SUM CONSTRUCTION OF AUTOMORPHIC BIBDS AND THEIR APPLICATIONS IN EXPERIMENTAL DESIGNS

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ABSTRACT: In this paper, a construction equivalent to “sum construction“ of BIBDs where BIBD is added to a BIBD that is automorphic to it is presented. The result is that we can get new BIBDs by forming the collection of a BIBD with its automorphic BIBDs. A new recursive technique has been developed for the construction of \( t-(v,k,\lambda_t) \) designs. It has also clearly shown that every \( t \) -design is also a BIB \( (v,k,\lambda) \) design. Therefore, this construction technique also generates BIBDs. Therefore, this note presents an alternative method that is simpler and unified for the construction of BIBDs that are very important in the experimental designs. As it provides designs for different values of \( k \), unlike many methods that provide designs for a single value of \( k \). Moreso, it provides both Steiner and non- Steiner designs.

Introduction

The following are properties of \( t-(v,k,\lambda_t) \) designs:

1. \( bk = \lambda_1 v \)
2. \( \lambda_t(v-(t-1)) = \lambda_{t-1}(k-(t-1)) \)

Indeed, number of ways of choosing \( k \) out \( v \) points is \( \binom{v}{k} \). And any point will occur in \( \binom{v-1}{k-1} \) blocks. But we require each point to be incident to \( \lambda_t \) blocks. Thus:

\[
b = \frac{\lambda_1 \binom{v}{k}}{\binom{v-1}{k-1}}
\]

Expanding R.H.S. we obtain:

\[
b = \lambda_1 \frac{v!}{k!(v-k)!} \left( \frac{v-1}{(k-1)!(v-1-(k-1))} \right) \left( \frac{k-1}{(v-1)!} \right) \left( \frac{v-1}{(k-1)!} \right) \left( \frac{k}{(v-k)!} \right) \left( \frac{v-1}{(v-1)!} \right) \\
= \lambda_1 v \frac{v!}{k!(v-k)!} \left( \frac{k-1}{(v-1)!} \right) \left( \frac{v-1}{(k-1)!} \right) \left( \frac{k}{(v-k)!} \right) \left( \frac{v-1}{(v-1)!} \right) \\
= \lambda_1 v \frac{v!}{k!(v-k)!} \left( \frac{k-1}{(v-1)!} \right) \left( \frac{v-1}{(k-1)!} \right) \left( \frac{k}{(v-k)!} \right) \left( \frac{v-1}{(v-1)!} \right) \\
= \lambda_1 v \frac{v!}{k!(v-k)!} \left( \frac{k-1}{(v-1)!} \right) \left( \frac{v-1}{(k-1)!} \right) \left( \frac{k}{(v-k)!} \right) \left( \frac{v-1}{(v-1)!} \right) \\
= \lambda_1 v \frac{v!}{k!(v-k)!} \left( \frac{k-1}{(v-1)!} \right) \left( \frac{v-1}{(k-1)!} \right) \left( \frac{k}{(v-k)!} \right) \left( \frac{v-1}{(v-1)!} \right)
\]

Hence, \( bk = \lambda_1 v \) as required

Similarly from the fact that \( t \) points are incident to \( \lambda_t \) blocks we have:
Also

Combining equation (i) and (ii) we have:

\[ \lambda_t \left( \frac{(v)}{(k-t)} \right) = \lambda_{t-1} \left( \frac{(v)}{(k-t-1)} \right) \tag{ii} \]

Expanding both sides we obtain:

\[
\lambda_t \frac{v!}{k! (v-1)!} \frac{(k-t)!}{(v-t)!(v-t-(k-t))!} = \lambda_{t-1} \frac{v!}{k! (v-k)!} \frac{(v-(t-1))!}{(k-(t-1))!(v-t-1-(k-(t-1)))!}
\]

This reduces to:

\[
\lambda_t \frac{v!}{k! (v-1)!} \frac{(k-t)!}{(v-t)!} = \lambda_{t-1} \frac{v!}{k!} \frac{(k-(t-1))!}{(v-(t-1))!}
\]

Multiplying both sides by \( \frac{k!}{v!} \) we have

\[
\lambda_t \frac{(k-t)!}{(v-t)!} = \lambda_{t-1} \frac{(k-(t-1))!}{(v-(t-1))!}
\]

Expanding R.H.S. and simplifying we have:

\[
\lambda_t \frac{(k-t)!}{(v-t)!} = \lambda_{t-1} \frac{(k-(t-1))!(k-t)!}{(v-(t-1))(v-t)!}
\]

Multiplying both sides by \( \frac{(v-t)!}{(k-t)!} \) we have:

\[
\lambda_t = \lambda_{t-1} \frac{(k-(t-1))}{(v-(t-1))}
\]

Hence,

\[
\lambda_t (v-(t-1)) = \lambda_{t-1} (k-(t-1))
\]

as required.

