SUM CONSTRUCTION OF AUTOMORPHIC BIBDS AND THEIR APPLICATIONS IN EXPERIMENTAL DESIGNS

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ABSTRACT: In this paper, a construction equivalent to “sum construction”, of BIBDs where BIBD is added to a BIBD that is automorphic to it is presented. The result is that we can get new BIBDs by forming the collection of a BIBD with its automorphic BIBDs. A new recursive technique has been developed for the construction of $t - (v, k, \lambda_t)$ designs. It has also clearly shown that every $t -$ design is also a BIB (v, k, λ) design. Therefore, this construction technique also generates BIBDs. Therefore, this note presents an alternative method that is simpler and unified for the construction of BIBDs that are very important in the experimental designs. As it provides designs for different values of k , unlike many methods that provide designs for a single value of k . Moreso, it provides both Steiner and non- Steiner designs.

Introduction

The following are properties of $t - (v, k, \lambda_t)$ designs:

1. $bk = \lambda_1 v$
2. $\lambda_t(v - (t - 1)) = \lambda_{t-1}(k - (t - 1))$

Indeed, number of ways of choosing k out of v points is $\binom{v}{k}$. And any point will occur in $\binom{v-1}{k-1}$ blocks. But we require each point to be incident to λ_t blocks. Thus:

$$b = \frac{\lambda_1 \binom{v}{k}}{\binom{v-1}{k-1}}$$

Expanding R.H.S. we obtain:

$$\begin{aligned} b &= \lambda_1 \frac{v!}{k!(v-k)!} \div \frac{(v-1)!}{(k-1)!(v-1-(k-1))!} \\ &= \lambda_1 \frac{v!}{k!(v-k)!} \times \frac{(k-1)!(v-k)!}{(v-1)!} \\ &= \lambda_1 \frac{v!}{k!} \times \frac{(k-1)!}{(v-1)!} \\ &= \lambda_1 v \frac{(v-1)!}{k(k-1)!} \times \frac{(k-1)!}{(v-1)!} \\ &= \lambda_1 \frac{v}{k} \end{aligned}$$

Hence, $bk = \lambda_1 v$ as required (3.1.1)

Similarly from the fact that t points are incident to λ_t blocks we have:

$$b = \lambda_t \frac{\binom{v}{k}}{\binom{v-t}{k-t}} \quad (i)$$

$$b = \lambda_{t-1} \frac{\binom{v}{k}}{\binom{v-(t-1)}{k-(t-1)}} \quad (ii)$$

Also

Combining equation (i) and (ii) we have:

$$\lambda_t \frac{\binom{v}{k}}{\binom{v-t}{k-t}} = \lambda_{t-1} \frac{\binom{v}{k}}{\binom{v-(t-1)}{k-(t-1)}}$$

Expanding both sides we obtain:

$$\begin{aligned} \lambda_t \frac{v!}{k!(v-1)!} \div \frac{(v-t)!}{(k-t)!(v-t-(k-t))!} \\ = \lambda_t \frac{v!}{k!(v-k)!} \div \frac{(v-(t-1))!}{(k-(t-1))!(v-(t-1)-(k-(t-1)))!} \end{aligned}$$

This reduces to:

$$\lambda_t \frac{v!}{k!} \times \frac{(k-t)!}{(v-t)!} = \lambda_{t-1} \frac{v!}{k!} \times \frac{(k-(t-1))!}{(v-(t-1))!}$$

Multiplying both sides by $\frac{k!}{v!}$ we have

$$\lambda_t \frac{(k-t)!}{(v-t)!} = \lambda_{t-1} \frac{(k-(t-1))!}{(v-(t-1))!}$$

Expanding R.H.S. and simplifying we have:

$$\lambda_t \frac{(k-t)!}{(v-t)!} = \frac{\lambda_{t-1}(k-(t-1))(k-t)!}{(v-(t-1))(v-t)!}$$

Multiplying both sides by $\frac{(v-t)!}{(k-t)!}$ we have:

$$\lambda_t = \frac{\lambda_{t-1}(k-(t-1))}{(v-(t-1))}$$

$$\text{Hence, } \lambda_t(v-(t-1)) = \lambda_{t-1}(k-(t-1))$$

as required.

