dimensional parameters (23) stream function reduces to

$$\psi^* = x^* - x^* \exp \left( - \beta^{-1} R_e y^* \right), \tag{44}$$

and velocity components become

$$u^* = \beta^{-1} R_e x^* \exp \left( - \beta^{-1} R_e y^* \right), \tag{45}$$

$$v^* = \exp \left( - \beta^{-1} R_e y^* \right) - 1. \tag{46}$$

Where  $\lambda = \frac{V^2}{\nu\beta}$ . If  $\beta \rightarrow 1$  the velocity components reduce to the viscous case solutions [10].

### 3.4 Flow into a Corner

The flow of an incompressible couple stress fluid along a corner is considered with suction at both the walls which are placed perpendicular to each other. The relevant boundary conditions are

$$\begin{aligned} u = v = 0, & \quad \text{on } x = 0, \quad \text{and } y = 0, \\ u, v \rightarrow -V, & \quad \text{as } x \rightarrow \infty, \quad \text{and } y \rightarrow \infty. \end{aligned} \quad (47)$$

The constants  $A_1, A_2$  and  $C$  in (15) and (16) are given the values  $-V, V$  and  $0$  respectively, where  $V$  is the uniform velocity then

$$f_1''(x) + \frac{V}{\nu\beta} f_1'(x) = 0, \quad (48)$$

$$f_2''(y) + \frac{V}{\nu\beta} f_2'(y) = 0. \quad (49)$$

With the help of (12), (14) and solutions of the above equations lead to the following form of stream function

$$\psi = V(x - y) + \left( E_9 + E_{10} \exp\left(-\frac{V}{\nu\beta} x\right) \right) \left( E_{11} + E_{12} \exp\left(-\frac{V}{\nu\beta} y\right) \right), \quad (50)$$

in which  $E_i (i = 9 - 12)$  are constants of integration. Using boundary conditions (47) we obtain expression for non-dimensional stream function as

$$\psi^* = x^* - y^* + \beta R_e^{-1} \left( \exp(-\beta^{-1} R_e x^*) - \exp(-\beta^{-1} R_e y^*) \right), \quad (51)$$

The velocity components become

$$u^* = -(1 - \exp(-\beta^{-1} R_e y)), \quad v^* = -(1 - \exp(-\beta^{-1} R_e x^*)), \quad (52)$$

we see that the no-slip conditions are satisfied at each boundaries, which are permeable and there is suction on both boundaries. For  $\beta \rightarrow 1$  the above solutions recover the viscous flow solution [10].

#### 4 Conclusions

Some exact solutions have been investigated for two dimensional equations of motion of the incompressible couple stress fluid by assuming vorticity as a function of stream. This study shows that each solution is strongly dependent on parameter  $\beta$ . Solutions for the stream functions and velocity distributions studied in this work demonstrate a good agreement with the already existing solutions for viscous fluid found in literature [10].

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