Equation (3.2) could be generalized as follows: starting with \( t = 6 \) we have:

\[
\lambda_6 (v-5) = \lambda_5 (k-5)
\]

\[
\lambda_5 (v-4) = \lambda_4 (k-4) , \quad t = 5
\]

\[
\lambda_4 (v-3) = \lambda_3 (k-3) , \quad t = 4
\]

\[
\lambda_3 (v-2) = \lambda_2 (k-2) , \quad t = 3
\]

\[
\lambda_2 (v-1) = \lambda_1 (k-1) , \quad t = 2
\]
Now:

\[ \lambda_2 = \frac{\lambda_1(k-1)}{v-1} \]

\[ \lambda_3 = \frac{\lambda_1(k-1)(k-2)}{(v-1)(v-2)} \]

\[ \lambda_4 = \frac{\lambda_1(k-1)(k-2)(k-3)}{(v-1)(v-2)(v-3)} \]

\[ \lambda_5 = \frac{\lambda_1(k-1)(k-2)(k-3)(k-4)}{(v-1)(v-2)(v-3)(v-4)} \]

\[ \lambda_6 = \frac{\lambda_1(k-1)(k-2)(k-3)(k-4)(k-5)}{(v-1)(v-2)(v-3)(v-4)(v-5)} \]

\[ \lambda_s(v-1)(v-2)(v-3)(v-4)(v-5) = \lambda_1(k-1)(k-2)(k-3)(k-4)(k-5) \]

For \( s = 1, 2, \ldots, t - 1 \) we get

\[ \lambda_s(v-(t-1))(v-(t-2))(v-(t-3))\ldots(v-(t-s)) = \lambda_{t-s}(k-(t-1))(k-(t-2))(k-(t-3))\ldots(k-(t-s)) \]

MAIN RESULTS CONSTRUCTION OF 3- \((v,k,1)\)-DESIGN

Here \( t = 3 \) and \( \lambda_t = 1 \) and putting this in equation (3.1.2) we have:

\[ v - 2 = \lambda_2(k - 2) \]

Hence

\[ \lambda_2 = \frac{v - 2}{k - 2} \] (3.2.1)

Now when \( t = 2 \) we have:

\[ \lambda_2(v-1) = \lambda_1(k-1) \]

That is

\[ \frac{\lambda_1}{\lambda_2} = \frac{v-1}{k-1} \] (3.2.2)

This implies

\[ \lambda_1 = \lambda_2 \frac{(v-1)}{(k-1)} \; ; \; \lambda_1 = \infty (v-1) \]

And

\[ \lambda_2 = \lambda_1 \frac{(k-1)}{(v-1)} \] (3.2.3)

Given that \( \lambda_1, \lambda_2v - 1 \) and \( k - 1 \) are all integers and \( \infty \) is a rational number which we will represent by \( x/y \) where \( x \) and \( y \) are positive integers. Thus the equations (3.5) become:

\[ \lambda_1 = \frac{x}{y}(v-1) \]

\[ y\lambda_1 = x(v-1) \]

And

\[ \lambda_2 = \frac{x}{y}(k-1) \]
\[ y\lambda_2 = x(k - 1) \]  

Case 1: \( x = 1 \)

Then (3.2.4) becomes

\[ y\lambda_1 = v - 1, \quad \implies v = y\lambda_1 + 1 \]

And

\[ y\lambda_2 = k - 1, \quad \implies k = y\lambda_2 + 1 \]

**THEOREM 3.2.1.** IF \( x = 1 \) and \( \lambda_2 - 1 \equiv 0 (\mod y) \), where \( y \) is an integer then there are only three non-trivial \( 3 - (v, k, 1) \) designs which are: \( 3 - (8, 4, 1), 3 - (22, 6, 1) \) and \( 3 - (112, 12, 1) \)

**Proof:** Now from (3.2.1) we know

\[ \lambda_2 = \frac{v - 2}{k - 2} \]

But \( v = y\lambda_1 + 1 \) and \( k = y\lambda_2 + 1 \)

Hence

\[ \lambda_2 = \frac{y\lambda_1 + 1 - 2}{y\lambda_2 + 1 - 2} = \frac{y\lambda_1 - 1}{y\lambda_2 - 1} \]

This implies

\[ \lambda_1 = \lambda_2^2 - \frac{\lambda_2 - 1}{y} \]

\[ v = y\lambda_2^2 - \lambda_2 + 2 \]

And

\[ k = y\lambda_2 + 1 \]

For this design to be design \( 3 - (v, k, 1) \) and from (3.1.1) it implies

\[ \left( \lambda_2^2 - \frac{\lambda_2 - 1}{y} \right)(y\lambda_2^2 - \lambda_2 + 2) \equiv 0 \mod(y\lambda_2 + 1) \]

That is

\[ \frac{(y\lambda_2^2 - \lambda_2 + 1)(y\lambda_2^2 - \lambda_2 + 2)}{y(y\lambda_2^2 + 1)} \]

Is a positive integer.