Equation (3.2) could be generalized as follows: starting with $t = 6$ we have :

$$\lambda_6(v-5) = \lambda_5(k-5)$$

$$\lambda_5(v-4) = \lambda_4(k-4) , \quad t = 5$$

$$\lambda_4(v-3) = \lambda_3(k-3) , \quad t = 4$$

$$\lambda_3(v-2) = \lambda_2(k-2) , \quad t = 3$$

$$\lambda_2(v-1) = \lambda_1(k-1) , \quad t = 2$$

Now:

$$\lambda_2 = \frac{\lambda_1(k-1)}{v-1}$$

$$\lambda_3 = \frac{\lambda_1(k-1)(k-2)}{(v-1)(v-2)}$$

$$\lambda_4 = \frac{\lambda_1(k-1)(k-2)(k-3)}{(v-1)(v-2)(v-3)}$$

$$\lambda_5 = \frac{\lambda_1(k-1)(k-2)(k-3)(k-4)}{(v-1)(v-2)(v-3)(v-4)}$$

$$\lambda_6 = \frac{\lambda_1(k-1)(k-2)(k-3)(k-4)(k-5)}{(v-1)(v-2)(v-3)(v-4)(v-5)}$$

$$\lambda_6(v-1)(v-2)(v-3)(v-4)(v-5) = \lambda_1(k-1)(k-2)(k-3)(k-4)(k-5)$$

for $s = 1, 2 \dots t-1$ we get

$$\begin{aligned} \lambda_t(v-(t-1))(v-(t-2))(v-(t-3)) \dots (v-(t-s)) \\ = \lambda_{t-s}(k-(t-1))(k-(t-2))(k-(t-3)) \dots (k-(t-s)) \end{aligned}$$

MAIN RESULTS CONSTRUCTION OF 3- $(v, k, 1)$ -DESIGN

Here $t = 3$ and $\lambda_t = 1$ and putting this in equation (3.1.2) we have:

$$v-2 = \lambda_2(k-2)$$

Hence

$$\lambda_2 = \frac{v-2}{k-2} \quad (3.2.1)$$

Now when $t = 2$ we have:

$$\lambda_2(v-1) = \lambda_1(k-1)$$

That is

$$\frac{\lambda_1}{\lambda_2} = \frac{v-1}{k-1} \quad (3.2.2)$$

This implies

$$\lambda_1 = \lambda_2 \frac{(v-1)}{(k-1)} ; \lambda_1 = \alpha (v-1)$$

And

$$\lambda_2 = \lambda_1 \frac{(k-1)}{(v-1)} \quad (3.2.3)$$

Given that $\lambda_1, \lambda_2, v-1$ and $k-1$ are all integers and α is a rational number which we will represent by x/y where x and y are positive integers. Thus the equations (3.5) become:

$$\lambda_1 = \frac{x}{y}(v-1)$$

$$y\lambda_1 = x(v-1)$$

And

$$\lambda_2 = \frac{x}{y}(k-1)$$

$$y\lambda_2 = x(k-1) \quad (3.2.4)$$

Casel $x = 1$

Then (3.2.4) becomes

$$y\lambda_1 = v-1, \Rightarrow v = y\lambda_1 + 1$$

And

$$y\lambda_2 = k-1, \Rightarrow k = y\lambda_2 + 1$$

THEOREM 3.2 .1. IF $x = 1$ and $\lambda_2 - 1 \equiv 0(mod y)$, where y is an integer then there are only three non- trivial $3 - (v, k, 1)$ designs which are: $3 - (8,4,1), 3 - (22,6,1)$ and $3 - (112,12,1)$

Proof: Now from (3.2.1) we know

$$\lambda_2 = \frac{v-2}{k-2}$$

But $v = y\lambda_1 + 1$ and $k = y\lambda_2 + 1$

Hence

$$\begin{aligned} \lambda_2 &= \frac{y\lambda_1 + 1 - 2}{y\lambda_2 + 1 - 2} \\ &= \frac{y\lambda_1 - 1}{y\lambda_2 - 1} \end{aligned}$$

This implies

$$\lambda_1 = \lambda_2^2 - \frac{\lambda_2 - 1}{y},$$

$$v = y\lambda_2^2 - \lambda_2 + 2$$

And

$$k = y\lambda_2 + 1$$

For this design to be design $3 - (v, k, 1)$ and from (3.1.1) it implies

$$\left(\lambda_2^2 - \frac{\lambda_2 - 1}{y}\right)(y\lambda_2^2 - \lambda_2 + 2) \equiv 0 \pmod{y\lambda_2 + 1}$$

That is

$$\frac{(y\lambda_2^2 - \lambda_2 + 1)(y\lambda_2^2 - \lambda_2 + 2)}{y(y\lambda_2 + 1)}$$

Is a positive integer.

Expanding and simplifying we have:

$$\begin{aligned} &\frac{y^2\lambda_2^4 - 2y\lambda_2^3 + \lambda_2^2(3y+1) - 3\lambda_2 + 2}{y^2\lambda_2 + y} \\ &= \lambda_2^3 - \frac{3\lambda_2^2}{y} + \frac{\lambda_2(3y+4)}{y^2} - \frac{6y+4}{y^3} \text{ rem } \frac{2y^2 + 6y + 4}{y^2} \end{aligned}$$

Which can be written as:

$$\lambda_2^3 - \frac{3\lambda_2^2}{y} + \frac{\lambda_2(3y+4)}{y^2} - \frac{6y+4}{y^3} + \frac{2y^2 + 6y + 4}{y^2(y^2\lambda_2 + y)} \quad (3.2.5)$$

Now

$$\frac{2y^2 + 6y + 4}{y^2(y^2\lambda_2 + y)} \quad (3.2.6)$$

will be an integer if y^2 divides $6y + 4$. The only values for in which this is possible are 1 and 2.

$$\frac{2y^2 + 6y + 4}{y^2(y^2\lambda_2 + y)} = \frac{3}{2\lambda_2 + 1}$$

$$\lambda_2 > 1$$

In this case (3.2.6) is not an integer.

For $y = 1$ (3.2.6) is

$$\frac{12}{\lambda_2 + 1}$$

Thus both (3.2.5) and (3.2.6) will be integers if λ_2 takes the values 2, 3, 5 and 11.

The table below gives corresponding values of k, λ_1, v and b .

Table 3.2.1. Case1; for $y = 1$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
2	3	3	4	4
3	4	7	8	14
5	6	21	22	77
11	12	111	112	1036

The required $3 - (v, k, 1)$ designs are; $3 - (8, 4, 1)$, $3 - (22, 6, 1)$, and $3 - (112, 12, 1)$.

A $3 - (4, 3, 1)$ is trivial given $t = k$, but it is required that $t < k$, hence is not included and our proof is completed

The above $3 - (v, k, 1)$ designs are related with the following BIB designs:

$2 - (4, 3, 2)$, $2 - (8, 4, 3)$, $2 - (22, 6, 5)$, $2 - (112, 12, 11)$. For example the configuration of

$2 - (8, 4, 3)$ which is $3 - (8, 4, 1)$ is:

$B1(1, 2, 3, 4), B2(1, 2, 5, 6), B3(1, 2, 7, 8), B4(1, 3, 5, 7), B5(1, 3, 6, 8), B6(1, 4, 5, 8), B7(1, 4, 6, 7), B8(2, 3, 5, 8), B9(2, 3, 6, 7), B10(2, 4, 5, 7), B11(2, 4, 6, 8), B12(3, 4, 5, 6), B13(3, 4, 7, 8), B14(5, 6, 7, 8)$.