Expanding and simplifying we have:

\[ \frac{y^2\lambda^4 - 2y\lambda^3 + \lambda^2 (3y + 1) - 3\lambda_2 + 2}{y^2\lambda_2 + y} \]

\[ = \lambda_2^3 - \frac{3\lambda_2^2}{y} + \frac{\lambda_2 (3y + 4)}{y^2} - \frac{6y + 4}{y^3} \rem \frac{2y^2 + 6y + 4}{y^2} \]

Which can be written as:

\[ \lambda_2^3 - \frac{3\lambda_2^2}{y} + \frac{\lambda_2 (3y + 4)}{y^2} - \frac{6y + 4}{y^3} + \frac{2y^2 + 6y + 4}{y^2 (y^2\lambda_2 + y)} \]  

\[ (3.2.5) \]

Now

\[ \frac{2y^2 + 6y + 4}{y^2 (y^2\lambda_2 + y)} \]  

\[ (3.2.6) \]
will be an integer if \( y^2 \text{ divides } 6y + 4 \). The only values for \( y \) in which this is possible are 1 and 2.

\[
\frac{2y^2 + 6y + 4}{y^2(\lambda_2 + y)} = \frac{3}{2\lambda_2 + 1}
\]

\( \lambda_2 > 1 \)

In this case (3.2.6) is not an integer.

For \( y = 1 \) (3.2.6) is

\[
\frac{12}{\lambda_2 + 1}
\]

Thus both (3.2.5) and (3.2.6) will be integers if \( \lambda_2 \) takes the values 2, 3, 5 and 11.

The table below gives corresponding values of \( k, \lambda_1, v \) and \( b \).

<table>
<thead>
<tr>
<th>( \lambda_2 )</th>
<th>( k )</th>
<th>( \lambda_1 )</th>
<th>( v )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>7</td>
<td>8</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>21</td>
<td>22</td>
<td>77</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td>111</td>
<td>112</td>
<td>1036</td>
</tr>
</tbody>
</table>

The required 3 – \((v, k, 1)\) designs are; 3 – (8,4,1), 3 – (22,6,1), and 3 – (112,12,1).

A 3 – (4,3,1) is trivial given \( t = k \), but it is required that \( t < k \), hence is not included and our proof is completed.

The above 3 – \((v, k, 1)\) designs are related with the following BIB designs: 2 – (4,3,2), 2 – (8,4,3), 2 – (22,6,5), 2 – (112,12,11). For example the configuration of 2 – (8,4,3) which is 3 – (8,4,1) is:

\[
B1(1,2,3,4), B2(1,2,5,6), B3(1,2,7,8), B4(1,3,5,7), B5(1,3,6,8), B6(1,4,5,8), B7(1,4,6,7), B8(2,3,5,8), B9(2,3,6,8), B10(2,4,5,7), B11(2,4,6,8), B12(3,4,5,6), B13(3,4,7,8), B14(5,6,7,8).
\]

### 3.2.1 Case 2, \( y = 1 \)

In this case (3.2.4) is

\[
\lambda_1 = x(v - 1), \quad \Rightarrow v = \frac{\lambda_1 + x}{x}
\]

\[
\lambda_2 = x(k - 1), \quad \Rightarrow k = \frac{\lambda_2 + x}{x}
\]

Where is \( x \) a positive integer and both \( \lambda_1 \) and \( \lambda_2 \) are divisible by \( x \).

Now from (3.2.1)

\[
\lambda_2 = \frac{v - 2}{k - 2}
\]

we get \( \lambda_1, v, and k \) as follows;

\[
\lambda_1 = \lambda_2^2 - x\lambda_2 + x,
\]

\[
v = \frac{\lambda_2^2 - x\lambda_2 + 2x}{x}
\]

and

\[
k = \frac{\lambda_2 + x}{x}
\]
Using \( b = \frac{\lambda_1 v}{k} \). for this design to be \( 3 - (v,k,1) \) design then:

\[
(\lambda^2 - x\lambda_2 + x) \frac{(\lambda^2 - x\lambda_2 + 2x)}{x} \equiv 0 \mod \left(\frac{\lambda_2 + x}{x}\right)
\]

That is

\[
\frac{(\lambda^2 - x\lambda_2 + x)(\lambda^2 - x\lambda_2 + 2x)}{\lambda_2 + x}
\]

is a positive integer.
Expanding and dividing we will have:

\[
\lambda^2 - 3x\lambda^2 + \lambda_2(3x + 4x^2) - (6x^2 + 4x^3)\text{rem} 2x^2 + 6x^3 + 4x^4
\]

That is

\[
\lambda^2 - 3x\lambda^2 + \lambda_2(3x + 4x^2) - (6x^2 + 4x^3) + \frac{2x^2 + 6x^3 + 4x^4}{\lambda_2 + x}
\]

Now \((3.2.1.1)\) will be integer if

\[
\frac{2x^2 + 6x^3 + 4x^4}{\lambda_2 + x}
\]

For \( x = 2 \) equation \((3.2.1.2)\) is:

\[
\frac{120}{\lambda_2 + 2}, \lambda_2 > 2
\]

Thus \( \lambda_2 \) can take any of the following values: 3, 4, 6, 8, 10, 18, 28, 32, 38, 58, and 118. But \( \lambda_2 \) must be divisible by 2. So 3 is not a possibility. We give corresponding values of \( k, \lambda_1, v \) and \( b \) in the table below.