3.2.1. Case 2, $y = 1$

In this case (3.2.4) is

$$\lambda_1 = x(v - 1), \Rightarrow v = \frac{\lambda_1 + x}{x}$$

$$\lambda_2 = x(k - 1), \Rightarrow k = \frac{\lambda_2 + x}{x}$$

Where is x a positive integer and both λ_1 and λ_2 are divisible by x .

Now from (3.2.1)

$$\lambda_2 = \frac{v - 2}{k - 2}$$

we get λ_1, v , and k as follows;

$$\lambda_1 = \lambda_2^2 - x\lambda_2 + x,$$

$$v = \frac{\lambda_2^2 - x\lambda_2 + 2x}{x}$$

and

$$k = \frac{\lambda_2 + x}{x}$$

Using $b = \frac{\lambda_1 v}{k}$. for this design to be $3 - (v, k, 1)$ design then:

$$(\lambda_2^2 - x\lambda_2 + x) \frac{(\lambda_2^2 - x\lambda_2 + 2x)}{x} \equiv 0 \pmod{\left(\frac{\lambda_2 + x}{x}\right)}$$

That is

$$\frac{(\lambda_2^2 - x\lambda_2 + x)(\lambda_2^2 - x\lambda_2 + 2x)}{\lambda_2 + x}$$

is a positive integer.

Expanding and dividing we will have:

$$\lambda_2^3 - 3x\lambda_2^2 + \lambda_2(3x + 4x^2) - (6x^2 + 4x^3) \text{rem } 2x^2 + 6x^3 + 4x^4$$

That is

$$\lambda_2^3 - 3x\lambda_2^2 + \lambda_2(3x + 4x^2) - (6x^2 + 4x^3) + \frac{2x^2 + 6x^3 + 4x^4}{\lambda_2 + x} \quad (3.2.1.1)$$

Now (3.2.1.1) will be integer if

$$\frac{2x^2 + 6x^3 + 4x^4}{\lambda_2 + x} \quad (3.2.1.2)$$

For $x = 2$ equation (3.2.1.2) is:

$$\frac{120}{\lambda_2 + 2}, \lambda_2 > 2$$

Thus λ_2 can take any of the following values 3,4,6,8,10,18,28,32,38,58, and 118. But λ_2 must be divisible by 2. So 3 is not a possibility. We give corresponding values of k, λ_1, v and b in the table below.

Table 3.2.1.1. Case 2; for $x = 2$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
4	3	10	6	20
6	4	26	14	91
8	5	50	26	260
10	6	82	42	574
18	10	290	146	4234
22	12	442	222	8177
28	15	730	366	17812
38	20	1370	686	46991
58	30	3250	1626	176150
118	60	13690	6846	1562029

The following designs $3 - (6,3,1), 3 - (14,4,1), 3 - (26,5,1), 3 - (42,6,1), 3 - (146,10,1), 3 - (222,12,1), 3 - (366,15,1), 3 - (686,20,1), 3 - (1676,30,1)$ and $3 - (6846,60,1)$. can then be obtain from $\text{BIB}(v, k, \lambda)$ designs given below

$2 - (6,3,4), 2 - (14,4,6), 2 - (26,5,8), 2 - (42,6,10), 2 - (146,10,18), 2 - (222,12,22), 2 - (366,15,28), 2 - (686,20,38), 2 - (1626,30,58),$ and $2 - (6846,60,118)$.

For $x = 3$ equation (3.2.1.2) is

$$\frac{504}{\lambda_2 + 3}, \lambda_2 > 3$$

thus the possible values of λ_2 where, λ_2 is divisible by 3 are: 6,9,15,18,21,33,39,60,69,81,123,165,249, and 501.

We give corresponding values of λ_1, v, k , and b in the table below.

Table 3.2.1.2. Case 2; for $x = 3$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
6	3	21	8	56
9	4	57	20	285
15	6	183	62	1891
18	7	273	92	3588
21	8	381	128	6096
33	12	993	332	27473
39	14	1407	470	47235
60	21	3423	1142	186146
69	24	4557	1520	288610
81	28	6321	2108	475881
165	56	26733	8912	4254366
249	84	61257	20420	14891285
501	168	249501	83168	123514876

Again we obtain $3 - (v, k, 1)$ designs from BIB designs below:

$2 - (8, 3, 6), 2 - (20, 4, 9), 2 - (67, 6, 15), 2 - (92, 7, 18), 2 - (128, 8, 21),$
 $2 - (332, 12, 33), \dots, \text{and } 2 - (83168, 168, 501)$

For $x = 4$, and using similar arguments as before we obtain possible values of λ_2 as follows: 8, 12, 16, 20, 28, 32, 36, which give the values of λ_1, k, v and b as follows:

Table 3.2.1.3. Case 2; for $x = 4$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	V	B
8	3	36	10	120
12	4	100	26	650
16	5	196	50	1960
20	6	324	82	4428
28	8	676	170	14365
32	9	900	226	22600
36	10	1156	290	33524

The desired $3 - (v, k, 1)$ designs are obtained from the following BIB designs:

$2 - (10, 3, 8), 2 - (26, 4, 12), 2 - (50, 5, 16), 2 - (82, 6, 20), 2 - (170, 8, 28), 2 - (226, 9, 32),$
 $\text{and } 2 - (290, 10, 36)$

3.2.2. Case 3, $x \neq y$ $x > 1, y > 1$

In this case (3.2.4) is

$$\lambda_1 = \frac{x}{y}(v-1) \Rightarrow v = \frac{y\lambda_1 + x}{x}$$

$$\lambda_2 = \frac{x}{y}(k-1) \Rightarrow k = \frac{y\lambda_2 + x}{x}$$

Where x and y are positive integers and λ_2 and y are divisible by x .

Now from (3.2.1)

$$\lambda_2 = \frac{v-2}{k-2}$$

We get λ_1, v , and k as follows:

$$\lambda_1 = \frac{y\lambda_2^2 - x\lambda_2 + x}{y}$$

$$v = \frac{y\lambda_2^2 - x\lambda_2 + 2x}{x}$$

And

$$k = \frac{y\lambda_2 + x}{x}$$

Using $b = \frac{\lambda_1 v}{k}$ for this to be $3 - (v, k, 1)$ design then

$$\frac{(y\lambda_2^2 - x\lambda_2 + x)}{y} \left(\frac{y\lambda_2^2 - x\lambda_2 + 2x}{x} \right) \equiv 0 \pmod{\left(\frac{y\lambda_2 + x}{x} \right)}$$

That is

$$\frac{(y\lambda_2^2 - x\lambda_2 + x)(y\lambda_2^2 - x\lambda_2 + 2x)}{y(y\lambda_2 + x)}$$

Is a positive integer.

Expanding and dividing we will obtain

$$y\lambda_2^3 - 3x\lambda_2^2 + \frac{\lambda_2^2(3xy + 4x^2)}{y} - \frac{(6x^2y + 4x^3)}{y^2} \text{ rem } \frac{2x^2y^2 + 6x^3y + 4x^4}{y^2}$$

That is:

$$y\lambda_2^3 - 3x\lambda_2^2 + \frac{\lambda_2^2(3xy + 4x^2)}{y} - \frac{(6x^2y + 4x^3)}{y^2} + \frac{2x^2y^2 + 6x^3y + 4x^4}{y^3(y\lambda_2 + x)} \quad (3.2.2.1)$$

Now (3.2.2.1) will be an integer if

$$\frac{2x^2y^2 + 6x^3y + 4x^4}{y^3(y\lambda_2 + x)} = \frac{2x^2 \left(1 + \frac{3x}{y} + 2\frac{x^2}{y^2} \right)}{y(y\lambda_2 + x)} \quad (3.2.2.2)$$

Is an integer.

For $x = 3$ and $y = 2$,

Equation (3.2.2.2) in this case is:

$$\frac{90}{2\lambda_2 + 3}$$

In this case there is only one non-trivial $3 - (v, k, 1)$ which we get as follows: the possible values of λ_2 are 3 or 21.