<table>
<thead>
<tr>
<th>( \lambda_2 )</th>
<th>( k )</th>
<th>( \lambda_1 )</th>
<th>( v )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
<td>10</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>26</td>
<td>14</td>
<td>91</td>
</tr>
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<td>8</td>
<td>5</td>
<td>50</td>
<td>26</td>
<td>260</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>82</td>
<td>42</td>
<td>574</td>
</tr>
<tr>
<td>18</td>
<td>10</td>
<td>290</td>
<td>146</td>
<td>4234</td>
</tr>
<tr>
<td>22</td>
<td>12</td>
<td>442</td>
<td>222</td>
<td>8177</td>
</tr>
<tr>
<td>28</td>
<td>15</td>
<td>730</td>
<td>366</td>
<td>17812</td>
</tr>
<tr>
<td>38</td>
<td>20</td>
<td>1370</td>
<td>686</td>
<td>46991</td>
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<td>58</td>
<td>30</td>
<td>3250</td>
<td>1626</td>
<td>176150</td>
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<td>118</td>
<td>60</td>
<td>13690</td>
<td>6846</td>
<td>1562029</td>
</tr>
</tbody>
</table>

The following designs \( 3 - (6,3,1), 3 - (14,1,1), 3 - (26,5,1), 3 - (42,6,1), 3 - (146,10,1) \), \( 3 - (222,12,1), 3 - (366,15,1), 3 - (686,20,1), 3 - (1676,30,1) \) and \( 3 - (6846,60,1) \) can then be obtained from \( BIB(v,k,\lambda) \) designs given below.

\[
2 - (6,3,4), 2 - (14,4,6), 2 - (26,5,8), 2 - (42,6,10), 2 - (146,10,18), 2 - (222,12,22), 2 - (366,15,28), 2 - (686,20,38), 2 - (1626,30,58) \text{ and } 2 - (6846,60,118).
\]

For \( x = 3 \) equation \((3.2.1.2)\) is

\[
\frac{504}{\lambda_2 + 3}, \lambda_2 > 3
\]

thus the possible values of \( \lambda_2 \) where, \( \lambda_2 \) is divisible by 3 are: 6, 9, 15, 18, 21, 33, 39, 60, 69, 81, 123, 165, 249, and 501.

We give corresponding values of \( \lambda_1, v, k, \) and \( b \) in the table below.
Table 3.2.1.2. Case 2; for $x = 3$ the possible cases of $3 - (v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
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<td>9</td>
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<td>249501</td>
<td>83168</td>
<td>123514876</td>
</tr>
</tbody>
</table>

Again we obtain $3 - (v, k, 1)$ designs from BIB designs below:
$2 - (8,3,6), 2 - (20,4,9), 2 - (67,6,15), 2 - (92,7,18), 2 - (128,8,21),$
$2 - (332,12,33), ..., and 2 - (83168,168,501)$

For $x = 4$, and using similar arguments as before we obtain possible values of $\lambda_2$ as follows: $8, 12, 16, 20, 28, 32, 36, ...$ which give the values of $\lambda_1, k, v$ and $b$ as follows:

Table 3.2.1.3. Case 2; for $x = 4$ the possible cases of $3 - (v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
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<td>36</td>
<td>10</td>
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</tr>
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<td>36</td>
<td>10</td>
<td>1156</td>
<td>290</td>
<td>33524</td>
</tr>
</tbody>
</table>

The desired $3 - (v, k, 1)$ designs are obtained from the following BIB designs:
$2 - (10,3,8), 2 - (26,4,12), 2 - (50,5,16), 2 - (82,6,20), 2 - (170,8,28), 2 - (226,9,32),$
$\text{and} 2 - (290,10,36)$

3.2.2. Case $3, x \neq y \quad x > 1, y > 1$

In this case (3.2.4) is

$$\lambda_1 = \frac{x}{y} (v - 1) \quad \Rightarrow v = \frac{y\lambda_1 + x}{x}$$
$$\lambda_2 = \frac{x}{y} (k - 1) \quad \Rightarrow k = \frac{y\lambda_2 + x}{x}$$

Where $x$ and $y$ are positive integers and $\lambda_2$ and $y$ are divisible by $x$.