We give corresponding values of λ_1, k, v , and b in the table below.

Table 3.2.2.1. Case 3; for $x = 3$ and $y = 2$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
3	3	6	5	10
21	15	411	275	7535

For $x = 5$, and $y = 2$

In this case the equation (3.2.2.2) becomes:

$$\frac{525}{2\lambda_2 + 5}$$

The corresponding values of $\lambda_1, \lambda_2, k, v$, and b are as in the table below:

Table 3.2.2.2. Case 3; for $x = 5$ and $y = 2$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
5	3	15	7	35
15	7	190	77	2090
35	15	1140	457	34732
85	35	7015	2807	562603

For $x = 7$, and $y = 2$ (3.2.2.2) becomes

$$\frac{1764}{2\lambda_2 + 7}$$

And we get the following table.

Table 3.2.2.3. Case 3; for $x = 7$ and $y = 2$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
7	3	28	9	84
21	7	371	107	5671
217	63	46333	13239	9736549

CONSTRUCTION OF $t - (v, k, 1)$ DESIGNS WITH $t = 3$ and $\lambda_i \geq 1$

In this section we extend the work in Onyango (2010) by constructing 3 - designs with , that $\lambda_i \geq 1$ is for general index and Steiner 4 – designs

3.3.1. CONSTRUCTION OF $3 - (v, k, 1)$

Here $t = 3$, $\lambda_t = c$ and from (3.1.2) we have:

$$\lambda_2 = \frac{c(v-2)}{k-2} \quad \text{for } \lambda_2 > 1 \quad (3.3.1.1)$$

Now when $t = 2$ we have:

$$\frac{\lambda_1}{\lambda_2} = \frac{v-1}{k-1} \quad (3.3.1.2)$$

Which implies;

$$\lambda_1 = \frac{\lambda_2(v-1)}{k-1} ; \Rightarrow \lambda_1 = \alpha (v-1)$$

And

$$\lambda_2 = \frac{\lambda_1(k-1)}{v-1} ; \Rightarrow \lambda_2 = \alpha (k-1) \quad (3.3.1.3)$$

As mentioned earlier, α is a rational number since $\lambda_1, \lambda_2, c, v-1$ and $k-1$ are all positive integers hence, we will represent it by $\frac{x}{y}$ where x and y are positive integers.

3.3.2. CASE I, $x = 1$

Then (3.3.1.3) becomes:

$$\lambda_1 = \frac{v-1}{y}, \Rightarrow v = y\lambda_1 + 1$$

And

$$\lambda_2 = \frac{k-1}{y}, \Rightarrow k = y\lambda_2 + 1$$

Now from (3.3.1.1) we have:

$$\lambda_2 = \frac{c(v-2)}{k-2}$$

But

$$v = y\lambda_1 + 1, \text{ and } k = y\lambda_2 + 1$$

Hence

$$\lambda_2 = \frac{c(y\lambda_1 - 1)}{y\lambda_2 - 1}$$

Solving for λ_1 we obtain:

$$\lambda_1 = \frac{y\lambda_2^2 - \lambda_2 + c}{cy}$$

And

$$v = \frac{y\lambda_2^2 - \lambda_2 + 2c}{c}$$

For this design to be $3 - (v, k, c)$ design and from $bk = v\lambda_1$; it implies

$$\frac{(y\lambda_2^2 - \lambda_2 + c)}{cy} \left(\frac{y\lambda_2^2 - \lambda_2 + 2c}{c} \right) \equiv 0 \pmod{y\lambda_2 + 1}$$

That is,

$$\frac{(y\lambda_2^2 - \lambda_2 + c)(y\lambda_2^2 - \lambda_2 + 2c)}{c^2 y (y\lambda_2 + 1)}$$

Is a positive integer.