Now from (3.2.1)

$$\lambda_2 = \frac{v - 2}{k - 2}$$

We get $\lambda_1, v, and k$ as follows:

$$\lambda_1 = \frac{y\lambda_2^2 - x\lambda_2 + x}{y}$$
\[ v = \frac{y\lambda_1^2 - x\lambda_2 + 2x}{x} \]

And
\[ k = \frac{y\lambda_2 + x}{x} \]

Using \( b = \frac{\lambda_1 v}{k} \) for this to be \( 3 - (v, k, 1) \) design then
\[
\frac{(y\lambda_2^2 - x\lambda_2 + x)(y\lambda_1^2 - x\lambda_2 + 2x)}{y} \equiv 0 \mod \left( \frac{y\lambda_2 + x}{x} \right)
\]

That is
\[
\frac{(y\lambda_2^2 - x\lambda_2 + x)(y\lambda_1^2 - x\lambda_2 + 2x)}{y(y\lambda_2 + x)}
\]

Is a positive integer.

Expanding and dividing we will obtain
\[
y\lambda_2^2 - 3x\lambda_2^2 + \frac{\lambda_2^2(3xy + 4x^2)}{y} - \frac{(6x^2y + 4x^3)}{y^2} \text{ rem } \frac{2x^2y^2 + 6x^3y + 4x^4}{y^2}
\]

That is:
\[
y\lambda_2^2 - 3x\lambda_2^2 + \frac{\lambda_2^2(3xy + 4x^2)}{y} - \frac{(6x^2y + 4x^3)}{y^2} + \frac{2x^2y^2 + 6x^3y + 4x^4}{y^3(y\lambda_2 + x)}
\]

Now (3.2.2.1) will be an integer if
\[
\frac{2x^2y^2 + 6x^3y + 4x^4}{y^3(y\lambda_2 + x)} = \frac{2x^2\left(1 + \frac{3x}{y} + \frac{2x^2}{y^2}\right)}{y(y\lambda_2 + x)}
\]

(3.2.2.2)

Is an integer.

For \( x = 3 \) and \( y = 2 \),

Equation (3.2.2.2) in this case is:
\[
\frac{90}{2\lambda_2 + 3}
\]

In this case there is only one non-trivial \( 3 - (v, k, 1) \) which we get as follows: the possible values of \( \lambda_2 \) are 3 or 21.

We give corresponding values of \( \lambda, k, v, \text{ and } b \) in the table below.

**Table 3.2.2.1.** Case 3; for \( x = 3 \) and \( y = 2 \) the possible cases of \( 3 - (v, k, 1) \) designs

<table>
<thead>
<tr>
<th>( \lambda_2 )</th>
<th>( k )</th>
<th>( \lambda_1 )</th>
<th>( v )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>21</td>
<td>15</td>
<td>411</td>
<td>275</td>
<td>7535</td>
</tr>
</tbody>
</table>

For \( x = 5, \text{ and } y = 2 \),

In this case the equation (3.2.2.2) becomes:
\[
\frac{525}{2\lambda_2 + 5}
\]

The corresponding values of \( \lambda_1, \lambda_2, k, v, \text{ and } b \) are as in the table below:
Table 3.2.2. Case 3; for $x = 5$ and $y = 2$ the possible cases of $3-(v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>15</td>
<td>7</td>
<td>35</td>
</tr>
<tr>
<td>15</td>
<td>7</td>
<td>190</td>
<td>77</td>
<td>2090</td>
</tr>
<tr>
<td>35</td>
<td>15</td>
<td>1140</td>
<td>457</td>
<td>34732</td>
</tr>
<tr>
<td>85</td>
<td>35</td>
<td>7015</td>
<td>2807</td>
<td>562603</td>
</tr>
</tbody>
</table>

For $x = 7$ and $y = 2$ (3.2.2.2) becomes

\[
\frac{1764}{2\lambda_2 + 7}
\]

And we get the following table.

Table 3.2.3. Case 3; for $x = 7$ and $y = 2$ the possible cases of $3-(v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>3</td>
<td>28</td>
<td>9</td>
<td>84</td>
</tr>
<tr>
<td>21</td>
<td>7</td>
<td>371</td>
<td>107</td>
<td>5671</td>
</tr>
<tr>
<td>217</td>
<td>63</td>
<td>46333</td>
<td>13239</td>
<td>9736549</td>
</tr>
</tbody>
</table>

CONSTRUCTION OF $t-(v,k,1)$ DESIGNS WITH $t = 3$ and $\lambda_t \geq 1$

In this section we extend the work in Onyango (2010) by constructing 3 - designs with $\lambda_t \geq 1$ is for general index and Steiner 4 – designs

3.3.1. CONSTRUCTION OF $3-(v,k,1)$

Here $t = 3$, $\lambda_t = c$ and from (3.1.2) we have:

\[
\lambda_2 = \frac{c(v-2)}{k-2} \quad \text{for } \lambda_2 > 1
\]  

(3.3.1.1)

Now when $t = 2$ we have:

\[
\frac{\lambda_1}{\lambda_2} = \frac{v-1}{k-1}
\]  

(3.3.1.2)

Which implies;

\[
\lambda_1 = \frac{\lambda_2(v-1)}{k-1} \implies \lambda_1 = \alpha (v - 1)
\]

And

\[
\lambda_2 = \frac{\lambda_1(k-1)}{v-1} \implies \lambda_2 = \alpha (k - 1)
\]  

(3.3.1.3)

As mentioned earlier, $\alpha$ is a rational number since $\lambda_1, \lambda_2, c, v - 1$ and $k - 1$ are all positive integers hence, we will represent it by $\frac{x}{y}$ where $x$ and $y$ are positive integers.

3.3.2. CASE 1, $x = 1$

Then (3.3.1.3) becomes:

\[
\lambda_1 = \frac{v - 1}{y}, \implies v = y\lambda_1 + 1
\]

And

\[
\lambda_2 = \frac{k - 1}{y}, \implies k = y\lambda_2 + 1
\]

Now from (3.3.1.1) we have:
\[ \lambda_2 = \frac{c(v - 2)}{k - 2} \]

But
\[ v = y\lambda_1 + 1, \text{ and } k = y\lambda_2 + 1 \]

Hence
\[ \lambda_2 = \frac{c(y\lambda_1 - 1)}{y\lambda_2 - 1} \]

Solving for \( \lambda_2 \) we obtain:
\[ \lambda_1 = \frac{y\lambda_2^2 - \lambda_2 + c}{cy} \]

And
\[ \nu = \frac{y\lambda_2^2 - \lambda_2 + 2c}{c} \]

For this design to be \( 3 - (v, k, c) \) design and from \( bk = \nu\lambda_1 \); it implies
\[ \frac{(y\lambda_2^2 - \lambda_2 + c)}{cy} \left( \frac{y\lambda_2^2 - \lambda_2 + 2c}{c} \right) \equiv 0 \text{ mod } (y\lambda_2 + 1) \]

That is,
\[ \frac{(y\lambda_2^2 - \lambda_2 + c)(y\lambda_2^2 - \lambda_2 + 2c)}{c^2y(y\lambda_2 + 1)} \]

Is a positive integer.

Expanding and dividing we have:
\[ \frac{\lambda_2^3}{c^2} - \frac{3\lambda_2^2}{c^2y} + \frac{\lambda_2(3cy + 4)}{c^2y^2} - \frac{(6cy + 4)}{c^2y^3} + \frac{2c^2y^2 + 6cy + 4}{c^2y^2(y^2\lambda_2 + y)} \]

(3.3.2.1)

Now (3.3.2.1) will be an integer if \( c^2y^2 \) divides \( 6cy + 4 \)

For \( c = 2 \), that is \( \lambda_2 = 2 \)

The only possible values for \( y \) in which this is possible are 1 and 2. Using;
\[ \frac{2c^2y^2 + 6cy + 4}{c^2y^2(y^2\lambda_2 + y)} \]

(3.3.2.2)

For \( y = 2 \) we have:
\[ \frac{15}{8(2\lambda_2 + 1)} \]

In this case (3.3.2.2.) is not an integer.

For \( y = 1 \) the equation (3.3.2.2) becomes:
\[ \frac{6}{\lambda_2 + 1} \]

Thus (3.3.2.2) will be an integer if \( \lambda_2 \) takes only of the following values; 2 and 5.

The table below gives corresponding values of \( \lambda_1, k, v \) and \( b \)

**Table 3.2.2.1.** Case 1: for \( y = 1 \) and \( c = 2 \) the possible cases of \( 3 - (v,k,1) \) designs

<table>
<thead>
<tr>
<th>( \lambda_2 )</th>
<th>( k )</th>
<th>( \lambda_1 )</th>
<th>( v )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>11</td>
<td>12</td>
<td>22</td>
</tr>
</tbody>
</table>
The first $3 - (3,3,2)$ does not exist hence we have only one $3 - (v,k,c)$ design in this case. This $3 - (12,6,2)$ is identified with this BIB design $B(12,6,5)$.

### 3.3.2. CASE I, $y = 1$

In this case (3.3.1.3) becomes:

$$\lambda_1 = x(v - 1), \quad \Rightarrow v = \frac{\lambda_1 + x}{x}$$

And

$$\lambda_2 = x(k - 1), \quad \Rightarrow k = \frac{\lambda_2 + x}{x}$$

Now from (3.3.1.1) we have:

$$\lambda_2 = \frac{c(\lambda_1 - x)}{\lambda_2 - x}$$

Solving for $\lambda_1$ we have:

$$\lambda_1 = \frac{\lambda_2^2 - x\lambda_2 + cx}{c}$$

Now

$$v = \frac{\lambda_2^2 - x\lambda_2 + 2cx}{cx}$$

Using similar argument as before:

$$\frac{(\lambda_2^2 - x\lambda_2 + xc)(\lambda_2^2 - x\lambda_2 + 2cx)}{c^2(\lambda_2 + x)}$$

Will be integer if

$$\frac{2c^2 x^2 + 6cx^3 + 4x^4}{c^2(\lambda_2 + x)}$$

(3.3.3.2)

Is an integer. Note that when $x = 1$ we get results of the previous case.