Expanding and dividing we have:

$$\frac{\lambda_2^3}{c^2} - \frac{3\lambda_2^2}{c^2 y} + \frac{\lambda_2(3cy + 4)}{c^2 y^2} - \frac{(6cy + 4)}{c^2 y^3} + \frac{2c^2 y^2 + 6cy + 4}{c^2 y^2 (y^2 \lambda_2 + y)} \quad (3.3.2.1)$$

Now (3.3.2.1) will be an integer if $c^2 y^2$ divides $6cy + 4$

For $c = 2$, that is $\lambda_c = 2$

The only possible values for y in which this is possible are 1 and 2. Using;

$$\frac{2c^2 y^2 + 6cy + 4}{c^2 y^2 (y^2 \lambda_2 + y)} \quad (3.3.2.2)$$

For $y = 2$ we have:

$$\frac{15}{8(2\lambda_2 + 1)}$$

In this case (3.3.2.2.) is not an integer.

For $y = 1$ the equation (3.3.2.2) becomes:

$$\frac{6}{\lambda_2 + 1}$$

Thus (3.3.2.2) will be an integer if λ_2 takes only of the following values; 2 and 5.

The table below gives corresponding values of λ_1, k, v and b

Table 3.2.2.1. Case 1; for $y = 1$ and $c = 2$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
2	3	2	3	2
5	6	11	12	22

The first $3 - (3,3,2)$ does not exist hence we have only one $3 - (v, k, c)$ design in this case. This $3 - (12,6,2)$ is identified with this BIB design $B(12,6,5)$.

3.3.2. CASE I, $y = 1$

In this case (3.3.1.3) becomes:

$$\lambda_1 = x(v-1), \quad \Rightarrow v = \frac{\lambda_1 + x}{x}$$

And

$$\lambda_2 = x(k-1), \quad \Rightarrow k = \frac{\lambda_2 + x}{x} \quad (3.3.3.1)$$

Now from (3.3.1.1) we have:

$$\lambda_2 = \frac{c(\lambda_1 - x)}{\lambda_2 - x}$$

Solving for λ_1 we have:

$$\lambda_1 = \frac{\lambda_2^2 - x\lambda_2 + cx}{c}$$

Now

$$v = \frac{\lambda_2^2 - x\lambda_2 + 2cx}{cx}$$

Using similar argument as before:

$$\frac{(\lambda_2^2 - x\lambda_2 + xc)(\lambda_2^2 - x\lambda_2 + 2cx)}{c^2(\lambda_2 + x)}$$

Will be integer if

$$\frac{2c^2x^2 + 6cx^3 + 4x^4}{c^2(\lambda_2 + x)} \quad (3.3.3.2)$$

Is an integer. Note that when $x = 1$ we get results of the previous case.

For $c = 2$ and $x = 2$ equation (3.3.3.2) is

$$\frac{48}{\lambda_2 + 2}, \quad \lambda_2 > 2$$

We give the corresponding values of $\lambda_2, \lambda_1, v, k$ and b in the table below;

Table 3.3.3.1. Case 2; for $x = 2$ and $c = 2$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
4	3	6	4	8
6	4	14	8	28
10	6	42	22	154
14	8	86	44	473
22	12	222	112	2072
46	24	1014	508	21463

The following designs;

$3 - (4,3,2), 3 - (8,4,2), 3 - (22,6,2), 3 - (44,8,2), 3 - (112,12,2),$ and $3 - (508,24,2)$ can be obtained from $BIB(v,k,\lambda)$ designs given below:

$B(4,3,4), B(8,4,6), B(22,6,10), B(44,8,14), B(112,12,22),$ and $B(508,24,46)$.

For $c = 2, x = 3$ and using similar arguments as before we give the values of $\lambda_2, \lambda_1, v, k$ and b as in the table below:

Table 3.3.3.2. Case 2; for $x = 3$ and $c = 2$ the possible cases of $3 - (v,k,1)$ designs

λ_2	k	λ_1	v	b
6	3	12	5	20
12	5	57	20	228
15	6	93	32	496
27	10	327	110	3597

We get the desired $3 - (v,k,c)$ designs from BIB designs below:

$B(5,3,6), B(20,5,12), B(32,6,15)$ and $(110,10,27)$.

For $c = 2, x = 4$ and using the same methods we give the table of values of; $\lambda_2, \lambda_1, k, v$ and b as follows:

Table 3.3.3.3. Case 2; for $x = 4$ and $c = 2$ the possible cases of $3 - (v,k,1)$ designs

λ_2	k	λ_1	v	b
8	3	20	6	40
12	4	52	14	182
16	5	100	26	520
20	6	168	42	1126
36	10	580	146	8468
44	12	884	222	16354
56	15	1460	366	35624

Similarly, for $c = 2, x = 5$ the values of $\lambda_1, \lambda_2, k, v$ and b we give them as in the table below:

Table 3.3.3.4. Case 2; for $x = 5$ and $c = 2$ the possible cases of $3 - (v,k,1)$ designs

λ_2	k	λ_1	v	b
10	3	30	7	70
20	5	155	32	992
25	6	255	52	2210
30	7	380	77	4180
65	14	1955	392	54740

Now, for $c = 3, x = 3$ and $x = 6$ and applying the same methods, we give values of $\lambda_2, \lambda_1, v, k$ and b in the tables below respectively.

Table 3.3.3.5. Case 2; for $x = 3$ and $c = 3$ the possible cases of $3 - (v,k,1)$ designs

λ_2	k	λ_1	v	b
6	3	9	4	12
9	4	21	8	42
15	6	63	22	231
24	9	171	58	1102
51	18	819	274	12467

Table 3.3.3.6. Case 2; for $x = 6$ and $c = 3$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
12	3	30	6	60
18	4	78	14	273
24	5	150	26	780
30	6	246	42	1722

Also for $c = 4, x = 2, x = 4$ and $x = 6$ and using similar arguments, we give values of $\lambda_2, \lambda_1, v, k$, and b in the tables below respectively.

Table 3.3.3.7. Case 2; for $x = 2$ and $c = 4$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
4	3	4	3	4
6	4	8	5	10
10	6	22	12	44
22	12	112	57	532

The first $3 - (3, 3, 4)$ design is trivial.

Table 3.3.3.8. Case 2; for $x = 4$ and $c = 4$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
12	4	28	8	56
20	6	84	22	308
28	8	172	44	946

Table 3.3.3.9. Case 2; for $x = 6$ and $c = 4$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
18	4	60	11	165
24	5	114	20	456
30	6	186	32	992

When $c = 5, x = 5$ and $x = 10$ and using similar methods the values of $\lambda_2, \lambda_1, v, k$ and b we give them in the tables below respectively.

Table 3.3.3.10. Case 2; for $x = 5$ and $c = 5$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
15	4	35	8	70
20	5	65	14	182
25	6	105	22	385
45	10	365	74	2701

Table 3.3.3.11. Case 2; for $x = 10$ and $c = 5$ the possible cases of $3 - (v, k, 1)$ designs

λ_2	k	λ_1	v	b
30	4	130	14	455
40	5	250	26	1300
50	6	410	42	2870

or $c = 6, x = 3$ and $x = 6$ we get the following tables respectively.

Conclusions:

In this study a new recursive technique has been developed for the construction of $t - (v, k, \lambda_t)$ designs. It has also clearly shown that every $t -$ design is also a BIB (v, k, λ_t) design. Therefore, this construction technique also generates BIBDs. Thus, the study has presented an alternative method that is simpler and unified for the construction of BIBDs that are very important in the experimental designs. As it provides designs for different values of k , unlike many methods that provide designs for a single value of k . Moreover, it provides both Steiner and non-Steiner designs.

Recommendation:

Although this study has provided a technique for the construction of $t -$ designs, it is still clear that construction method of $t -$ designs is not known in general. In fact, it is not clear how one might construct $t -$ designs with arbitrary block size. We therefore invite researchers to come up with "additive theorems" for this construction to make it general for any value of t as this may bring in new techniques and ideas. There is also need for obtaining a theorem which would give all values of x and y for the case three in this construction in order to see new Steiner $t -$ designs. Lastly, if there is a computer package that could be incorporated in the method to aid in calculations

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