For $c = 2$ and $x = 2$ equation (3.3.3.2) is

$$\frac{48}{\lambda_2 + 2}, \quad \lambda_2 > 2$$

We give the corresponding values of $\lambda_2, \lambda_1, v, k$ and $b$ in the table below:

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>14</td>
<td>8</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>42</td>
<td>22</td>
<td>154</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
<td>86</td>
<td>44</td>
<td>473</td>
</tr>
<tr>
<td>22</td>
<td>12</td>
<td>222</td>
<td>112</td>
<td>2072</td>
</tr>
<tr>
<td>46</td>
<td>24</td>
<td>1014</td>
<td>508</td>
<td>21463</td>
</tr>
</tbody>
</table>
The following designs;  
\[3 - (4,3,2), 3 - (8,4,2), 3 - (22,6,2), 3 - (44,8,2), 3 - (112,12,2), \text{and} 3 - (508,24,2)\]  
can be obtained from BIB\((v,k,\lambda)\) designs given below;  
\(B(4,3,4), B(8,4,6), B(22,6,10), B(44,8,14), B(112,12,22), \text{and} B(508,24,46).\)  
For \(c = 2, x = 3\) and using similar arguments as before we give the values of \(\lambda_2, \lambda_1, v, k \text{ and } b\) as in the table below;  

**Table 3.3.3.2.** Case 2; for \(x = 3\) and \(c = 2\) the possible cases of \(3 - (v,k,1)\) designs

<table>
<thead>
<tr>
<th>(\lambda_2)</th>
<th>(k)</th>
<th>(\lambda_1)</th>
<th>(v)</th>
<th>(B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>3</td>
<td>12</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>57</td>
<td>20</td>
<td>228</td>
</tr>
<tr>
<td>15</td>
<td>6</td>
<td>93</td>
<td>32</td>
<td>496</td>
</tr>
<tr>
<td>27</td>
<td>10</td>
<td>327</td>
<td>110</td>
<td>3597</td>
</tr>
</tbody>
</table>

We get the desired \(3 - (v,k,c)\) designs from BIB designs below;  
\(B(5,3,6), B(20,5,12), B(32,6,15)\) and \((110,10,27).\)  
For \(c = 2, x = 4\) and using the same methods we give the table of values of \(\lambda_2, \lambda_1, k, v \text{ and } b\) as follows;  

**Table 3.3.3.3.** Case 2; for \(x = 4\) and \(c = 2\) the possible cases of \(3 - (v,k,1)\) designs

<table>
<thead>
<tr>
<th>(\lambda_2)</th>
<th>(k)</th>
<th>(\lambda_1)</th>
<th>(v)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>3</td>
<td>20</td>
<td>6</td>
<td>40</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>52</td>
<td>14</td>
<td>182</td>
</tr>
<tr>
<td>16</td>
<td>5</td>
<td>100</td>
<td>26</td>
<td>520</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>168</td>
<td>42</td>
<td>1126</td>
</tr>
<tr>
<td>36</td>
<td>10</td>
<td>580</td>
<td>146</td>
<td>8468</td>
</tr>
<tr>
<td>44</td>
<td>12</td>
<td>884</td>
<td>222</td>
<td>16354</td>
</tr>
<tr>
<td>56</td>
<td>15</td>
<td>1460</td>
<td>35624</td>
<td></td>
</tr>
</tbody>
</table>

Similarly, for \(c = 2, x = 5\) the values of \(\lambda_1, \lambda_2, k, v \text{ and } b\) we give them as in the table below;  

**Table 3.3.3.4.** Case 2; for \(x = 5\) and \(c = 2\) the possible cases of \(3 - (v,k,1)\) designs

<table>
<thead>
<tr>
<th>(\lambda_2)</th>
<th>(k)</th>
<th>(\lambda_1)</th>
<th>(v)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3</td>
<td>30</td>
<td>7</td>
<td>70</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>155</td>
<td>32</td>
<td>992</td>
</tr>
<tr>
<td>25</td>
<td>6</td>
<td>255</td>
<td>52</td>
<td>2210</td>
</tr>
<tr>
<td>30</td>
<td>7</td>
<td>380</td>
<td>77</td>
<td>4180</td>
</tr>
<tr>
<td>65</td>
<td>14</td>
<td>1955</td>
<td>392</td>
<td>54740</td>
</tr>
</tbody>
</table>

Now, for \(c = 3, x = 3\) and \(x = 6\) and applying the same methods, we give values of \(\lambda_2, \lambda_1, v, k \text{ and } b\) in the tables below respectively.  

**Table 3.3.3.5.** Case 2; for \(x = 3\) and \(c = 3\) the possible cases of \(3 - (v,k,1)\) designs

<table>
<thead>
<tr>
<th>(\lambda_2)</th>
<th>(k)</th>
<th>(\lambda_1)</th>
<th>(v)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>3</td>
<td>9</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>21</td>
<td>8</td>
<td>42</td>
</tr>
<tr>
<td>15</td>
<td>6</td>
<td>63</td>
<td>22</td>
<td>231</td>
</tr>
<tr>
<td>24</td>
<td>9</td>
<td>171</td>
<td>58</td>
<td>1102</td>
</tr>
<tr>
<td>51</td>
<td>18</td>
<td>819</td>
<td>274</td>
<td>12467</td>
</tr>
</tbody>
</table>
Table 3.3.6.  Case 2; for $x = 6$ and $c = 3$ the possible cases of $3 - (v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>3</td>
<td>30</td>
<td>6</td>
<td>60</td>
</tr>
<tr>
<td>18</td>
<td>4</td>
<td>78</td>
<td>14</td>
<td>273</td>
</tr>
<tr>
<td>24</td>
<td>5</td>
<td>150</td>
<td>26</td>
<td>780</td>
</tr>
<tr>
<td>30</td>
<td>6</td>
<td>246</td>
<td>42</td>
<td>1722</td>
</tr>
</tbody>
</table>

Also for $c = 4, x = 2, x = 4$ and $x = 6$ and using similar arguments, we give values of $\lambda_2, \lambda_1, v, k, and b$ in the tables below respectively.

Table 3.3.7.  Case 2; for $x = 2$ and $c = 4$ the possible cases of $3 - (v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>8</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>22</td>
<td>12</td>
<td>44</td>
</tr>
<tr>
<td>22</td>
<td>12</td>
<td>112</td>
<td>57</td>
<td>532</td>
</tr>
</tbody>
</table>

The first $3 - (3,3,4)$ design is trivial.

Table 3.3.8.  Case 2; for $x = 4$ and $c = 4$ the possible cases of $3 - (v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>4</td>
<td>28</td>
<td>8</td>
<td>56</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>84</td>
<td>22</td>
<td>308</td>
</tr>
<tr>
<td>28</td>
<td>8</td>
<td>172</td>
<td>44</td>
<td>946</td>
</tr>
</tbody>
</table>

Table 3.3.9.  Case 2; for $x = 6$ and $c = 4$ the possible cases of $3 - (v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>4</td>
<td>60</td>
<td>11</td>
<td>165</td>
</tr>
<tr>
<td>24</td>
<td>5</td>
<td>114</td>
<td>20</td>
<td>456</td>
</tr>
<tr>
<td>30</td>
<td>6</td>
<td>186</td>
<td>32</td>
<td>992</td>
</tr>
</tbody>
</table>

When $c = 5, x = 5$ and $x = 10$ and using similar methods the values of $\lambda_2, \lambda_1, v, k$ and $b$ we give them in the tables below respectively.

Table 3.3.10.  Case 2; for $x = 5$ and $c = 5$ the possible cases of $3 - (v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>4</td>
<td>35</td>
<td>8</td>
<td>70</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>65</td>
<td>14</td>
<td>182</td>
</tr>
<tr>
<td>25</td>
<td>6</td>
<td>105</td>
<td>22</td>
<td>385</td>
</tr>
<tr>
<td>45</td>
<td>10</td>
<td>365</td>
<td>74</td>
<td>2701</td>
</tr>
</tbody>
</table>

Table 3.3.11.  Case 2; for $x = 10$ and $c = 5$ the possible cases of $3 - (v,k,1)$ designs

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\lambda_1$</th>
<th>$v$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>4</td>
<td>130</td>
<td>14</td>
<td>455</td>
</tr>
<tr>
<td>40</td>
<td>5</td>
<td>250</td>
<td>26</td>
<td>1300</td>
</tr>
<tr>
<td>50</td>
<td>6</td>
<td>410</td>
<td>42</td>
<td>2870</td>
</tr>
</tbody>
</table>

or $c = 6, x = 3$ and $x = 6$ we get the following tables respectively.
Conclusions:

In this study a new recursive technique has been developed for the construction of \( t-(v,k,\lambda_t) \) designs. It has also clearly shown that every \( t \) design is also a BIB \( (v,k,\lambda_t) \) design. Therefore, this construction technique also generates BIBDs. Thus, the study has presented an alternative method that is simpler and unified for the construction of BIBDs that are very important in the experimental designs. As it provides designs for different values of \( k \), unlike many methods that provide designs for a single value of \( k \). Moreso, it provides both Steiner and non-Steiner designs.

Recommendation:

Although this study has provided a technique for the construction of \( t \) designs, it is still clear that construction method of \( t \) designs is not known in general. In fact, it is not clear how one might construct \( t \) designs with arbitrary block size. We therefore invite researchers to come up with “additive theorems “for this construction to make it general for any value of.\( x \) as this may bring in new techniques and ideas. There is also need for obtaining a theorem which would give all values of \( x \) and \( y \) for the case three in this construction in order to see new Steiner \( t \) designs. Lastly, if there is a computer package that could be incorporated in the method to aid in calculations.

REFERENCES: